CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$02, (61–74). July 2008

$C^{(n)}$-Almost Automorphic Solutions of Some Nonautonomous Differential Equations

Khalil Ezzinbi
Université Cadi Ayyad, Faculté des Sciences Semlalia,
Département de Mathématiques, BP. 2390, Marrakech, Morocco
email: ezzinbi@ucam.ac.ma

Valerie Nelson
Department of Mathematics, Morgan State University,
1700 E. Cold Spring Lane, Baltimore, MD 21251, USA
email: valerie.nelson@morgan.edu
and
Gaston N'Guérékata
Department of Mathematics, Morgan State University,
1700 E. Cold Spring Lane, Baltimore, MD 21251, USA
email: gaston.n'guerekata@morgan.edu

ABSTRACT

This paper is concerned with the study of properties of $C^{(n)}$-almost automorphic functions and their uniform spectra. We apply the obtained results to prove Massera type theorems for the nonautonomous differential equation in $\mathbb{C}^k$: $x'(t) = A(t)x(t)+f(t),\ t \in \mathbb{R}$ and $A(t)$ is τ periodic and the equation $x'(t) = Ax(t) + f(t),\ t \in \mathbb{R}$ where the operator A generates a quasi-compact semigroup in a Banach space, and in both cases f is $C^{(n)}$-almost automorphic.

RESUMEN

En este artículo estudiamos las propriedades de funciones $C^{(n)}$-casi automoficas. Aplicamos los resultados obtenidos para provar teoremas de tipo Massera para la ecuación diferencial no autonoma en $\mathbb{C}^k$: $x'(t) = A(t)x(t) + f(t),\ t \in \mathbb{R}$, $A(t)$ es τ-periódica y para la ecuación $x'(t) = Ax(t) + f(t),\ t \in \mathbb{R}$ donde el operador A genera un semigrupo casi compacto en un espacio de Banach, en ambos casos f es una función $C^{(n)}$-casi automorfica.

Key words and phrases: *Evolution equation, mild solution, almost automorphy, uniform spectrum.*

Math. Subj. Class.: *47D06, 34G10, 45M05*

1 Introduction

Let us consider in $\mathbb{C}^k$ equations of the form

$$\frac{dx}{dt} = A(t)x + f(t), \tag{1.1}$$

where $A(t)$ is a (generally unbounded) linear operator which is τ-periodic, and f is a $C^{(n)}$-almost automorphic) function on $\mathbb{R}$. We will prove a Massera type result for the above differential equation and present conditions under which every bounded solution of this equation is $C^{(n+1)}$-almost automorphic.

The concept of $C^{(n)}$-almost automorphic functions was introduced by Ezzinbi, Fatajou and N'Guérékata in [9] as a generalization of $C^{(n)}$-almost periodicity (see for instance [1, 2, 3, 5, 13]).

In their work [9] , the authors study the existence of $C^{(n)}$-almost automorphic solutions, $(n \geq 1)$, for the following partial neutral functional differential equation

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_{_t} + L(u_t) + f(t) \text{ for } t \in \mathbb{R} \tag{1.2}$$

where A is a linear operator on a Banach space X satisfying the following well-known Hille-Yosida condition

$(\mathbf{H_0})$ there exist $\bar{M} \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{\bar{M}}{(\lambda - \omega)^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$. $\mathcal{D} : C \to X$ is a bounded linear operator, where $C = C([-r, 0]\,; X)$ is the space of continuous functions from $[-r, 0]$

to X endowed with the uniform norm topology. For the well posedness of equation (1.2), we assume that $\mathcal{D}$ has the following form

$$\mathcal{D}\varphi = \varphi(0) - \int_{-r}^{0} [d\eta(\theta)]\,\varphi(\theta) \text{ for } \varphi \in C,$$

for a mapping $\eta : [-r, 0] \to \mathcal{L}(X)$ of bounded variation and non atomic at zero, which means that there exists a continuous nondecreasing function $\delta : [0, r] \to [0, +\infty)$ such that $\delta(0) = 0$ and

$$\left| \int_{-s}^{0} [d\eta(\theta)]\,\varphi(\theta) \right| \leq \delta(s) \sup_{-r\leq\theta\leq 0} |\varphi(\theta)| \text{ for } \varphi \in C \text{ and } s \in [0, r]\,,$$

where $\mathcal{L}(X)$ denotes the space of bounded linear operators from X to X. For every $t \geq \sigma$, the history function $u_t \in C$ is defined by

$$u_t(\theta) = u(t + \theta) \text{ for } \theta \in [-r, 0]\,.$$

L is a bounded linear operator from C to X and f is a continuous function from $\mathbb{R}$ to X.

