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Dynamical Inverse Problem for the Equation $u_{tt} - \Delta u - \nabla \ln \rho \cdot \nabla u = 0$ (the BC Method)

M.I. Belishev
Saint-Petersburg Department of the Steklov Mathematical Institute (POMI),
27 Fontanka, St. Petersburg 191023, Russia,
email: `belishev@pdmi.ras.ru`

ABSTRACT

A dynamical system of the form

$$\begin{aligned} &u_{tt} - \Delta u - \nabla \ln \rho \cdot \nabla u = 0, && \text{in} \quad \mathbb{R}^n_+ \times (0,T) \\ &u|_{t=0} = u_t|_{t=0} = 0, && \text{in} \quad \overline{\mathbb{R}^n_+} \\ &u_{x^n} = f && \text{on} \quad \partial\mathbb{R}^n_+ \times [0,T], \end{aligned}$$

is considered, where $\mathbb{R}^n_+ := \{x = \{x^1, \dots, x^n\} |\ x^n > 0\}$; $\rho = \rho(x)$ is a smooth positive function (density) such that $\rho, \frac{1}{\rho}$ are bounded in $\overline{\mathbb{R}^n_+}$; f is a (Neumann) boundary control of the class $L_2(\partial\mathbb{R}^n_+ \times [0,T])$; $u = u^f(x,t)$ is a solution (wave). With the system one associates a response operator $R^T : f \mapsto u^f|_{\partial\mathbb{R}^n_+ \times [0,T]}$. A dynamical inverse problem is to determine the density from the given response operator.

Fix an open subset $\sigma \subset \partial\mathbb{R}^n_+$; let $L_2(\sigma \times [0,T])$ be the subspace of controls supported on σ. A *partial response operator* R^T_σ acts in this subspace by the rule $R^T_\sigma f = u^f|_{\sigma \times [0,T]}$; let R^{2T}_σ be the operator corresponding to the same system considered on the doubled time interval $[0, 2T]$. Denote $B^T_\sigma := \{x \in \mathbb{R}^n_+ |\ \{x^1, \dots, x^{n-1}, 0\} \in \sigma,\ 0 < x^n < T\}$ and assume $\rho|_\sigma$ to be known. We show that R^{2T}_σ determines $\rho|_{B^T_\sigma}$ and propose an efficient procedure recovering the density. The procedure is available for constructing numerical algorithms.

The instrument for solving the problem is the boundary control method which is an approach to inverse problems based on their relations with control theory (Belishev, 1986). Our presentation is elementary and can serve as introduction to the BC method.

RESUMEN

Consideramos el sistema dinámico

$$\begin{aligned} &u_{tt} - \Delta u - \nabla \ln \rho \cdot \nabla u = 0, && \text{en} \quad \mathbb{R}^n_+ \times (0,T) \\ &u|_{t=0} = u_t|_{t=0} = 0, && \text{en} \quad \overline{\mathbb{R}^n_+} \\ &u_{x^n} = f && \text{sobre} \quad \partial\mathbb{R}^n_+ \times [0,T], \end{aligned}$$

donde $\mathbb{R}^n_+ := \{x = \{x^1, \dots, x^n\} |\ x^n > 0\}$; $\rho = \rho(x)$ es una función positiva suave (densidad) tal que $\rho, \frac{1}{\rho}$ son limitada en $\overline{\mathbb{R}^n_+}$; f es un control en la frontera (Neumann) de clase $L_2(\partial\mathbb{R}^n_+ \times [0,T])$; $u = u^f(x,t)$ es la solución (Onda). Con el sistema asociamos un operador respuesta $R^T : f \mapsto u^f|_{\partial\mathbb{R}^n_+ \times [0,T]}$. Un problema dinámico inverso consiste en determinar la densidad desde el operador respuesta.

Fije un subconjunto abierto $\sigma \subset \partial\mathbb{R}^n_+$; sea $L_2(\sigma \times [0,T])$ el subespacio de los controles soportados em σ. Un operador respuesta parcial R^T_σ actua en este subespacio mediante la regla $R^T_\sigma f = u^f|_{\sigma \times [0,T]}$; sea R^{2T}_σ el operador correspondiente al mismo sistema considerado en el intervalo de tiempo $[0, 2T]$. Denote $B^T_\sigma := \{x \in \mathbb{R}^n_+ |\ \{x^1, \dots, x^{n-1}, 0\} \in \sigma,$ $0 < x^n < T\}$ y suponga que $\rho|_\sigma$ es conocido. Nosotros mostramos que R^{2T}_σ determina $\rho|_{B^T_\sigma}$ y es propuesto un procedimento eficiente de recuperar la densidad. El procedimiento es encontrado por construción de algoritmos númericos.

El instrumento de resolver el problema es el método de control en la frontera este es un abordage para problemas inversos basado en sus relaciones con teoria de control (Belishev, 1986). Nuestra presentación es elemental y puede servir como introducción al método BC.

