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Continuous or Discontinuous Deformations of C^*-Algebras

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ABSTRACT

We study deformations of C^*-algebras that become continuous or discontinuous.

RESUMEN

Estudiamos deformación de C^*-algebras que son continuas o discontinuas.

Key words and phrases: *C^*-algebra, Continuous field, Crossed product.*

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INTRODUCTION

Continuous fields of C^*-algebras have been of interest in the theory of C^*-algebras (see Dixmier [5, Chapter 10]). In particular, continuous field C^*-algebras of continuous trace with Hausdorff spectrums are well studied to classify them. In this case the continuous fields of C^*-algebras become locally trivial and they are built up by trivial continuous field C^*-algebras that are tensor products of the C^*-algebras of continuous functions on their base spaces with some fixed fibers. Continuous deformations of C^*-algebras are in a particular case of continuous fields of C^*-algebras in the sense that their base spaces are the closed interval $[0,1]$ and the fibers on the half open interval $(0,1]$ are the same (cf. E-theory in Blackadar [1]). It has been known that continuous deformations of C^*-algebras may have non-Hausdorff spacetrums in general ([5, 10]).

It is first obtained in [10] that there exists no continuous deformation from a C^*-algebra generated by isometries to a C^*-algebra generated by unitaries, in particular, no continuous deformation from Cuntz and Toeplitz algebras to the C^*-algebras of continuous functions on the tori. In this paper we investigate some interesting properties for continuous or discontinuous deformations of C^*-algebras beyond the result of [10], but using its ideas. We find it convenient to divide continuous deformations of C^*-algebras into two classes. One consists of degenerate continuous deformations of C^*-algebras and the other does of nondegenerate continuous deformations of C^*-algebras, that we define later. We find that it is easy to have degenerate continuous deformations of C^*-algebras, some of which are useful to provide some examples with non-Hausdorff spectrums, and it is not easy to construct nondegenerate continuous deformations of C^*-algebras. Indeed, we find that there exists no nondegenerate continuous deformations in some cases as given below.

In Section 1 we forcus on degenerate or nondegenerate continuous deformations of C^*-algebras. In Secion 2 we give some nondegenerate discontinuous deformations of C^*-algebras by considering crossed product C^*-algebras by the integer group $\mathbb{Z}$ and the real group $\mathbb{R}$ and by semigroup crossed product C^*-algebras by the semigroup(s) of natural numbers, which would be of interest.

Refer to Dixmier [5], Pedersen [8] and Murphy [7] for details of the C^*-algebra theory.

1 Continuous deformations of C^*-algebras

Recall that a continuous deformation from a C^*-algebra $\mathfrak{A}$ to another $\mathfrak{B}$ means a continuous field C^*-algebra $\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})$ on the closed interval $[0,1]$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_0 = \mathfrak{B}$ and $\mathfrak{A}_t = \mathfrak{A}$ for $0 < t \leq 1$, where the continuous field C^*-algebra is defined and generated by giving continuous operator fields on $[0,1]$ such that their norm at fibers are continuous and the set of (or generated by) their evaluations at each point $t \in [0,1]$ is dense in $\mathfrak{A}_t$. Refer to [5] for details of continuous fields of C^*-algebras.

Definition 1.1 We say that a continuous deformation from a C^*-algebra $\mathfrak{A}$ to another $\mathfrak{B}$ is degenerate if there exist continuous operator fields coming from some generators of $\mathfrak{A}$ that are zero at $0 \in [0,1]$. We say that a continuous deformation from a C^*-algebra $\mathfrak{A}$ to another $\mathfrak{B}$ is nondegenerate if it is not degenerate, i.e., there exist no continuous operator fields coming from generators of $\mathfrak{A}$ that are zero at $0 \in [0,1]$.

Proposition 1.2 *Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras. Assume that we have the following splitting exact sequence*: $0 \to C_0((0,1],\mathfrak{A}) \to E \to \mathfrak{B} \to 0$, *where $C_0((0,1],\mathfrak{A})$ is the C^*-algebra of continuous $\mathfrak{A}$-valued functions on the half open interval* $(0,1]$. *Then the extension E is a continuous deformation from $\mathfrak{A}$ to $\mathfrak{B}$.*

Remark. A continuous deformation from $\mathfrak{A}$ to $\mathfrak{B}$ has the same decomposition as the extension E above, but its extension is not necessarily splitting.

Example 1.3 Let $\mathfrak{A}$ be a unital C^*-algebra. Then we have the following natural splitting exact sequence: $0 \to C_0((0,1],\mathfrak{A}) \to E \to \mathbb{C} \to 0$, where the unit operator field f defined by $f(t) = 1 \in \mathfrak{A}$ for $(0,1]$ and $f(0) = 1 \in \mathbb{C}$ is continuous in E. This continuous deformation is degenerate if $\mathfrak{A} \neq \mathbb{C}$ and nondegenerate if $\mathfrak{A} = \mathbb{C}$.