Another important problem studied in [9] is the following Massera type result.

Consider the differential equations

$$\frac{dx}{dt} = Dx(t) + e(t), \tag{1.3}$$

where D is a constant $d \times d$ matrix and $e :\to \mathbb{R}^d$ is $C^{(n)}$-almost automorphic function. Then if Equ. (1.3) has a bounded solution on $\mathbb{R}^+$, it has a $C^{(n+1)}$-almost automorphic solution. Moreover every bounded solution on $\mathbb{R}$ is $C^{(n+1)}$-almost automorphic.

In the present paper we continue the study of elementary properties of $C^{(n)}$-almost automorphic functions and apply them to investigate the $C^{(n)}$-almost automorphic functions solutions to the non autonomous periodic equation (1.1).

The work is organized as follows. In Section 2, we review the concept of $C^{(n)}$-almost periodic functions and present further properties of $C^{(n)}$-almost automorphic functions with values in a Hilbert space. In Section 3, we discuss some results related to the uniform spectrum of $C^{(n)}$-almost automorphic functions. Our main results (Theorem 4.2 and 4.11)are presented in Section 4.

2 $C^{(n)}$-almost periodic and $C^{(n)}$-almost automorphic functions

We recall some properties about $C^{(n)}$-almost periodic and $C^{(n)}$-almost automorphic functions. Let $BC(\mathbb{R}, X)$ be the space of all bounded and continuous functions from $\mathbb{R}$ to X, equipped with the uniform norm topology. Let $h \in BC(\mathbb{R}, X)$ and $\tau \in \mathbb{R}$, we define the function h_τ by

$$h_\tau(s) = h(\tau + s) \text{ for } s \in \mathbb{R}.$$

Let $C^n(\mathbb{R}, X)$ be the space of all continuous function which have a continuous n-th derivative on $\mathbb{R}$ and $C_b^n(\mathbb{R}, X)$ be the subspace of $C^n(\mathbb{R}, X)$ of functions satisfying

$$\sup_{t\in\mathbb{R}} \sum_{i=0}^{n} \|h^{(i)}(t)\| < \infty,$$

$h^{(i)}$ denotes the i-the derivative of h. Then $C_b^n(\mathbb{R}, X)$ is a Banach space provided with the following norm

$$\|h\|_n = \sup_{t\in\mathbb{R}} \sum_{i=0}^{n} \|h^{(i)}(t)\|.$$

Definition 2.1. A bounded continuous function $h : \mathbb{R} \to X$ is said to be almost periodic if

$$\{h_\tau : \tau \in \mathbb{R}\} \text{ is relatively compact in } BC(\mathbb{R}, X).$$

Definition 2.2. A continuous function $\theta : \mathbb{R}\times X \to X$ is said to be almost periodic in t uniformly in x if for any compact K in X and for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ such that

$$\lim_{n\to\infty} \theta(t + s_n, x) \text{ exists uniformly in } (t, x) \in \mathbb{R}\times K.$$

Definition 2.3. [3] Let $\varepsilon > 0$ and $h \in C_b^n(\mathbb{R}, X)$. A number $\tau \in \mathbb{R}$ is said to be a $\|\cdot\|_n - \varepsilon$ almost period of the function f if

$$\|h_\tau - h\|_n < \varepsilon.$$

The set of all $\|\cdot\|_n - \varepsilon$ almost period of the function h is denoted by $E^{(n)}(\varepsilon, f)$.

Definition 2.4. [3] A function $h \in C_b^n(\mathbb{R}, X)$ is said to be a almost periodic function if for every $\varepsilon > 0$, the set $E^{(n)}(\varepsilon, h)$ is relatively dense in $\mathbb{R}$.

Definition 2.5. $AP^{(n)}(X)$ is the space of the C^n-almost periodic functions.

Since it is well known that for any almost periodic functions h_1 and h_2 and $\varepsilon > 0$, there exists a relatively dense set of their common ε almost period. Consequently, we get the following result.

Proposition 2.6. $h \in AP^{(n)}(X)$ if and only if $h^{(i)} \in AP(X)$ for $i = 0, 1, 2, ..., n$.