Key words and phrases: *Dynamical inverse problem, response operator, determination of density, boundary control method.*

Math. Subj. Class.: *35Bxx, 35R30*

1 About the paper

The problem under consideration comes from geophysics. We deal with a dynamical system of the form

$$u_{tt} - \Delta u - \nabla \ln \rho \cdot \nabla u = 0 \qquad \text{in} \quad \mathbb{R}^n_+ \times (0,T) \tag{1}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in} \quad \overline{\mathbb{R}^n_+} \tag{2}$$

$$u_{x^n} = f \qquad \text{on} \quad \Gamma \times [0,T], \tag{3}$$

where $\mathbb{R}^n_+ := \left\{x = \{x^1, \dots, x^n\} |\ x^n > 0\right\}$ simulates the Earth, $\Gamma := \partial\mathbb{R}^n_+$ is the Earth surface, $\rho = \rho(x)$ is a smooth function (density) satisfying $0 < \rho_* \leq \rho(\cdot) \leq \rho^*$, f is a (Neumann) boundary control of the class $L_2(\Gamma \times [0,T])$, $u = u^f(x,t)$ is a solution. The solution describes a wave initiated by the control and propagating into the Earth from the surface. With the system one associates a *response operator* $R^T : f \mapsto u^f|_{\partial\mathbb{R}^n_+ \times [0,T]}$. From the physical viewpoint, f is a force applied at the surface, whereas $u^f|_{\partial\mathbb{R}^n_+ \times [0,T]}$ is a displacement measured at the same surface. Thus, R^T is an "input $\mapsto$ output" map representing the measurements, which the external observer implements at the surface. The dynamical inverse problem, which the paper deals with, is to determine the density from the given response operator.

2 Results

Begin with certain of the notations. With a point $x \in \overline{\mathbb{R}^n_+}$ we associate a pair $(\gamma, \tau) : \gamma := \{x^1, \dots, x^{n-1}, 0\} \in \Gamma$, $\tau := x^n \geq 0$ of its semigeodesic coordinates and write $x = x(\gamma, \tau)$. Fix $\xi > 0$, let $\sigma \subset \Gamma$ be an open subset at the surface; the set $B^\xi_\sigma := \{x \in \mathbb{R}^n_+ |\ x = x(\gamma,\tau),\ \gamma \in \sigma,\ \tau \in (0,\xi)\}$ is called a *tube* with a bottom σ and a top $\sigma^\xi = \{x(\gamma,\tau) |\ \gamma \in \sigma, \tau = \xi\}$. Also, introduce a subdomain $\Omega^\xi_\sigma := \{x \in \overline{\mathbb{R}^n_+} |\ \text{dist}\ (x,\sigma) < \xi\}$ [1] (contoured with $cdef$ on Fig 1). The tube (shadowed on Fig 1) is the part of the subdomain illuminated with rays emanating from σ in normal direction to the surface.

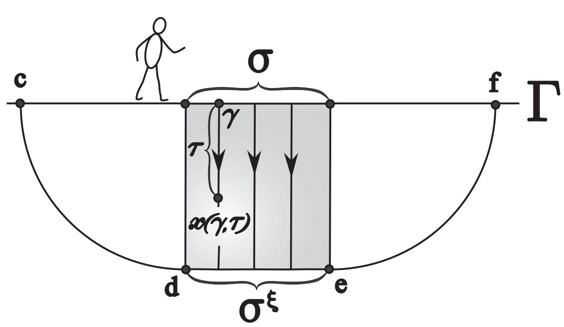

Figure 1: *The tube B^ξ_σ and the subdomain Ω^ξ_σ*

[1] dist is the standard Euclidean distance in $\mathbb{R}^n_+$

Let $\sigma \subset \Gamma$ and $T > 0$ be fixed. Consider system (1)–(3) with the final moment $t = 2T$ and introduce a *partial response operator* R_σ^{2T} acting in the (sub)space $L_2(\sigma \times [0, 2T])$ of controls supported on σ by the rule $R_\sigma^{2T} f := u^f \mid_{\sigma \times [0,2T]}$ [2]. As is well-known, the waves in system (1)–(3) propagate with the unit velocity. By this, the response operator depends on the density locally: R_σ^{2T} is determined by the behavior of ρ in the subdomain Ω_σ^T only [3]. Such a locality motivates the setup of the inverse problem: *given operator* R_σ^{2T} *to recover* $\rho|_{\Omega_\sigma^T}$. However, since the substitution $\rho \to c\rho$ with a constant $c > 0$ does not change system (1)–(2), the unique determination of density is impossible. Avoiding such a nonuniqueness, it is natural to assume the boundary values of ρ to be known. The main result of the paper is

Theorem 1 *Let $T > 0$ be fixed, the operator R_σ^{2T} given, and the function $\rho|_\sigma$ known. Then these data uniquely determine $\rho|_{B_\sigma^T}$.*

The proof is constructive: we propose an efficient procedure recovering the density in the tube. Moreover, the procedure is provided with an additional option that is visualization of waves: given f we recover $u^f|_{B_\sigma^T}$.

3 Motivation and comments

There are two reasons to deal namely with the version (1) of the general wave equation with variable coefficients. First, the interest is motivated by possible applications in geophysics (see, e.g., [6]). The second reason is the following. The instrument for solving the problem is the boundary control method (BCm), which is an approach to inverse problems based on their relations with control theory (Belishev, 1986). In comparison with another versions (see [1] – [4]), the variant of the BCm available for equation (1) is the simplest one. As such, it has good chances for numerical realization. Our presentation is elementary: along with the paper [3], this one can serve as an introduction to the BCm. The plan of the paper is as follows:

- in section 4, the basic notions and objects (spaces, operators etc) are introduced
- in section 5, we present the so-called amplitude formula (AF) which solves the inverse problem: it recovers $\rho|_{B_\sigma^T}$ via R_σ^{2T}
- sections 6–10 are devoted to the derivation of the AF
- in section 11, a certain additional option of the BCm is described: the AF enables one to recover the solutions u^f in the tube B_σ^T that is what we call a *wave visualization*
- section 12 contains the concluding remarks; also, the extension of our results is shortly discussed.