Degenerate continuous deformations

Theorem 1.4 *Let $\mathfrak{A}$ be a C^*-algebra. Suppose that $\mathfrak{A}$ has a non-trivial projection p, and let $p\mathfrak{A}p$ denote the C^*-subalgebra of $\mathfrak{A}$ generated by the elements pap for $a \in \mathfrak{A}$. Then there exists a continuous deformation from $\mathfrak{A}$ to $p\mathfrak{A}p$. Also, if $\mathfrak{A}$ is unital, then there exists a continuous deformation from $\mathfrak{A}$ to $p\mathfrak{A}p \oplus (1-p)\mathfrak{A}(1-p)$, where $1-p$ can be replaced with a projection of $\mathfrak{A}$ orthogonal to p.*

Proof. We construct a continuous field C^*-algebra $\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_t = \mathfrak{A}$ for $0 < t \leq 1$ and $\mathfrak{A}_0 = p\mathfrak{A}p$ as follows. Assume that constant continuous operator fields f on $p\mathfrak{A}p$ such as $f(t) = f(s) \in p\mathfrak{A}p$ for $t, s \in [0,1]$ are contained in $\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})$. And assume that other continuous operator fields of $\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})$ vanish at zero.

More concretely, we can take the other way to prove the statement in the case that $\mathfrak{A}$ is a unital C^*-algebra as follows. Then any element $a \in \mathfrak{A}$ can be viewed as the following matrix:

$$a \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

for $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, and $a_{22} = (1-p)a(1-p)$. Thus, we take the following matrix functions as continuous operator fields of $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$:

$$a(t) \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \quad \text{with} \quad a(0) \begin{pmatrix} pap & 0 \\ 0 & 0 \end{pmatrix}$$

for $t \in [0,1]$ such that $a(1) = a$.

For the second assertion, we just replace $a_{22}(0) = 0$ with $a_{22}(0) = (1-p)a(1-p)$. □

Example 1.5 There exists a continuous deformation from the matrix algebra $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$ for $n \geq m \geq 1$ by Theorem 1.4 since $M_m(\mathbb{C}) \cong pM_n(\mathbb{C})p$ for p a rank m projection of $M_n(\mathbb{C})$. Also, there exists a continuous deformation from the matrix algebra $M_n(\mathbb{C})$ to $\mathbb{C}^k$, where $1 \leq k \leq n$ by choosing k orthogonal rank 1 projections of $M_n(\mathbb{C})$. Note that this continuous deformation has non Hausdorff spectrum if $k \geq 2$.

There exists a continuous deformation from the C^*-algebra $\mathbb{K}$ of compact operators to $M_m(\mathbb{C})$ for any $m \geq 1$ by Theorem 1.4 since $M_m(\mathbb{C}) \cong p(\mathbb{K})p$ for p a rank m projection of $\mathbb{K}$. Also, there exists a continuous deformation from the C^*-algebra $\mathbb{K}$ to $\mathbb{C}^k$ $(k \geq 1)$ and to $C_0(\mathbb{N})$ the C^*-algebra of sequences vanishing at infinity.

Let $\mathfrak{A}$ be an AF algebra, i.e., an inductive limit of finite dimensional C^*-algebras (or finite direct sums of matrix algebras over $\mathbb{C}$). Then, as shown in Theorem 1.4 there exists a continuous deformation from $\mathfrak{A}$ to its C^*-subalgebra $M_m(\mathbb{C})$ for some $m \geq 1$.

Let $\mathfrak{A} \oplus \mathfrak{B}$ be the direct sum of C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$. Then there exists a continuous deformation from $\mathfrak{A} \oplus \mathfrak{B}$ to $\mathfrak{A}$.

Theorem 1.6 *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{B}$ a unital C^*-algebra. Then there exists a continuous deformation from the C^*-tensor product $\mathfrak{A} \otimes \mathfrak{B}$ with a C^*-norm to $\mathfrak{A}$.*

Proof. Note that any C^*-tensor product $\mathfrak{A} \otimes \mathfrak{B}$ with a certain C^*-norm is generated by simple tensors $a \otimes b$ for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. We construct a continuous field C^*-algebra $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_t = \mathfrak{A} \otimes \mathfrak{B}$ for $t \in (0,1]$ and $\mathfrak{A}_0 = \mathfrak{A}$ as follows. Since $\mathfrak{B}$ is unital, we assume that the constant operator fields on $\mathfrak{A} \cong \mathfrak{A} \otimes \mathbb{C}$ in $\mathfrak{A} \otimes \mathfrak{B}$ are continuous and other continuous operator fields vanish at zero. □

Example 1.7 Let $C(\mathbb{T}^n)$ be the C^*-algebra of continuous functions on the n-torus $\mathbb{T}^n$ $(n \geq 0)$, where $C(\mathbb{T}^0) = \mathbb{C}$. Then there exists a continuous deformation from $C(\mathbb{T}^n)$ to $C(\mathbb{T}^m)$ for $n > m \geq 0$ since $C(\mathbb{T}^n) \cong C(\mathbb{T}^m) \otimes C(\mathbb{T}^{n-m})$.