Since $AP(X)$ equipped with uniform norm topology is a Banach space, then we get the following result.

Proposition 2.7. $AP^{(n)}(X)$ provided with the norm $\|\cdot\|_n$ is a Banach space.

Example. The following example of a C^n-almost periodic function has been given in [5]. Let

$$g(t) = \sin(\alpha t) + \sin(\beta t),$$

where $\dfrac{\alpha}{\beta} \notin \mathbb{Q}$. Then the function $h(t) = e^{g(t)}$ is C^n-almost periodic for any $n \geq 1$. In [5], one can find example of function which is C^n-almost periodic but not C^{n+1}-almost periodic.

Definition 2.8. [18] A continuous function $h : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ such that

$$k(t) = \lim_{n\to\infty} h(t + s_n) \text{ exists for all } t \text{ in } \mathbb{R}$$

and

$$\lim_{n\to\infty} k(t - s_n) = h(t) \text{ for all } t \text{ in } \mathbb{R}.$$

Remark. By the pointwise convergence, the function k is just measurable and not necessarily continuous. If the convergence in both limits is uniform, then h is almost periodic. The concept of almost automorphy is then larger than the one of the almost periodicity. If h is almost automorphic, then its range is relatively compact, thus bounded in norm. Let $p(t) = 2 + cost + cos\sqrt{2}t$ and $h : \mathbb{R} \to \mathbb{R}$ such that $h = sin\frac{1}{p}$. Then h is almost automorphic, but h is not uniformly continuous on $\mathbb{R}$, it follows that h is not almost periodic.

Definition 2.9. [18] A continuous function $h : \mathbb{R} \to X$ is said to be compact almost automorphic if for every sequence of real numbers $(s'_n)_n$, there exists a subsequence $(s_n)_n$ such that

$$\lim_{m\to\infty} \lim_{n\to\infty} h(t + s_n - s_m) = h(t) \text{ uniformly on any compact set in } \mathbb{R}.$$

Theorem 2.10. *[18] If we equip $AA_c(X)$, the space of compact almost automorphic X-valued functions, with the sup norm, then $AA_c(X)$ is a Banach space.*

Theorem 2.11. *[18] If we equip $AA(X)$, the space of almost automorphic X-valued functions, with the sup norm, then $AA(X)$ turns out to be a Banach space.*

Definition 2.12. A continuous function $\theta : \mathbb{R}\times X \to X$ is said to be almost automorphic in t with respect to x if for every sequence of real numbers $(s'_n)_n$, there exists a subsequence $(s_n)_n$ such that

$$\lim_{m\to\infty} \lim_{n\to\infty} \theta(t + s_n - s_m, x) = \theta(t, x) \text{ for } t \in \mathbb{R} \text{ and } x \in X.$$

Now we recall the concept of C^n-almost automorphic functions recently introduced in [9] as a generalization of the one of C^n-almost periodic functions.

Definition 2.13. A continuous function $h : \mathbb{R} \to X$ is said to be C^n- almost automorphic for $n \geq 1$ if for $i = 0, 1, ..., n$, the i-th derivative $h^{(i)}$ of h is almost automorphic.

We will denote by $AA^{(n)}(X)$ the space of all C^n-almost automorphic X-valued functions.

Definition 2.14. ([9]) A continuous function $h : \mathbb{R} \to X$ is said to be C^n-compact almost automorphic if for $i = 0, 1, ..., n$, the i-th derivative $h^{(i)}$ of h is compact almost automorphic.

We denote by $AA_c^{(n)}(X)$ the space of all C^n-compact almost automorphic X-valued functions. Since $AA(X)$ and $AA_c(X)$ are Banach spaces, then we get also the following result.

Proposition 2.15. ([9]) $AA^{(n)}(X)$ and $AA_c^{(n)}(X)$ provided with the norm $|.|_n$ are Banach spaces.

The following superposition result is easy to prove.

Proposition 2.16. Let $f \in AA^{(n)}(\mathbb{X})$ and $A \in B(\mathbb{X})$. Then $Af \in AA^{(n)}(\mathbb{X})$.

Proposition 2.17. Let $\lambda \in AA^{(n)}(\mathbb{R}, K)$ and $f \in A^{(n)}(X)$ where X is a Banach space over the field K. Then $(\lambda f)(t) := \lambda(t)f(t)$ is in $AA^{(n)}(X)$.