[2] this operator represents the measurements implemented at the part of the Earth surface

[3] in other words, R_σ^{2T} does not depend on $\rho|_{\mathbb{R}_+^n \setminus \Omega_\sigma^T}$

Simplifying the notations, we accept the convention: unless the otherwise is specified, the subset $\sigma \subset \Gamma$ is assumed fixed and we write R^{2T} instead of R^{2T}_σ, Ω^ξ instead of Ω^ξ_σ, B^ξ instead of B^ξ_σ, etc. Also, without loss of generality, we assume σ to be bounded and $\partial\sigma$ smooth.

4 Dynamical system

With system (1)–(3) one associates

(i) an *outer space* $\mathcal{F}^T := \{f \in L_{2,\rho_0}(\Gamma \times [0,T]) |\ \text{supp}\ f \subset \overline{\sigma} \times [0,T]\}$ of controls acting from σ with the inner product

$$(f,g)_{\mathcal{F}^T} = \int_{\sigma\times[0,T]} f(\gamma,t)\, g(\gamma,t)\, \rho_0(\gamma) d\Gamma dt\,,$$

where $\rho_0 := \rho|_\Gamma$, $d\Gamma$ is the Euclidean surface element on $\partial\mathbb{R}^n_+$. In $\mathcal{F}^T$ we single out a family of subspaces

$$\mathcal{F}^{T,\xi} := \left\{f \in \mathcal{F}^T |\ \text{supp}\ f \subset \overline{\sigma} \times [T-\xi, T]\right\}, \qquad \xi \in [0,T]$$

consisting of the delayed controls ($T-\xi$ is the value of delay, ξ is the action time; $\mathcal{F}^{T,0} = \{0\}$, $\mathcal{F}^{T,T} = \mathcal{F}^T$);

(ii) an *inner space* of states (waves) $\mathcal{H}^T := L_{2,\rho}(\Omega^T)$ with the product

$$(y,w)_{\mathcal{H}^T} = \int_{\Omega^T} y(x)\, w(x)\, \rho(x) dx$$

and the family of its subspaces [4]

$$\mathcal{H}^\xi := \left\{y \in \mathcal{H}^T |\ \text{supp}\ y \subset \overline{\Omega^\xi}\right\}, \quad \xi \in [0,T]\,.$$

Since the waves described by a hyperbolic system (1)–(3) propagate (from σ into $\mathbb{R}^n_+$) with the speed 1, the inclusion supp $u^f(\cdot,t) \subset \overline{\Omega^t}$ holds for all $t \in [0,T]$, whereas Ω^T is the subdomain filled with waves at the final moment $t = T$. Correspondingly, we consider the waves as time depended elements of the space $\mathcal{H}^T$;

(iii) a *control operator* $W^T : \mathcal{F}^T \to \mathcal{F}^T$, $W^T f := u^f(\cdot,T)$. This operator is continuous [7] and injective for any $T > 0$ [1]. By the above-mentioned hyperbolicity, for $f \in \mathcal{F}^{T,\xi}$ one has supp $u^f(\cdot,T) \subset \overline{\Omega^\xi}$ that yields the embedding $W^T \mathcal{F}^{T,\xi} \subset \mathcal{H}^\xi$;

(iv) a *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $R^T f := u^f\,|_{\sigma\times[0,T]}$, which is also a continuous map [5];

(v) a *connecting operator* $C^T : \mathcal{F}^T \to \mathcal{F}^T$, $C^T := (W^T)^* W^T$. For $f,g \in \mathcal{F}^T$, one has

$$\big(u^f(\cdot,T), u^g(\cdot,T)\big)_{\mathcal{H}^T} = (W^T f, W^T g)_{\mathcal{H}^T} = (C^T f, g)_{\mathcal{F}^T}\,, \tag{4}$$

[4] recall that $\Omega^\xi = \{x \in \overline{\mathbb{R}^n_+}\ |\ \text{dist}\ (x,\sigma) < \xi\}$
[5] moreover, R^T is a compact operator [7]

so that C^T connects the metrics of the outer and inner spaces. By injectivity of W^T, the operator C^T is also injective.

One of central points of the BCm is an explicit relation between the response and connecting operators. Denote by $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ the operator extending controls from $\sigma \times [0,T]$ to $\sigma \times [0,2T]$ as odd (with respect to $t = T$) functions of time; let $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$ be the integration: $(J^{2T} f)(\cdot,t) := \int_0^t f(\cdot,s)\,ds$; $R^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$ the response operator of system (1)–(3) with the final moment $t = 2T$.

Lemma 1 *The relation*

$$C^T \;=\; \frac{1}{2}\,(S^T)^* J^{2T} R^{2T} S^T \tag{5}$$

holds.

Proof Choose $f, g \in \mathcal{F}^T$ and denote $f_- := S^T f$. Assume f, g to be such that u^{f_-}, u^g are the classical solutions. Blagoveschenskii's function

$$b(s,t) := \int_{\mathbb{R}^n_+} u^{f_-}(\cdot,s)\,u^g(\cdot,t)\,\rho dx \qquad s,t \in [0,2T]\times[0,T]$$

is well defined and satisfies

$$\begin{aligned}
&b_{tt}(s,t) - b_{ss}(s,t) = \int_\Omega \left[u^{f_-}(\cdot,s)\,u^g_{tt}(\cdot,t) - u^{f_-}_{ss}(\cdot,s)\,u^g(\cdot,t)\right]\,\rho dx = \\
&\int_\Omega \left[u^{f_-}(\cdot,s)\,\mathrm{div}\rho\nabla u^g(\cdot,t) - \mathrm{div}\rho\nabla u^{f_-}(\cdot,s)\,u^g(\cdot,t)\right]\,dx = \\
&\int_\Gamma \left[-u^{f_-}(\cdot,s)\,u^g_{x^n}(\cdot,t) + u^{f_-}_{x^n}(\cdot,s)\,u^g(\cdot,t)\right]\,\rho_0 d\Gamma = \\
&-\int_\Gamma \left[(R^{2T}f_-)(\cdot,s)\,g(\cdot,t) - f_-(\cdot,s)\,(R^T g)(\cdot,t)\right]\,\rho_0 d\Gamma \;=:\; F(s,t)
\end{aligned}$$