As for crossed product C^*-algebras by groups,

Theorem 1.8 *Let $\mathfrak{A}$ be a unital C^*-algebra, Γ a discrete group and $\mathfrak{A} \rtimes_\alpha \Gamma$ the full crossed product C^*-algebra by an action α of Γ on $\mathfrak{A}$. Then there exists a continuous deformation from $\mathfrak{A} \rtimes_\alpha \Gamma$ to either $\mathfrak{A}$ or the full group C^*-algebra $C^*(\Gamma)$ of Γ. Moreover, there exists a continuous deformation from the reduced crossed product C^*-algebra $\mathfrak{A} \rtimes_{\alpha,r} \Gamma$ to either $\mathfrak{A}$ or the reduced group C^*-algebra $C_r^*(\Gamma)$ of Γ.*

Proof. Note that the full crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha \Gamma$ is generated by $\mathfrak{A}$ and $C^*(\Gamma)$, and $\mathfrak{A}$ and $C^*(\Gamma)$ are C^*-subalgebras of $\mathfrak{A} \rtimes_\alpha \Gamma$. We assume that the constant operator fields on $\mathfrak{A}$ (or $C^*(\Gamma)$) in $\mathfrak{A} \rtimes_\alpha \Gamma$ are continuous and other continuous operator fields vanish at zero. Also, we can replace $\mathfrak{A} \rtimes_\alpha \Gamma$ with $\mathfrak{A} \rtimes_{\alpha,r} \Gamma$ and $C^*(\Gamma)$ with $C_r^*(\Gamma)$ respectively. □

Theorem 1.9 *Let $\mathfrak{A}$ be a unital C^*-algebra, G a locally compact group and $\mathfrak{A} \rtimes_\alpha G$ the full crossed product C^*-algebra by an action α of G on $\mathfrak{A}$. Then there exists a continuous deformation from $\mathfrak{A} \rtimes_\alpha G$ to the full group C^*-algebra $C^*(G)$ of G. Moreover, there exists a continuous deformation from the reduced crossed product C^*-algebra $\mathfrak{A} \rtimes_{\alpha,r} G$ to the reduced group C^*-algebra $C_r^*(G)$ of G.*

Proof. Note that the full crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha G$ is generated by elements af for $a \in \mathfrak{A}$ and $f \in C^*(G)$, and $C^*(G)$ is a C^*-subalgebra of $\mathfrak{A} \rtimes_\alpha G$. We assume that the constant operator fields on $C^*(G)$ in $\mathfrak{A} \rtimes_\alpha G$ are continuous and other continuous operator fields vanish at zero. Also, we can replace $\mathfrak{A} \rtimes_\alpha G$ with $\mathfrak{A} \rtimes_{\alpha,r} G$ and $C^*(G)$ with $C_r^*(G)$ respectively. □

As for free products of C^*-algebras,

Theorem 1.10 *Let $\mathfrak{A}$, $\mathfrak{B}$ be unital C^*-algebras. Then there exists a continuous deformation from the (full or reduced) unital free product C^*-algebra $\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}$ (an amalgam over $\mathbb{C}$) to $\mathfrak{A}$.*

Proof. Note that the (full or reduced) unital free product C^*-algebra $\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}$ is generated by $\mathfrak{A}$ and $\mathfrak{B}$, where the unit of $\mathfrak{A}$ is identified with that of $\mathfrak{B}$. We construct a continuous field C^*-algebra $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_t = \mathfrak{A} *_{\mathbb{C}} \mathfrak{B}$ for $t \in (0,1]$ and $\mathfrak{A}_0 = \mathfrak{A}$ by assuming the constant operator fields on $\mathfrak{A}$ in $\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}$ are continuous and other continuous operator fields vanish at zero. □

Example 1.11 Let $C^*(F_2)$ be the full group C^*-algebra of the free group F_2 with two generators (see Davidson [4]). Then there exists a continuous deformation from $C^*(F_2)$ to $C(\mathbb{T})$ since $C^*(F_2) \cong C^*(\mathbb{Z}) *_{\mathbb{C}} C^*(\mathbb{Z})$ and $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ by the Fourier transform.

Nondegenerate continuous deformations

Example 1.12 Let H_3 be the real 3-dimensional Heisenberg Lie group and $C^*(H_3)$ its group C^*-algebra. Since H_3 is isomorphic to a semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$, we have $C^*(H_3) \cong C^*(\mathbb{R}^2) \rtimes \mathbb{R} \cong C_0(\mathbb{R}^2) \rtimes \mathbb{R}$ by the Fourier transform. Then it is known that $C^*(H_3)$ can be viewed as the continuous field C^*-algebra $\Gamma_0(\mathbb{R}, \{\mathfrak{A}_t\}_{t\in\mathbb{R}})$ with fibers $\mathfrak{A}_t = \mathbb{K}$ for $t \neq 0$ and $\mathfrak{A}_0 = C_0(\mathbb{R}^2)$ since $\mathfrak{A}_t \cong C_0(\mathbb{R}) \rtimes_{\alpha^t} \mathbb{R} \cong \mathbb{K}$ for $t \neq 0$ where the action α^t of $\mathbb{R}$ on $\mathbb{R}$ is a shift and $\mathfrak{A}_0 \cong C_0(\mathbb{R}) \rtimes_{\alpha^0} \mathbb{R} \cong C_0(\mathbb{R}^2)$ since the action α^0 of $\mathbb{R}$ on $\mathbb{R}$ is trivial. Therefore, the restriction of this continuous field C^*-algebra to $[0,1]$ gives a continuous deformation from $\mathbb{K}$ to $C_0(\mathbb{R}^2)$.