We also have the following results

Theorem 2.18. *Let $\mathbb{X}$ be a Hilbert space and $f \in AA^{(n)}(\mathbb{X})$. Then the function $F(t) = \int_0^t f(s)ds \in AA^{(n+1)}(\mathbb{X})$ iff $\mathcal{R}_F$ is bounded in $\mathbb{X}$.*

Proof. We have just to prove the only if part. It comes by induction. The case $n = 0$ is known ([18] Theorem 2.4.6). Assume now that f is in $AA^{(n)}(\mathbb{X})$, and that the theorem is true for $n-1$; then $F \in AA^{(n)}(\mathbb{X})$. But we have $F' = f$ and so $F' \in AA^{(n)}(\mathbb{X})$, from which we conclude that $F \in AA^{(n+1)}(\mathbb{X})$. □

Theorem 2.19. *Let $\nu \in AA^{(n)}(\mathbb{R}, \mathcal{L}_s(X,Y))$ and $f \in AA^{(n)}(\mathbb{R}, X)$. Then $\nu f \in AA^{(n)}(\mathbb{R}, Y)$ for two Banach spaces X and Y.*

Proof. It suffices to observe that $\nu^{(i)} f^{(n-i)} : \mathbb{R} \to Y$ is almost automorphic, for each $i = 0, 1, ...n$. □

3 Uniform spectrum of a function in $BC(\mathbb{R}, X)$

Let us consider the following simple ordinary differential equation in a complex Banach space $\mathbb{X}$

$$x'(t) - \lambda x = f(t), \tag{3.1}$$

where $f \in BC(\mathbb{X})$. If $Re\lambda \neq 0$, the homogeneous equation associated with this has an exponential dichotomy; so, (3.1) has a unique bounded solution which we denote by $x_{f,\lambda}(\cdot)$. Moreover, from the theory of ordinary differential equations, it follows that for every fixed $\xi \in \mathbb{R}$,

$$x_{f,\lambda}(\xi) := \begin{cases} \int_{-\infty}^{\xi} e^{\lambda(\xi-t)} f(t)dt & (\text{if } Re\lambda < 0) \\ -\int_{\xi}^{+\infty} e^{\lambda(\xi-t)} f(t)dt & (\text{if } Re\lambda > 0). \end{cases} \tag{3.2}$$

$$= \begin{cases} \int_{-\infty}^{0} e^{-\lambda\eta} f(\xi+\eta)d\eta & (\text{if } Re\lambda < 0) \\ -\int_{0}^{+\infty} e^{-\lambda\eta} f(\xi+\eta)d\eta & (\text{if } Re\lambda > 0). \end{cases} \tag{3.3}$$

As is well known, the differentiation operator $\mathcal{D}$ is a closed operator on $BC(\mathbb{R}, \mathbb{X})$. The above argument shows that $\rho(\mathcal{D}) \supset \mathbb{C}\backslash i\mathbb{R}$ and $x_{f,\lambda} = (\mathcal{D} - \lambda)^{-1} f$ for every $\lambda \in \mathbb{C}\backslash i\mathbb{R}$ and $f \in BC(\mathbb{R}, \mathbb{X})$.

Hence, for every $\lambda \in \mathbb{C}$ with $Re\lambda \neq 0$ and $f \in BC(\mathbb{R}, \mathbb{X})$ the function $[(\lambda - \mathcal{D})^{-1} f](t) = \widehat{S(t)f}(\lambda) \in BC(\mathbb{R}, \mathbb{X})$. Moreover, $(\lambda - \mathcal{D})^{-1} f$ is analytic on $\mathbb{C} \backslash i\mathbb{R}$.

Definition 3.1. Let f be in $BC(\mathbb{R}, \mathbb{X})$. Then,

i) $\alpha \in \mathbb{R}$ is said to be *uniformly regular* with respect to f if there exists a neighborhood $\mathcal{U}$ of $i\alpha$ in $\mathbb{C}$ such that the function $(\lambda - \mathcal{D})^{-1} f$, as a complex function of λ with $Re\lambda \neq 0$, has an analytic continuation into $\mathcal{U}$.

ii) The set of $\xi \in \mathbb{R}$ such that ξ is not uniformly regular with respect to $f \in BC(\mathbb{R}, \mathbb{X})$ is called *uniform spectrum* of f and is denoted by $sp_u(f)$.