(in the second equality we have used equation (1) in the form $\rho u_{tt} = \mathrm{div}(\rho\nabla u)$. Finding b by the D'Alembert formula (with regard to the initial conditions $b(\cdot,0) = b_t(\cdot,0) = 0$), putting $t = T$, and taking into account the oddness of f_-, we get

$$\begin{aligned}
&b(T,T) \;=\; \frac{1}{2}\int_0^T dt \int_t^{2T-t} F(s,t)\,ds \;= \\
&-\;\frac{1}{2}\int_0^T dt \int_t^{2T-t} ds \int_\Gamma (R^{2T}f_-)(\cdot,s)\,g(\cdot,t)\,\rho_0 d\Gamma := \\
&\int_{\Gamma\times[0,T]} \frac{1}{2}\left\{\int_0^t (R^{2T}f_-)(\cdot,s)\,ds \;-\; \int_0^{2T-t}(R^{2T}f_-)(\cdot,s)\,ds\right\}\,g(\cdot,t)\,\rho_0 d\Gamma dt
\end{aligned}$$

that can be easily transformed to

$$b(T,T) = \left(\frac{1}{2}\,(S^T)^* J^{2T} R^{2T} S^T f, g\right)_{\mathcal{F}^T}\;. \tag{6}$$

On the other hand, we have

$$b(T,T) = \left(u^{f_-}(\cdot,T), u^g(\cdot,T)\right)_{\mathcal{H}^T} =$$
$$\left(u^f(\cdot,T), u^g(\cdot,T)\right)_{\mathcal{H}^T} = \langle \text{see } (4)\rangle = (C^T f, g)_{\mathcal{F}^T}\,. \tag{7}$$

Comparing (6) with (7) and taking into account the density (in $\mathcal{F}^T$) of f, g used, we arrive at (5). □

5 Amplitude formula

Here we present a relation that determines the density from the response operator. Recall that $x(\gamma,\tau)$ denotes the point in $\mathbb{R}^n_+$ with the semigeodesic coordinates γ and τ, the subset $\sigma \ni \gamma$ is fixed, $\rho_0 = \rho|_\Gamma$.

Fix $\xi \in (0,T)$; let $\check{f}^\xi := \{f_k^\xi\}_{k=1}^\infty \subset \mathcal{F}^{T,\,\xi}$ be a linearly independent complete system [6] of delayed controls acting from σ and such that the corresponding solutions $u^{f_k^\xi}$ are classical. With the system we associate its Gram matrix $\{G_{ik}\}_{i,k=1}^\infty$:

$$G_{ik} := (C^T f_i^\xi, f_k^\xi)_{\mathcal{F}^T} = \int_{\sigma\times[T-\xi,T]} (C^T f_i^\xi)(\gamma,t)\, f_k^\xi(\gamma,t)\,\rho_0(\gamma) d\Gamma dt \tag{8}$$

and a sequence of numbers $\{\beta_i\}_{i=1}^\infty$:

$$\beta_i := -\,(\varkappa^T, f_i^\xi)_{\mathcal{F}^T} = -\int_{\sigma\times[T-\xi,T]} (T-t)\, f_i^\xi(\gamma,t)\,\rho_0(\gamma) d\Gamma dt\,, \tag{9}$$

where $\varkappa^T = \varkappa^T(\gamma,t) := T-t$. For any integer $N \geq 1$, a linear algebraic system

$$\sum_{k=1}^N G_{ik}\,\alpha_k^N \;=\; \beta_i, \quad i = 1,\ldots,N \tag{10}$$

is uniquely solvable (w.r.t. $\alpha_1^N,\ldots,\alpha_N^N$) by injectivity of the operator C^T.

Lemma 2 *For a fixed $(\gamma,\xi) \in \sigma\times(0,T)$, the relation*

$$\rho\left(x(\gamma,\xi)\right) \;=\; \rho_0(\gamma) \left\{ \left[\frac{\partial}{\partial t} \lim_{N\to\infty} \sum_{k=1}^N \alpha_k^N \left(C^T f_k^\xi\right)\right](\gamma,t)\Big|_{t=T-\xi-0}^{t=T-\xi+0} \right\}^2 \tag{11}$$

holds.

Relation (11) is a relevant version of the so-called *amplitude formula* (AF), which is one of the main tools for solving inverse problems by the BCm: see [1], [2], [3]. The assertion of Theorem 1 easily follows from the AF. Indeed, if ρ_0 is known and R^{2T} is given, then one can find C^T by (5) and, choosing a system $\check{f}^\xi$, determine the r.h.s. of (11). As (γ,ξ) runs over $\sigma\times(0,T)$, the points $x(\gamma,\xi)$ exhaust the tube B^T. Hence, $\rho|_{B^T}$ is determined by R^{2T}. In sections 6–10 we derive the AF.

[6] the completeness means that $\mathrm{clos}_{\mathcal{F}^T}\,\mathrm{span}\,\check{f}^\xi = \mathcal{F}^{T,\,\xi}$

6 Propagation of jumps

The well-known and typical for hyperbolic problems fact is that discontinuity of a control f in system (1)–(3) implies discontinuity of the corresponding wave u^f, the latter discontinuity (singularity) propagating along the space-time rays and being supported on the characteristic surfaces. Below we recall certain details.