Let H_{2n+1} be the real $(2n+1)$-dimensional generalized Heisenberg Lie group and $C^*(H_{2n+1})$ its group C^*-algebra. Since H_{2n+1} is isomorphic to a semi-direct product $\mathbb{R}^{n+1} \rtimes \mathbb{R}^n$, we have $C^*(H_{n+1}) \cong C^*(\mathbb{R}^{n+1}) \rtimes \mathbb{R}^n \cong C_0(\mathbb{R}^{n+1}) \rtimes \mathbb{R}^n$ by the Fourier transform. Then it is known that $C^*(H_{2n+1})$ can be viewed as the continuous field C^*-algebra $\Gamma_0(\mathbb{R}, \{\mathfrak{A}_t\}_{t\in\mathbb{R}})$ with fibers $\mathfrak{A}_t = \mathbb{K}$ for $t \neq 0$ and $\mathfrak{A}_0 = C_0(\mathbb{R}^{2n})$ since $\mathfrak{A}_t \cong C_0(\mathbb{R}^n) \rtimes_{\alpha^t} \mathbb{R}^n \cong \mathbb{K}$ for $t \neq 0$ where the action α^t of $\mathbb{R}^n$ on $\mathbb{R}^n$ is a shift and $\mathfrak{A}_0 \cong C_0(\mathbb{R}^n) \rtimes_{\alpha^0} \mathbb{R}^n \cong C_0(\mathbb{R}^{2n})$ since the action α^0 of $\mathbb{R}^n$ on $\mathbb{R}^n$ is trivial. Therefore, the restriction of this continuous field C^*-algebra to $[0,1]$ gives a continuous deformation from $\mathbb{K}$ to $C_0(\mathbb{R}^{2n})$.

More generally,

Proposition 1.13 *Let $\mathfrak{A}$ be a C^*-algebra, G a locally compact group and $\mathfrak{A} \rtimes_{\alpha^t} G$ the full crossed product C^*-algebras by actions α^t of G on $\mathfrak{A}$ for $t \in [0,1]$. Suppose that the actions $\{\alpha^t\}_{t\in[0,1]}$ are continuous in the sense that the maps from $t \in [0,1]$ to $\alpha_t(a)$ for $a \in \mathfrak{A}$ are continuous and that $\mathfrak{A} \rtimes_{\alpha^t} G \cong \mathfrak{A} \rtimes_{\alpha^s} G$ for $t, s \in (0,1]$ and α^0 is trivial. Then there exists a continuous deformation from $\mathfrak{A} \rtimes_{\alpha^1} G$ to $\mathfrak{A} \otimes C^*(G)$. Furthermore, similarly we can replace $\mathfrak{A} \rtimes_{\alpha^t} G$ with their reduced crossed product C^*-algebras and $C^*(G)$ with its reduced group C^*-algebra respectively.*

Remark. Even if $G = \mathbb{R}$, the assumption $\mathfrak{A} \rtimes_{\alpha^t} \mathbb{R} \cong \mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$ for $t, s \in (0,1]$ are not true in general. For instance, let $C(\mathbb{T}^2) \rtimes_\theta \mathbb{R}$ be the crossed product C^*-algebra by the action θ of $\mathbb{R}$ on $\mathbb{T}^2$ defined by $\theta_t(z,w) = (e^{2\pi it}z, e^{2\pi i\theta t}w) \in \mathbb{T}^2$ where $\theta \in \mathbb{R}$, which is also called the foliation C^*-algebra of $C(\mathbb{T}^2)$ by $\mathbb{R}$ of Connes [2]. Then it is known that $C(\mathbb{T}^2) \rtimes_\theta \mathbb{R} \cong \mathbb{K} \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z})$, where $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is the rotation algebra corresponding to θ. Moreover, it is known that $C(\mathbb{T}) \rtimes_\theta \mathbb{Z} \cong C(\mathbb{T}) \rtimes_{\theta'} \mathbb{Z}$ if and only if $\theta = \theta'$ or $\theta = 1 - \theta'$ (mod 1).

The proposition above gives a general procedure to construct nondegenerate continuous fields by crossed products C^*-algebras, but it is not easy to have continuous actions

$\{\alpha^t\}_{t\in[0,1]}$ in the sense above and check the isomorphisms of their crossed product C^*-algebras for $t \in (0, 1]$.

As for tensor products of C^*-algebras,

Proposition 1.14 *Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras. Suppose that the C^*-tensor product $\mathfrak{A} \otimes \mathfrak{B}$ with a C^*-norm is isomorphic to $\mathfrak{A}$. Then there exists a continuous deformation from $\mathfrak{A}\otimes\mathfrak{B}$ to $\mathfrak{A}$.*

Example 1.15 We have $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$. A C^*-algebra $\mathfrak{A}$ is stable if $\mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{A}$.

Let $\mathfrak{A}$ be a simple separable nuclear C^*-algebra. Then $\mathfrak{A} \cong \mathfrak{A} \otimes O_\infty$ if and only if $\mathfrak{A}$ is purely infinite, where O_∞ is the Cuntz algebra generated by a sequence of othogonal isometries. A C^*-algebra $\mathfrak{A}$ is simple, separable, unital and nuclear if and only if $\mathfrak{A}\otimes O_2 \cong O_2$, where O_2 is the Cuntz algebra generated by two orthogonal isometires with the sum of their range projections equal to the identity. See Rørdam [9] for these significant results.