Observe that, if $f \in BUC(\mathbb{R}, \mathbb{X})$, then $\alpha \in \mathbb{R}$ is uniformly regular if and only if it is regular with respect to f (cf. [15]).

We now list some properties of uniform spectra of functions in $BC(\mathbb{R}, \mathbb{X})$.

Proposition 3.2. Let $g, f, f_n \in BC(\mathbb{R}, \mathbb{X})$ such that $f_n \to f$ as $n \to +\infty$ and let $\Lambda \subset \mathbb{R}$ be a closed subset satisfying $sp_u(f_n) \subset \Lambda$ for all $n \in \mathbb{N}$. Then the following assertions hold:

i) $sp_u(f) = sp_u(f(h + \cdot))$;

ii) $sp_u(\alpha f(\cdot)) \subset sp_u(f),\ \alpha \in \mathbb{C}$;

iii) $sp(f) \subset sp_u(f)$;

iv) $sp_u(Bf(\cdot)) \subset sp_u(f),\ B \in L(\mathbb{X})$;

v) $sp_u(f + g) \subset sp_u(f) \cup sp_u(g)$;

vi) $sp_u(f) \subset \Lambda$.

We also recall the following important result (see [15] for the proof).

Proposition 3.3. Let $f \in BC(\mathbb{R}, \mathbb{X})$. Then

$$sp_u(f) = sp_c(f),$$

where $sp_c(f)$ denotes the Carleman spectrum of f.

From the above properties, the following is obtained:

Proposition 3.4. ([3]) Let $f \in C_b^{(n)}(\mathbb{X})$. Then

$$sp_u(f^{(i)}) \subset sp_u(f^{(i-1)}),\ for\ every\ i = 1, 2, ..., n.$$

Now we can state and prove.

Lemma 3.5. *Let $f \in AA^{(n)}(\mathbb{X})$ and $\phi \in L^1(\mathbb{R})$ whose Fourier transform has compact support $supp(\phi)$. Then the function $g := \phi * f \in AA^{(n)}(\mathbb{X})$; moreover $sp_u(g) \subset sp_u(f) \cap supp(\phi)$.*

Proof. Let's assume $n = 0$. And let (s'_n) be an arbitrary sequence of real numbers. Since $f \in AA(X)$, there exists a subsequence (s_n) such that

$$h(t-s) := \lim_{n\to\infty} f(t-s+s_n)$$

is well-defined for each $t, s \in \mathbb{R}$, and

$$\lim_{n\to\infty} h(t-s-s_n) = f(t-s)$$

each $t, s \in \mathbb{R}$.

Note that $\|f(t-s+s_n)\phi(s)\| \le \|f\|_\infty \|\phi(s)\|$. And since $\phi \in L^1(\mathbb{R})$, we may deduce by the Lebesgue' dominated convergence theorem that

$$\lim_{n\to\infty} g(t+s_n) = \int_{\mathbb{R}} \lim_{n\to\infty} f(t-s+s_n)\phi(s)ds = \int_{\mathbb{R}} h(t-s)\phi(s)ds = (h*\phi)(t)$$

for each $t \in \mathbb{R}$.

Similarly we can prove that

$$\lim_{n\to\infty} (h*\phi)(t-s_n) = (\phi * f)(t)$$

for each $t \in \mathbb{R}$.

Thus $\phi \star f \in AA(X)$. Now we know that g is C^n with derivatives: $g^{(k)} = \phi * f^{(k)}$ (if $k \le n$). So, for each $k \le n$, $g^{(k)} \in AA(X)$, and the lemma follows. □

4 Applications to Differential Equations

Consider in a (complex) Banach space X the linear equation

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where $A : D(A) \subset X \to X$ is a linear operator, and $f \in C(\mathbb{R}, X)$.

We first generalize [9] Theorem 3.20 as follows.

Lemma 4.1. *Suppose $f \in AA^{(n)}(X)$ and $A \in L(X)$. Then every bounded solution of Eq.(4.2) is in $AA^{(n+1)}(X)$.*

Proof. It suffices to observe that since A is bounded, then

$$x^{(n+1)}(t) = Ax^{(n)}(t) + f^{(n)}(t).$$

□

We have the following Massera type result.

Theorem 4.2. *Let $f \in AA^{(n)}(\mathbb{C}^k)$. If Eq. (4.1) has a bounded solution on R^+, then it has a $AA^{(n+1)}(\mathbb{C}^k)$ solution. Moreover every bounded solution of the differential equation*

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R}, \tag{4.2}$$

where $A(t) : \mathbb{R} \to \mathcal{M}_k(\mathbb{C})$ is τ-periodic, is in $AA^{(n+1)}(\mathbb{C}^k)$.