Let $\theta_0(t) := \frac{1}{2}\left[1 + \operatorname{sign} t\right]$, $t \in \mathbb{R}$ be the Heavyside function; the sequence $\{\theta_k(\cdot)\}_{k=-\infty}^{k=\infty} : \frac{d\theta_k}{dt} = \theta_{k-1}$ is usually referred to as a smoothness scale [7].

Choose a smooth control $f \in \mathcal{F}^T$ and fix $\xi \in (0, T)$; by $f_\xi(\gamma, t) := f(\gamma, t)\theta_0\left(t - (T - \xi)\right)$ we denote its cut-off function on $\sigma \times [T - \xi, T]$, so that f_ξ is an element of the subspace $\mathcal{F}^{T,\,\xi}$. In the generic case, the control f_ξ has a jump at the cross-section $\sigma \times \{t = T - \xi\}$ (see ab on Fig 2), the value (amplitude) of the jump being equal to $f_\xi(\,\cdot\,, t)|_{t=T-\xi-0}^{t=T-\xi+0} = f(\,\cdot\,, T - \xi) - 0 = f(\,\cdot\,, T - \xi)$.

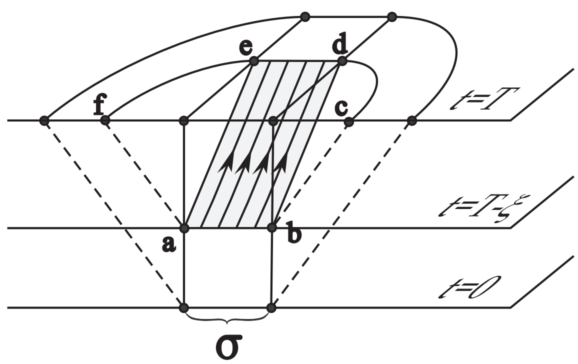

Figure 2: *Jumps in system (1)–(3)*

Jumps of a Neumann control induce jumps of a wave velocity. Namely, the velocity $u_t^{f_\xi}$ turns out to be discontinuous; its jump is supported on the characteristic surface $\{(x, t)|\ t = (T - \xi) + \operatorname{dist}(x, \sigma)\}$. This surface consists of the plane part *abde* and two conic parts *bcd* and *aef* (see Fig 2; the arrows pick up the space-time rays, which the discontinuity propagates along). The jump on the conic parts is weaker than the one on the plane part and plays no role in the further considerations. The jump on the plane part is described as follows. In a (space-time) neighborhood of *abde*, the solution is sought in the form

$$u^{f_\xi} = \mathcal{A}_p + w_p\,, \qquad \mathcal{A}_p(x, t) := \sum_{k=1}^{p} A_k(\gamma, \tau)\,\theta_k\left(t - (T - \xi) - \tau\right)\,, \tag{12}$$

where x in the l.h.s. is $x(\gamma, \tau)$; $\mathcal{A}_p$ is an *ansatz*, which is a function of the class $C^{p-1}_{\rm loc}$; A_k are the so-called *amplitude functions*; $w_p \in C^p_{\rm loc}$ is a smoother reminder. Substituting such a representation in (1)–(3), one derives a recurrent system of ODE's [8] for the amplitude functions. As result, one

[7] θ_k with negative k's are understood in the sense of distributions
[8] the so-called transport equations

arrives at the representation (for $p=1$) of the form

$$
\begin{aligned}
u^{f_\xi}(x,t) \;=& -\left(\frac{\rho\,(x(\gamma,\tau))}{\rho_0(\gamma,\tau)}\right)^{-\frac{1}{2}} f(\gamma,T-\xi)\,\theta_1\,(t-(T-\xi)-\tau)\;+\\
& w(x,t)\,\theta_2\,(t-(T-\xi)-\tau)\;,
\end{aligned}
\tag{13}
$$

where $x=x(\gamma,\tau)$ in the l.h.s. and w is a smooth function. For details of this technique see, e.g., [5].

At the final moment $t=T$, the wave $u^{f_\xi}(\cdot\,,T)$ is supported in the subdomain Ω^ξ contoured with $cdef$ on Fig 2. The surface $cdef$ coincides with the forward front of the wave. By (13), in a neighborhood of the plane part ed of the front, the representation

$$
\begin{aligned}
& u^{f_\xi}(x,T) \;=\; (W^T f_\xi)(x) \;=\\
& -\left(\frac{\rho\,(x(\gamma,\tau))}{\rho_0(\gamma,\tau)}\right)^{-\frac{1}{2}} f(\gamma,T-\xi)\,\theta_1\,(\xi-\tau)\;+\;w(x)\,\theta_2(\xi-\tau)
\end{aligned}
\tag{14}
$$

holds with a smooth w. Correspondingly, the velocity of the wave has a jump at ed:

$$
\begin{aligned}
& u_t^{f_\xi}\,(x(\gamma,\tau),T)\Big|_{\tau=\xi-0}^{\tau=\xi+0}=0-u_t^{f_\xi}\,(x(\gamma,\xi-0))=\langle\text{see (13)}\rangle=\\
& -\left(\frac{\rho\,(x(\gamma,\xi))}{\rho_0(\gamma,\xi)}\right)^{-\frac{1}{2}} f(\gamma,T-\xi)\,,\qquad \gamma\in\sigma\,.
\end{aligned}
\tag{15}
$$

So, up to the factor $-(\frac{\rho}{\rho_0})^{-\frac{1}{2}}$, the shape of the velocity jump reproduces the shape of the control jump.