2 Discontinuous deformations of C^*-algebras

Nondegenerate discontinuous deformations

Theorem 2.1 *Let $\mathfrak{A}$ be a unital commutative C^*-algebra and $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ the crossed product C^*-algebra of $\mathfrak{A}$ by a non trivial action α of $\mathbb{Z}$. Then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ to $\mathfrak{A}$. If $\mathfrak{A}$ is nonunital and commutative, then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ to $\mathfrak{A}^+$ the unitization of $\mathfrak{A}$ by $\mathbb{C}$.*

Proof. Note that $\mathfrak{A}$ is a C^*-subalgebra of $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ and $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is generated by $\mathfrak{A}$ and a unitary corresponding to the action α of $\mathbb{Z}$. Let U be such a unitary. Then we have the covariance relation: $UaU^* = \alpha_1(a)$ for $a \in \mathfrak{A}$. Suppose that we had a continuous field C^*-algebra $\Gamma([0, 1], \{\mathfrak{A}_t\}_{t\in[0,1]})$ such that $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{A}_t = \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ for $0 < t \le 1$. We may assume that (certain) constant continuous operator fields on $\mathfrak{A}$ (or $\mathfrak{A}^+$ if $\mathfrak{A}$ is non unital) are contained in $\Gamma([0, 1], \{\mathfrak{A}_t\}_{t\in[0,1]})$ (where the argument below is applicable to the case without constant continuous operator fields). Also, we may assume that the operator field f defined by $f(0) = u$ a unitary of $\mathfrak{A}$ (or u a unitary of $\mathfrak{A}^+$ if $\mathfrak{A}$ is nonunital) and $f(t) = U$ for $0 < t \le 1$ is also contained in it. Then the operator field fbf^* for (certain) $b \in \mathfrak{A}$ defined by $fbf^*(t) = f(t)bf^*(t) = UbU^* = \alpha_1(b)$ and $fbf^*(0) = ubu^* = uu^*b = b$ must be continuous. But this is impossible in general since $b \ne \alpha_1(b)$ for some $b \in \mathfrak{A}$ since α is non trivial so that $(b - fbf^*)(t) = b - \alpha_1(b) \ne 0$ for $t \in (0, 1]$ but $(b - fbf^*)(0) = b - b = 0$. □

Example 2.2 Let $C(\mathbb{T})$ be the C^*-algebra of continuous functions on the torus $\mathbb{T}$ and $C(\mathbb{T}) \rtimes_{\alpha^\theta} \mathbb{Z}$ the crossed product C^*-algebra that is called a rotation algebra, where α^θ is induced from the action of $\mathbb{Z}$ on $\mathbb{T}$ by the multiplication $e^{2\pi i\theta t}$ for $t \in \mathbb{Z}$ (see Wegge-Olsen [11]). By Theorem 2.1, there exists no nondegenerate continuous deformation from $C(\mathbb{T}) \rtimes_{\alpha^\theta} \mathbb{Z}$ to $C(\mathbb{T})$.

Moreover, let $C(\mathbb{T}^k) \rtimes_{\alpha^\Theta} \mathbb{Z}$ be the crossed product C^*-algebra (which is one of noncommutative tori) by an action α^Θ by $\mathbb{Z}$ on $C(\mathbb{T}^k)$, where $\Theta = (\theta_j)_{j=1}^k$ and $\alpha_t^\Theta(z_j) = (e^{2\pi i\theta_j t} z_j) \in \mathbb{T}^k$ for $t \in \mathbb{Z}$. Then there exists no nondegenerate continuous deformation from $C(\mathbb{T}^k) \rtimes_{\alpha^\theta} \mathbb{Z}$ to $C(\mathbb{T}^k)$.

Furthermore,

Theorem 2.3 *Let $\mathfrak{A}$ be a unital simple C^*-algebra and $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ the crossed product C^*-algebra of $\mathfrak{A}$ by a non trivial action α of $\mathbb{Z}$. Then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ to $\mathfrak{A}$. If $\mathfrak{A}$ is nonunital and simple, then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ to $\mathfrak{A}^+$ the unitization of $\mathfrak{A}$ by $\mathbb{C}$.*

Proof. Let U be a unitary corresponding to α. Suppose that we had a continuous field C^*-algebra $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$ such that $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{A}_t = \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ for $0 < t \leq 1$. We may assume that the operator field f defined by $f(0) = u$ a unitary of $\mathfrak{A}$ (or u a unitary of $\mathfrak{A}^+$ if $\mathfrak{A}$ is nonunital) and $f(t) = U$ for $0 < t \leq 1$ is also contained in it. Then the operator field fuf^* defined by $fuf^*(t) = f(t)uf^*(t) = UuU^* = \alpha_1(u)$ and $fuf^*(0) = uuu^* = u$ must be continuous. Hence it follows that $\alpha_1(u) = u$ since the operator field $fuf^* - \alpha_1(u)$ is continuous and $(fuf^* - \alpha_1(u))(t) = 0$ for $t \in (0,1]$ so that $(fuf^* - \alpha_1(u))(0) = 0$. Thus, u is fixed under α. Therefore, the C^*-algebra $C^*(u)$ generated by u is fixed under α. Then $\mathfrak{A}$ must have $C^*(u)$ as a nontrivial quotient C^*-algebra, which contradicts to that $\mathfrak{A}$ is simple.