Proof. The proof is similar to Theorem 3.1 [14]. First let us note that by Floquet's theory and without loss of generality we may assume that $A(t) = A$ is independent of t. Next we will show that the problem can be reduced to the one-dimensional case. In fact, if A is independent of t, by a change of variable if necessary, we may assume that A is of Jordan normal form. In this direction we can go further with assumption that A has only one Jordan box. That is, we have to prove the theorem for equations of the form

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_k'(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_k(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_k(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

Now if x is a bounded solution of the above system on $\mathbb{R}^+$, then by Theorem 3.14 [9], it has an almost almost automorphic solution on $\mathbb{R}$. Since $f \in AA^{(n)}(\mathbb{C})$, then by Lemma 4.1 above, we deduce that $x \in AA^{(n+1)}(\mathbb{C})$. □

The following is easy to establish.

Corollary 4.3. Consider the Differential Equation

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R} \tag{4.3}$$

where $f \in AA^{(n)}(\mathbb{R}^k)$, and $A \in B(\mathbb{R}^k)$ such that the real part of each of its eigenvalues is negative. Then Eq.(4.3) has a unique solution in $x \in AA^{(n+1)}(\mathbb{R}^k)$.

We also have the following result.

Theorem 4.4. *Let $A \in B(\mathbb{R}^k)$ and suppose that Eq.(4.3) has a unique $AA^{(1)}(\mathbb{R}^k)$ solution for each $f \in AA^k$. Then the map $T : AA(\mathbb{R}^k) \to AA^{(1)}(\mathbb{R}^k)$, $f \to x$ is linear and continuous, that is there exists $c > 0$ such that*

$$\|x\|_1 \le c\|f\|_0$$

where $\|\cdot\|_0$ denotes the usual sup norm in $AA(\mathbb{R}^k)$

Proof. Linearity of T is obvious. Let us prove its continuity.

First, let us consider the map $S : AA^{(1)}(\mathbb{R}^k) \to AA(\mathbb{R}^k)$ given by

$$Sx(t) = f(t).$$

That is, x is the unique $AA^{(1)}(\mathbb{R}^k)$ solution to ACP. S is defined as

$$(Sx)(t) = x'(t) - Ax(t) = f(t),$$

thus $Sx = f$ so $STf = f$. Also $TSx = Tf = x$. We deduce that $S = T^{-1}$.

On another hand we have

$$\|Sx\|_0 \leq \|x'\|_0 + K\|x\|_0 \leq K_1(\|x'\|_0 + \|x\|_0)$$

where K_1=max $(1, K)$. Thus we have

$$\|Sx\|_0 \leq K_1\|x\|_1.$$

That means S is continuous. And since S is injective, then $S^{-1} = T$ is continuous ([16] 1.6.6 Corollary page 44) This ends our proof. □

Now we investigate the existence of $C^{(n)}$ almost automorphic solutions for the following equation

$$x'(t) = Ax(t) + f(t) \text{ for } t \in \mathbb{R} \tag{4.4}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ in a Banach space X.

Definition 4.5. We say that a functon is a mild solution of equation (4.4) if for any σ and $t \geq \sigma$, we have

$$x(t) = T(t-\sigma)x(\sigma) + \int_\sigma^t T(t-s)f(s)ds.$$

For simplicity, mild solution will be called solution in the sequel.

We need to recall some preliminary results on quasi compact semigroups.We first introduce the Kuratowski measure of noncompactness $\alpha(.)$ of bounded sets K in a Banach space X by

$$\alpha(K) = \inf\{\varepsilon > 0 : K \text{ has a finite cover of balls of diameter} < \varepsilon\}.$$

For a bounded linear operator B on X, $|B|_\alpha$ is defined by

$$|B|_\alpha = \inf\{\varepsilon > 0 : \alpha(B(K)) \leq \varepsilon\alpha(K) \text{ for any bounded set } K \text{ of } X\}.$$

The essential growth bound $\omega_{ess}(T)$ of the semigroup $(T(t))_{t\geq 0}$ is defined by

$$\omega_{ess}(T) = \lim_{t\to+\infty} \frac{1}{t}\log|T(t)|_\alpha,$$
$$= \inf_{t>0} \frac{1}{t}\log|T(t)|_\alpha.$$

Definition 4.6. The essential spectrum $\sigma_{ess}(A)$ of A is the set of $\lambda \in \sigma(A)$: the spectrum of A, such that one of the following conditions holds:

(i) $\mathrm{Im}(\lambda I - A)$ is not closed,

(ii) the generalized eigenspace $M_\lambda(A) = \bigcup_{k\geq 1} Ker(\lambda I - A)^k$ is of infinite dimension,

(iii) λ is a limit point of $\sigma(A) \setminus \{\lambda\}$.