7 Dual system

A dynamical system

$$
\begin{aligned}
& v_{tt}-\Delta v-\nabla\ln\rho\cdot\nabla v=0, && \text{in}\quad \mathbb{R}^n_+\times(0,T) && (16)\\
& v|_{t=0}=0,\quad v_t|_{t=0}=y, && \text{in}\quad \overline{\mathbb{R}^n_+} && (17)\\
& v_{x^n}=0 && \text{on}\quad \Gamma\times[0,T], && (18)
\end{aligned}
$$

is said to be *dual* to system (1)–(3); its solution $v=v^y(x,t)$ describes a wave initiated by a velocity perturbation y and propagating (in the reversed time) into $\mathbb{R}^n_+$. The term 'dual' is motivated by the following relation between solutions of these systems.

Lemma 3 *For any square summable f and y compactly supported in $\Gamma\times[0,T]$ and $\mathbb{R}^n_+$ respectively, the equality*

$$
\int_{\mathbb{R}^n_+} u^f(\cdot\,,T)\,y\,\rho dx \;=\; \int_{\Gamma\times[0,T]} f\,v^y\,\rho_0 d\Gamma dt
\tag{19}
$$

is valid.

Proof Let f and y be compactly supported and such that the solutions u^f and v^y are classical. The relations

$$0=\int_{\mathbb{R}^n_+\times[0,T]}\left[u^f_{tt}-\Delta u^f-\nabla\rho\cdot\nabla u^f\right]\,v^y\,\rho dx\,dt=$$
$$\int_{\mathbb{R}^n_+\times[0,T]}\left[\rho u^f_{tt}-\operatorname{div}\,\rho\nabla u^f\right]\,v^y\,\rho dx\,dt=$$
$$\int_{\mathbb{R}^n_+}\left[u^f_t\,v^y-u^f\,v^y_t\right]\Big|_{t=0}^{t=T}\rho dx+\int_0^T dt\int_\Gamma\left[u^f_{x^n}\,v^y-u^f\,v^y_{x^n}\right]\,\rho_0 d\Gamma+$$
$$\int_{\mathbb{R}^n_+\times[0,T]}u^f\,\left[v^y_{tt}-\Delta v^y-\nabla\rho\cdot\nabla v^y\right]\,\rho dx\,dt\;=\;\langle\,\text{see }(2),(17)\,\rangle\;=$$
$$-\int_{\mathbb{R}^n_+}u^f(\cdot,T)\,y\,\rho dx\;+\;\int_{\Gamma\times[0,T]}f\,v^y\,\rho_0 d\Gamma dt$$

imply (19). By the density of the chosen f's and y's in the corresponding L_2- spaces, the passage to the limit in the proper sense leads to the assertion of the lemma. □

Taking in (17) $y\in\mathcal{H}^T$, introduce an *observation operator* $O^T:\mathcal{H}^T\to\mathcal{F}^T$, $O^Ty:=v^y|_{\sigma\times[0,T]}$. For $f\in\mathcal{F}^T$, relation (19) can be written in the form $(W^Tf,y)_{\mathcal{H}^T}=(f,O^Ty)_{\mathcal{F}^T}$, which yields

$$O^T\;=\;\left(W^T\right)^* \tag{20}$$

and clarifies the duality of the systems.

8 Jumps in dual system

Here we consider the dual system provided with the specific Cauchy data (17): the velocity perturbation y is assumed to be discontinuous. Recall the notations: $\Omega^\xi=\{x\in\mathbb{R}^n_+|\ \text{dist}\,(x,\sigma)<\xi\}$, $B^\xi=\{x\in\mathbb{R}^n_+|\ x=x(\gamma,\tau),\ \gamma\in\sigma,\tau\in(0,\xi)\},\quad \xi>0$.

Fix $\xi\in(0,T)$ and take a function $y\in C^\infty(\overline{\Omega^T})\subset\mathcal{H}^T$; denote by

$$y_\xi:=\begin{cases} y & \text{in }\Omega^\xi\\ 0 & \text{in }\overline{\Omega^T}\setminus\Omega^\xi\end{cases}$$

its cut-off function onto the subdomain Ω^ξ. In the generic case, the function y_ξ has a jump at a surface $\partial\Omega^\xi\cap\mathbb{R}^n_+$ (see $cdef$ on Fig 3).

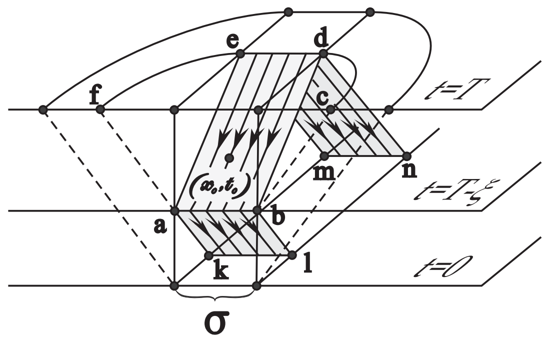

Figure 3: *Jumps in the dual system (16)-(18)*

Return to the dual system and replace y by y_ξ in (17). The well-known fact is that discontinuous data produce discontinuous waves. Namely, the jump of the data at $cdef$ implies the jump of the wave velocity, the latter one being supported on the characteristic surfaces. In particular, these surfaces contain the plane parts $abde$ and $ednm$ consisting of the space-time rays (see the arrows) emanating from the set $\sigma^\xi \times \{t = T\}$ (see ed). The amplitude of the jumps at these parts can be found by standard geometrical optics technique, i.e., by the use of the relevant analog of representation (12) for the solution v^{y_ξ}. Omitting the details, the result is as follows.