We use the similar argument for the case of $\mathfrak{A}$ nonunital and simple. □

Example 2.4 Let O_n be the Cuntz algebra generated by n orthogonal isometries $\{S_j\}_{j=1}^n$ such that $\sum_{j=1}^n S_j S_j^* = 1$ (see Cuntz [3] or the text books Davidson [4] or Wegge-Olsen [11]). Then by Theorem 2.3 there exists no nondegenerate continuous deformation from $O_n \otimes \mathbb{K}$ to $M_{n^\infty} \otimes \mathbb{K}$, where M_{n^∞} is the UHF algebra. It is known that the C^*-tensor product $O_n \otimes \mathbb{K}$ isomorphic to the crossed product C^*-algebra $(M_{n^\infty} \otimes \mathbb{K}) \rtimes_\alpha \mathbb{Z}$ (see Rørdam [9]).

Moreover,

Theorem 2.5 *Let $\mathfrak{A}$ be an either commutative or simple, unital C^*-algebra and $\mathfrak{A} \rtimes_\alpha \Gamma$ the (reduced or full) crossed product C^*-algebra of $\mathfrak{A}$ by a non trivial action α of Γ a discrete* *group. Then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \Gamma$ to $\mathfrak{A}$. If $\mathfrak{A}$ is nonunital and either commutative or simple, then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \Gamma$ to $\mathfrak{A}^+$ the unitization of $\mathfrak{A}$ by $\mathbb{C}$.*

Proof. Note that the (full or reduced) crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha \Gamma$ is generated by $\mathfrak{A}$ and the unitaries corresponding to generators of Γ and $\mathfrak{A}$ is a C^*-subalgebra of $\mathfrak{A} \rtimes_\alpha \Gamma$. Let U be one of the unitaries. We apply the arguments given in the proofs of Theorems 2.1 and 2.3 for the C^*-algebra generated by $\mathfrak{A}$ and U. Note that U may have torsion in the arguments. □

As for crossed product C^*-algebras by continuous groups,

Theorem 2.6 *Let $\mathfrak{A}$ be an either commutative or simple, unital (or non unital) C^*-algebra and $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ the crossed product C^*-algebra of $\mathfrak{A}$ by a non trivial action α of $\mathbb{R}$. Then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ to $\mathfrak{A}$.*

Proof. Note that the crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ is generated by elements af for $a \in \mathfrak{A}$ and $f \in C^*(\mathbb{R})$. Since $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$ by the Fourier transform, we identify elements of $C^*(\mathbb{R})$ with those of $C_0(\mathbb{R})$. Note that the unitization $C_0(\mathbb{R})^+$ by $\mathbb{C}$ is isomorphic to $C(\mathbb{T})$. Now suppose that we had a continuous field C^*-algebra $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$ such that $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{A}_t = \mathfrak{A} \rtimes_\alpha \mathbb{R}$ for $0 < t \leq 1$. Then we can have a extended continuous field C^*-algebra $\Gamma([0,1], \{\mathfrak{B}_t\}_{t\in[0,1]})$ such that $\mathfrak{B}_0 = \mathfrak{A}$ and $\mathfrak{B}_t$ the C^*-algebra generated by $\mathfrak{A}$ and that $C(\mathbb{T})$ for $0 < t \leq 1$ by assuming that the operator field from the unit of $C(\mathbb{T})$ to the unit of $\mathfrak{A}$ (or of $\mathfrak{A}^+$ if $\mathfrak{A}$ nonunital) is continuous.

Suppose that $\mathfrak{A}$ is commutative. Since α is nontrivial, there exists $b \in \mathfrak{A}$ such that $UbU^* \neq b$. Indeed, if $UbU^* = b$ for any $b \in \mathfrak{A}$, then $\mathfrak{A}$ and $C(\mathbb{T})$ commute. Hence $\mathfrak{A}$ and $C_0(\mathbb{R})$ commute. Thus, $\mathfrak{A} \rtimes_\alpha \mathbb{R} \cong \mathfrak{A} \otimes C^*(\mathbb{R})$ so that α must be trivial. Therefore, we can adopt the argument given in the proof of Theorem 2.1.

Suppose that $\mathfrak{A}$ is simple. On the other hand, by the argument given in the proof of Theorem 2.3, we have $UuU^* = u$, where the operator field from U to $u \in \mathfrak{A}$ is continuous. Thus, the C^*-algebra $C^*(u)$ generated by u commutes with $C(\mathbb{T})$ generated by U. Hence $C^*(u)$ commutes with $C^*(\mathbb{R})$. Then $\mathfrak{A}$ has $C^*(u)$ as a nontrivial quotient C^*-algebra, which is the contradiction. □

Remark. We can replace with $\mathbb{R}$ with $\mathbb{T}$ in the statement above. Note that $C^*(\mathbb{T}) \cong C_0(\mathbb{Z})$ by the Fourier transform and $C_0(\mathbb{Z})^+ \cong C((\mathbb{Z})^+)$, where $(\mathbb{Z})^+$ is the one point compactification of $\mathbb{Z}$ and it is identified with a closed subset of $\mathbb{T}$.