The essential radius of any bounded operator $\mathcal{T}$ in Y is defined by

$$r_{ess}(\mathcal{T}) = \sup\{|\lambda| : \lambda \in \sigma_{ess}(\mathcal{T})\}.$$

Definition 4.7. We say that the semigroup $(T(t))_{t\geq 0}$ is quasi compact if

$$\omega_{ess}(T) < 0.$$

Theorem 4.8. *The semigroup $(T(t))_{t\geq 0}$ is quasi compact if for some $t_0 > 0$, we have*

$$r_{ess}(T(t_0)) < 1.$$

Lemma 4.9. *If the semigroup $(T(t))_{t\geq 0}$ is quasi compact. Then,*

$$\sigma^+(A) = \{\lambda \in \sigma(A) : \mathrm{Re}(\lambda) \geq 0\}$$

is a finite set of the eigenvalues of A which are not in the essential spectrum.

Theorem 4.10. *[9] Assume that the semigroup $(T(t))_{t\geq 0}$ is quasi compact. Then X is decomposed as follows*

$$X = S \oplus V,$$

where X is $T-$invariant and there are positive constants α and N such that

$$|T(t)x| \leq Ne^{-\alpha t}|x| \quad \text{for } t \geq 0 \text{ and } x \in S. \tag{4.5}$$

Moreover V is a finite dimensional space and the restriction of $\mathcal{T}$ to V becomes a group.

Let P^- and P^+ denote respectively the projection operators respectively of X into S and V.

Theorem 4.11. *Assume that the semigroup $(T(t))_{t\geq 0}$ is quasi compact and the input function f is $C^{(n)}$-almost automorphic. If equation (4.4) has a bounded solution on $\mathbb{R}^+$, then it has a $C^{(n)}$-almost automorphic solution. Moreover every bounded solution of equation (4.4) on $\mathbb{R}$ is a $C^{(n)}$-almost automorphic solution.*

Proof of Theorem. Let B be a matrix be such that

$$T(t) = e^{tB} \text{ in } V.$$

Let u be a bounded solution of equation (4.4) on $\mathbb{R}^+$. The function $z(t) = P^+u(t)$ is a bounded solution on $\mathbb{R}^+$ of the following ordinary differential equation

$$z'(t) = Bz(t) + P^+f(t) \text{ for } t \geq 0. \tag{4.6}$$

Moreover, the function $t \to P^+f(t)$ is $C^{(n)}$-almost automorphic from $\mathbb{R}$ to $\mathbb{R}^d$. By Theorem 4.2 we get that the reduced system (4.6) has a $C^{(n)}$-almost automorphic solution $\widetilde{z}$ and the function v defined by

$$v(t) = \widetilde{z}\,(t) + \int_{-\infty}^{t} T\,(t-s)\,P^-f\,(s)\,ds \text{ for } t \in \mathbb{R},$$

is a bounded solution of equation (4.4) on $\mathbb{R}$. We claim that v is $C^{(n)}$-almost automorphic. In fact, let y be defined by

$$y(t) = \int_{-\infty}^{t} T\,(t-s)\,P^-f\,(s)\,ds \text{ for } t \in \mathbb{R}.$$

Then $y \in C_b^{(n)}(\mathbb{R}, X)$. Clearly y is $a.a.$ by [19]. Also we have $y'(t) = P^-f(t) + y(t)$. So y' is $a.a.$ In general $y^{(i)} = P^-f^{(i-1)}(t) + y^{(i-1)}(t),\ i = 1, 2, ..., n$, which implies that y is $C^{(n)}$ almost automorphic. Let u be a bounded solution on $\mathbb{R}$, then u is given by the following formula

$$u\,(t) = z\,(t) + \int_{-\infty}^{t} T\,(t-s)\,P^-f\,(s)\,ds \text{ for } t \in \mathbb{R},$$

where

$$z(t) = P^+u(t) \text{ for } t \in \mathbb{R}$$

is a solution of the reduced system (4.6), which is $C^{(n)}$-almost automorphic by Theorem 4.2 and arguing as above, one can prove that the function

$$t \to \int_{-\infty}^{t} T\,(t-s)\,P^-f\,(s)\,ds \text{ for } t \in \mathbb{R},$$

is also $C^{(n)}$-almost automorphic.