For a point $(x_0, t_0) \in abde$ such that $x_0 = x(\gamma, \tau)$, $\gamma \in \sigma, 0 < \tau < \xi$ and $t_0 = T - (\tau - \xi)$, one has

$$v_t^{y_\xi}(x_0, t)\Big|_{t=t_0-0}^{t=t_0+0} = \frac{1}{2}\left(\frac{\rho\left(x(\gamma,\xi)\right)}{\rho\left(x(\gamma,\tau)\right)}\right)^{\frac{1}{2}} y\left(x(\gamma,\xi)\right) . \tag{21}$$

The meaning of this formula is quite transparent: the jump of data at the point $x(\gamma, \xi)$ initiates the jump of velocity, which propagates (in the reversed time) along the ray $\{(x(\gamma, \tau), t) \mid t = T - (\tau - \xi),\ \tau \in [0, \xi]\}$, its amplitude being proportional to the jump of data up to the factor $\frac{1}{2}\left(\frac{\rho}{\rho}\right)^{\frac{1}{2}}$.[9]

Further, at the moment $t = T - \xi$ the jump reaches the boundary Γ and is reflected back into $\mathbb{R}^n_+$ (see the rays constituting the part $ablk$). As result, a trace of the velocity $v_t^{y_\xi}$ on $\sigma \times [0, T]$ turns out to be discontinuous, its jump being supported at the cross-section $\sigma \times \{t = T - \xi\}$ (see ab). The amplitude of this jump can be found from (21): the equality

$$v_t^{y_\xi}(\gamma, t)\Big|_{t=T-\xi-0}^{t=T-\xi+0} = \left(\frac{\rho\left(x(\gamma,\xi)\right)}{\rho\left(x(\gamma,0)\right)}\right)^{\frac{1}{2}} y\left(x(\gamma,\xi)\right)$$

[9]the part $ednm$ also supports the jump moving into $\mathbb{R}^n_+$ in opposite direction. This jump plays no role in the further considerations.

holds [10]. By the definition of the observation operator, this equality can be written as

$$\left(\frac{\partial}{\partial t}\,O^T y_\xi\right)(\gamma,t)\Big|_{t=T-\xi-0}^{t=T-\xi+0} = \left(\frac{\rho\,(x(\gamma,\xi))}{\rho_0(\gamma)}\right)^{\frac{1}{2}}\,y\,(x(\gamma,\xi))\ . \tag{22}$$

At last, putting $y=1$ (so that 1_ξ is an *indicator* of the subdomain Ω^ξ), we obtain the important auxiliary relation

$$\left(\frac{\rho\,(x(\gamma,\xi))}{\rho_0(\gamma)}\right)^{\frac{1}{2}} = \left(\frac{\partial}{\partial t}\,O^T 1_\xi\right)(\gamma,t)\Big|_{t=T-\xi-0}^{t=T-\xi+0}, \quad (\gamma,\xi)\in\sigma\times(0,T)\,. \tag{23}$$

Completing the derivation of (11) in the next two sections, we show how to express the r.h.s. of (23) through the inverse data (operator R^{2T}).

9 Controllability. Wave basis

Fix $\xi\in(0,T)$. In control theory, the set $\mathcal{U}^\xi := \mathrm{Ran}\,W^T = \{u^f(\cdot,T)\,|\;f\in\mathcal{F}^{T,\,\xi}\}$ of waves produced by controls acting from σ (the action time is ξ) is said to be reachable (at the moment $t=\xi$). Since the wave propagation speed is equal to 1, the waves constituting $\mathcal{U}^\xi$ are supported in the metric neighborhood Ω^ξ of σ; as result, the embedding $\mathcal{U}^\xi\subset\mathcal{H}^\xi$ holds. The property of system (1)–(3), which plays the key role in solving inverse problems, is that for any σ and ξ this embedding is dense: $\mathrm{clos}\,\mathcal{U}^\xi=\mathcal{H}^\xi$. Control theory interprets this fact as *a local approximate boundary controllability* [11] of the system. Roughly speaking, it means that the reachable set is rich enough: any function supported in the subdomain Ω^ξ filled with waves, can be approximated (in L_2- metric) by a wave $u^f(\cdot,T)\in\mathcal{U}^\xi$ with the properly chosen control $f\in\mathcal{F}^{T,\,\xi}$. The proof of this property relays on the fundamental Holmgren–John–Tataru uniqueness theorem (see [1] for detail).

Let $\check f^\xi=\{f_k^\xi\}_{k=1}^\infty\subset\mathcal{F}^{T,\,\xi}$ be a linearly independent complete system of controls acting from σ; denote by $\check u^\xi=\{u_k^\xi\}_{k=1}^\infty\subset\mathcal{H}^\xi,\ u_k^\xi:=u^{f_k^\xi}(\cdot,T)=W^Tf_k^\xi$ a system of the corresponding waves. By the controllability, the latter system is also complete: $\mathrm{clos\,span}\,\check u^\xi=\mathcal{H}^\xi$. With a slight abuse of terms, we call $\check u^\xi$ a *wave basis*.

Denote by P_N^ξ the orthogonal projection in $\mathcal{H}^T$ onto $\mathrm{span}\,\{u_k^\xi\}_{k=1}^N$. Since the waves form a complete system, one has $\text{s-lim}_{N\to\infty}P_N^\xi=P^\xi$, where P^ξ projects in $\mathcal{H}^T$ onto $\mathcal{H}^\xi$, i.e., cuts off functions supported in Ω^T onto Ω^ξ. Recall that 1_ξ denotes the indicator of Ω^ξ. Representing

$$1_\xi\;=\;P^\xi 1_T\;=\;\lim_{N\to\infty}P_N^\xi 1_T\,,\qquad P_N^\xi\,1_T\;=\;\sum_{k=1}^N\alpha_k^N\,u_k^\xi\,, \tag{24}$$

[10] this equality can be also derived from (14), (15) by the use of duality relation (19). The factor $\frac{1}{2}$ is doubled owing to the contribution of the reflected rays