Example 2.7 Let $C^*(H_3)$ be the group C^*-algebra of the real 3-dimensional Heisenberg Lie group H_3. Then $C^*(H_3) \cong C^*(\mathbb{R}^2) \rtimes \mathbb{R} \cong C_0(\mathbb{R}^2) \rtimes \mathbb{R}$ since $H_3 \cong \mathbb{R}^2 \rtimes \mathbb{R}$. Hence there exists no nondegenerate continuous deformation from $C^*(H_3)$ to $C_0(\mathbb{R}^2)$ of $C_0(\mathbb{R}^2) \rtimes \mathbb{R}$.

Furthermore,

Theorem 2.8 *Let $\mathfrak{A}$ be an either commutative or simple, unital (or non unital) C^*-algebra and $\mathfrak{A} \rtimes_\alpha \mathbb{R}^n$ the crossed product C^*-algebra of $\mathfrak{A}$ by an action α of $\mathbb{R}^n$ such that the restriction of α to any factor $\mathbb{R}$ of $\mathbb{R}^n$ is non trivial. Then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{R}^n$ to $\mathfrak{A}$.*

Proof. We use the same process as given in the proof of the theorem above. Note that $\mathfrak{A} \rtimes_\alpha \mathbb{R}^n$ is generated by elements af for $a \in \mathfrak{A}$ and $f \in C^*(\mathbb{R}^n)$, and $C^*(\mathbb{R}^n) \cong C_0(\mathbb{R}^n)$ so that $C_0(\mathbb{R}^n)^+ \cong C((\mathbb{R}^n)^+) \cong C(S^n)$, where $(\mathbb{R}^n)^+$ is the one point compactification of $\mathbb{R}^n$ and S^n is the n-dimensional sphere. Take a unitary U of $C(S^n)$ that corresponds to a coordinate projection from S^n $(n \geq 2)$ to $\mathbb{T}$ and gives a nontrivial action on $\mathfrak{A}$. □

Remark. We can replace with $\mathbb{R}^n$ with $\mathbb{T}^n$ in the statement above. Note that $C^*(\mathbb{T}^n) \cong C_0(\mathbb{Z}^n)$ by the Fourier transform and $C_0(\mathbb{Z}^n)^+ \cong C((\mathbb{Z}^n)^+)$, where $(\mathbb{Z}^n)^+$ is the one point compactification of $\mathbb{Z}^n$ and it is identified with a closed subset of $\mathbb{T}$.

Example 2.9 Let $C^*(H_{2n+1})$ be the group C^*-algebra of the real $(2n+1)$-dimensional Heisenberg Lie group H_{2n+1}. Then $C^*(H_{2n+1}) \cong C^*(\mathbb{R}^{n+1}) \rtimes \mathbb{R}^n \cong C_0(\mathbb{R}^{n+1}) \rtimes \mathbb{R}^n$ since $H_{2n+1} \cong \mathbb{R}^{n+1} \rtimes \mathbb{R}^n$. Hence there exists no nondegenerate continuous deformation from $C^*(H_{2n+1})$ to $C_0(\mathbb{R}^{n+1})$ of $C_0(\mathbb{R}^{n+1}) \rtimes \mathbb{R}^n$.

As for crossed product C^*-algebras by semigroups.

Theorem 2.10 *Let $\mathfrak{A}$ be a unital C^*-algebra with no proper isometries and $\mathfrak{A} \rtimes_\alpha \mathbb{N}$ the semigroup crossed product C^*-algebra of $\mathfrak{A}$ by an action α of the additive semigroup $\mathbb{N}$ of natural numbers by proper isometries. Then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{N}$ to $\mathfrak{A}$. If $\mathfrak{A}$ is non unital and without proper isometries, then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{N}$ to the unitization $\mathfrak{A}^+$ by $\mathbb{C}$.*

Proof. Suppose that we had a continuous field C^*-algebra $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{A}_t = \mathfrak{A} \rtimes_\alpha \mathbb{N}$ for $t \in (0,1]$. Note that $\mathfrak{A} \rtimes_\alpha \mathbb{N}$ is generated by $\mathfrak{A}$ and a proper isometry. Let S be such a isometry. Then we have the covariance relation: $SaS^* = \alpha_1(a)$ for $a \in \mathfrak{A}$. Since $S^*S = 1$ the unit of $\mathfrak{A}$ (and $\mathfrak{A} \rtimes_\alpha \mathbb{N}$) (or $1 \in \mathbb{C}$ of $\mathfrak{A}^+$ if $\mathfrak{A}$ is non unital) the operator field f defined by $f(t) = S^*S$ and $f(0) = 1$ in $\mathfrak{A}$ is continuous. We may assume that the operator field g defined by $g(t) = S$ for $t \in (0,1]$ and $g(0) = a$ an element of $\mathfrak{A}$ is continuous. Then it follows that $a^*a = 1$.

If $a \neq 1$, then the last equation is the contradiction since $\mathfrak{A}$ has no proper isometries.

If $a = 1$, then note that the operator field h defined by $h(t) = SS^*$ for $t \in (0,1]$ and $h(0) = 1$ is continuous since the operator field g is so. Hence, the operator field $f - h$ is also continuous, which is impossible because $f(t) - h(t) = 1 - SS^* \neq 0$ for $t \in (0,1]$ but $f(0) - h(0) = 1 - 1 = 0$. □

Example 2.11 It is known that $O_n \cong M_{n^\infty} \rtimes_\alpha \mathbb{N}$ (see [9]). Since the UHF algebra M_{n^∞} has no proper isometries, we obtain by the theorem above that there exists no nondegenerate continuous deformation from O_n to M_{n^∞}.