Received: December 2007. Revised: February 2008.

References

[1] M. Adamczak, *$C^{(n)}$-almost periodic functions*, Comment. Math. Prace Mat. **37** (1997), 1–12.

[2] M. Adamczak and S. Stoínski, *On the ($NC^{(n)}$)-almost periodic functions*, Proceedings of the 6th. Conference on Functions Spaces (R. Grzáslewicz, Cz. Ryll-Nardzewski, H. Hudzik, and J. Musielak, eds), World Scientific Publishing, New Jersey, 2003, 39–48.

[3] J.B. Baillon, J. Blot, G.M. N'Guérékata and D. Pennequin, *On C^n-almost periodic solutions to some nonautonomous differential equations in Banach spaces*, Annales Societatis Mathematicae Polonae, Serie 1, XLVI (2), 263–273, (2006).

[4] D. Bugajewski and T. Diagana, *Almost automorphy of the convolution operator and applications to differential and functional differential equations*, Nonlinear Studies, Vol. 13, No. **2** (2006), 129–140.

[5] D. Bugajewski and G. N'Guérékata, *On some classes of almost periodic functions in abstract spaces*, Intern. J. Math. and Math. Sci. 2004 (61)(2004), 3237–3247.

[6] C. Corduneanu, *Almost periodic solutions for some class of functional differential equations*, Funct. Diff. Equ., Vol. 14, No. **2-3-4** (2007), 223–229.

[7] T. Diagana, G. N'Guérékata and Nguyen Van Minh, *Almost automorphic solutions of evolution equations*, Proc. Amer. Math. Soc. **132** (2004), 3289–3298.

[8] K. Ezzinbi, S. Fatajou and G.M. N'Guérékata, *Massera type theorem for the existence of $C^{(n)}$-almost periodic solutions for partial functional differential equations with infinite delay*, Nonlinear Analysis, Theory, Methods and Applications, to appear.

[9] K. Ezzinbi, S. Fatajou and G.M. N'Guérékata, *$C^{(n)}$-almost automorphic solutions for partial neutral functional differential equations*, Applicable Analysis, Vol. 86, Issue **9** (2007), 1127–1146.

[10] Y. Hino, and S. Murakami, *Almost automorphic solutions for abstract functional differential equations*, J. Math. Anal. Appl. **286** (2003), 741–752.

[11] Y. Hino, T. Naito, N.V. Minh, and J.S. Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, Taylor & Francis, London - New York, 2002.

[12] B.M. Levitan and V.V. Zhikov, *Almost Periodic Functions and Differential Equations*, Moscow Univ. Publ. House 1978. English translation by Cambridge University Press 1982.

[13] J. Liang, L. Maniar, G. N'Guérékata and Ti-Jun Xiao, *Existence and uniqueness of $C^{(n)}$-almost periodic solution to some ordinary differential equations*, Nonlinear Analysis, **66** (2007), 1899–1910.

[14] J. Liu, G. N'Guérékata and Nguyen van Minh, *A Massera type theorem for almost automorphic solutions of differential equations*, J. Math. Anal. Appl. **299** (2004), no. 2, 587–599.

[15] J. Liu, G. N'Guérékata, Nguyen Van Minh and Vu Quoc Phong, *Bounded solutions of parabolic equations in continuous function spaces*, Funkciolaj Ekvacioj.

[16] R.E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics, **183** Springer, New York-Berlin-Milan-London.

[17] G.M. N'Guérékata, *Almost automorphic functions and applications to abstract evolution equations*, Contemporary Math. **252** (1999), 71–76.

[18] G.M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer, Amsterdam, 2001.

[19] G.M. N'Guérékata, *Existence and uniqueness of almost automorphic mild solutions to some semilinear abstract differential equations*, Semigroup Forum, Vol. **69** (2004), No. 1, 80–89.

[20] G.M. N'Guérékata, *Topics in Almost Automorphy*, Springer, New York, 2005.