Theorem 2.12 *Let $\mathfrak{A}$ be a unital C^*-algebra with no proper isometries and $\mathfrak{A} \rtimes_\alpha \mathbb{N}^\times$ the semigroup crossed product C^*-algebra of $\mathfrak{A}$ by an action α of the multiplicative semigroup $\mathbb{N}^\times$ of natural numbers by proper isometries. Then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{N}^\times$ to $\mathfrak{A}$. If $\mathfrak{A}$ is non unital and without proper isometries, then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{N}^\times$ to the unitization $\mathfrak{A}^+$ by $\mathbb{C}$.*

Proof. Note that the semigroup crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha \mathbb{N}^\times$ is generated by $\mathfrak{A}$ and $C^*(\mathbb{N}^\times)$, and $C^*(\mathbb{N}^\times)$ is isomorphic to the infinite tensor product of $C^*(\mathbb{N})$ over prime numbers since $\mathbb{N}^\times \cong \oplus\mathbb{N}$ over prime numbers, where $C^*(\mathbb{N})$ is the C^*-algebra generated by a proper isometry, which is just the usual Toeplitz algebra. Thus, $\mathfrak{A} \rtimes_\alpha \mathbb{N}$ corresponding to $\mathfrak{A}$ and a certain proper isometry in $C^*(\mathbb{N}^\times)$ is regarded as a C^*-subalgebra of $\mathfrak{A} \rtimes_\alpha \mathbb{N}^\times$. Therefore, we can use the arguments as given in the proof of the theorem above. □

Example 2.13 Following Laca-Raeburn [6], the Hecke C^*-algebra of Bost-Connes is realized as the semigroup crossed product C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^\times$. Thus, we obtain by the theorem above that there exists no nondegenerate continuous deformation from $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^\times$ to $C^*(\mathbb{Q}/\mathbb{Z})$.

Moreover,

Theorem 2.14 *Let $\mathfrak{A}$ be a unital C^*-algebra with no proper isometries and $\mathfrak{A} \rtimes_\alpha N$ the (reduced or full) semigroup crossed product C^*-algebra of $\mathfrak{A}$ by an action α of a discrete semigroup N by proper isometries. Then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha N$ to $\mathfrak{A}$. If $\mathfrak{A}$ is non unital and without proper isometries, then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha N$ to the unitization $\mathfrak{A}^+$ by $\mathbb{C}$.*

Proof. Note that the (reduced or full) semigroup crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha N$ is generated by $\mathfrak{A}$ and isometries corresponding to generators of N. Let S be one of the isometries. We apply the argument given in the proof of Theorem 2.10 for the C^*-algebra generated by $\mathfrak{A}$ and S. □

As for free products of C^*-algebras,

Theorem 2.15 *Let $\mathfrak{A}$ be a C^*-algebra that contains an either unitary or isometry generator. Then there exists no nondegenerate continuous deformation from the (full or reduced) unital free product C^*-algebra $\mathfrak{A} *_{\mathbb{C}} C(\mathbb{T})$ to $C(\mathbb{T})$.*

Proof. Let U be a unitary generator of $\mathfrak{A}$ and V the generating unitary of $C(\mathbb{T})$. We assume that we had a nondegenerate continuous deformation from (full or reduced) free product C^*-algebra $\mathfrak{A} *_{\mathbb{C}} C(\mathbb{T})$ to $C(\mathbb{T})$. Then we may assume that the constant operator field f by V is continuous and the operator field g from U to a certain unitary W of $C(\mathbb{T})$ is also continuous. Then $(fg - gf)(t) = f(t)g(t) - g(t)f(t) = VU - UV \neq 0$ for $t \in (0, 1]$ but $(fg - gf)(0) = f(0)g(0) - g(0)f(0) = VW - WV = 0$ since $C(\mathbb{T})$ is commutative, which leads to the contradiction.

In the argument above we can replace U with a isometry generator S of $\mathfrak{A}$ since we can assume that the operator field from S to a unitary of $C(\mathbb{T})$ is continuous. □

Example 2.16 Since $C^*(F_2) \cong C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$, there exists no nondegenerate continuous deformation from the full group C^*-algebra $C^*(F_2)$ of F_2 to $C(\mathbb{T})$.

Similarly,

Theorem 2.17 *Let $\mathfrak{A}$ be a C^*-algebra that contains an either unitary or isometry generator U, and $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ be the crossed product C^*-algebra by a non trivial action α of $\mathbb{Z}$ on $\mathfrak{A}$. Suppose that $VUV^* \neq U$, where V is the generaing unitary corresponding to α. Then there exists no nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ to $C(\mathbb{T})$.*

Proof. Consider the operator field from $VUV^* - U \neq 0$ to $VWV^* - W = VV^*W - W = 0$, where W is a certain unitary of $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ (by the Fourier transform). If we had a nondegenerate continuous deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ to $C(\mathbb{T})$, this operator field should be continuous but it is impossible. □

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