[11] this motivates the name "boundary control method"

one can find the coefficients α^ξ_N as follows. Multiplying the last equality in (24) by u^ξ_i, one gets

$$\sum_{k=1}^{N} G_{ik}\,\alpha^N_k \;=\; \beta_i, \quad i=1,\dots,N\,, \tag{25}$$

where

$$\begin{aligned} &G_{ik} := (u^\xi_i, u^\xi_k)_{\mathcal{H}^T} = (W^T f^\xi_i, W^T f^\xi_k)_{\mathcal{H}^T} = \langle \text{see } (4)\rangle = \\ &(C^T f^\xi_i, f^\xi_k)_{\mathcal{F}^T} = \int_{\sigma\times[T-\xi,T]} (C^T f^\xi_i)(\gamma,t)\, f^\xi_k(\gamma,t)\,\rho_0(\gamma) d\Gamma dt \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\beta_i := (P^\xi_N 1_T, u^\xi_i)_{\mathcal{H}^T} = (1_T, P^\xi_N u^\xi_i)_{\mathcal{H}^T} = (1_T, u^\xi_i)_{\mathcal{H}^T} = \\ &\int_{\Omega^T} u^{f^\xi_i}(\cdot,T)\,\rho dx = \langle \text{see } (2)\rangle = \int_{\Omega^T}\rho dx \int_0^T (T-t)\, u^{f^\xi_i}_{tt}(\cdot,T)\,dt \;= \\ &\int_0^T dt\,(T-t)\int_{\Omega^T} \operatorname{div}\rho\nabla u^{f^\xi_i}(\cdot,T)\,dx = -\int_{\Gamma\times[0,T]} (T-t)\, u^{f^\xi_i}_{x^n}(\cdot,T)\,\rho_0 d\Gamma dt = \\ &\langle \text{see } (3)\rangle = -\int_{\sigma\times[T-\xi,T]} (T-t)\, f^\xi_i(\gamma,t)\,\rho_0 d\Gamma dt = \,-\,(\varkappa^T, f^\xi_i)_{\mathcal{F}^T} \end{aligned} \tag{27}$$

with $\varkappa^T(\gamma,t) := T-t$. In the course of integration by parts, the integral over $\partial\Omega^T\cap\mathbb{R}^n_+$ vanishes because the wave $u^{f^\xi_i}(\cdot,T)$ is supported into the smaller subdomain Ω^ξ. Also, note that the linear independence of the system $\check{f}^\xi$ and the injectivity of the of the operator W^T provide the linear independence of the system $\check{u}^\xi$ in $\mathcal{H}^\xi$. By the latter, the Gram matrix G is invertible and, hence, system (25) is uniquely solvable w.r.t. $\alpha^N_1,\dots,\alpha^N_N$.

10 Completing the proof.

Applying O^T to the both sides of (24), with regard to $u^\xi_k = W^T f^\xi_k$, we have

$$O^T 1_\xi = \lim_{N\to\infty}\sum_{k=1}^{N}\alpha^N_k\, O^T W^T f^\xi_k = \langle \text{see } (20)\rangle = \lim_{N\to\infty}\sum_{k=1}^{N}\alpha^N_k\, C^T f^\xi_k\,. \tag{28}$$

At last, substituting (28) into (23), we arrive at (11). The amplitude formula is derived and Theorem 1 is proven. □

The obtained results can be presented in the form of a procedure recovering the density. If the values of $\rho_0|_\sigma$ are known and the response operator R^{2T} is given (as result of measurements on the part σ of the Earth surface), the external observer can determine the density ρ in the tube B^T as follows:

Step 1 Find the operator C^T by (5). Fix $\xi\in(0,T)$ and choose a complete system of controls $\check{f}^\xi\subset\mathcal{F}^{T,\xi}$. Find the Gram matrix $\{G_{ik}\}$ and the numbers β_i by (26), (27).

Step 2 Solve system (25) and find $\alpha_1^N, \dots, \alpha_N^N$ for $N = 1, 2, \dots$. Fixing a $\gamma \in \sigma$, determine $\rho\,(x(\gamma, \xi))$ by the AF (11).

Step 3 Varying $(\gamma, \xi) \in \sigma \times (0, T)$, recover $\rho|_{B^T}$.

11 Visualization of waves

The amplitude formula enables one to recover not only the density but the waves themselves. Choose a control $f \in \mathcal{F}^T$ providing the wave u^f to be a classical solution of (1)–(3). For a fixed $(\gamma, \xi) \in \sigma \times (0, T)$, the representation

$$u^f\,(x(\gamma,\xi),T) \;= \\ \left(\frac{\rho\,(x(\gamma,\xi))}{\rho_0(\gamma)}\right)^{-\frac{1}{2}} \left\{ \left[\frac{\partial}{\partial t} \lim_{N\to\infty} \sum_{k=1}^N \eta_k^N C^T f_k^\xi\right](\gamma,t)\Big|_{t=T-\xi-0}^{t=T-\xi+0} \right\} \tag{29}$$

holds, where the coefficients $\eta_1^N, \dots, \eta_N^N$ satisfy the system

$$\sum_{k=1}^N G_{ik}\,\eta_k^N \;=\; \theta_i, \quad i = 1, \dots, N$$

with $\theta_i := (C^T f, f_i^\xi)_{\mathcal{F}^T}$.

Indeed, putting $y = u^f(\cdot, T)$ in (22), we have

$$\left(\frac{\rho\,(x(\gamma,\xi))}{\rho_0(\gamma)}\right)^{\frac{1}{2}} u^f\,(x(\gamma,\xi),T) \;=\; \left[\frac{\partial}{\partial t} O^T \left(u^f\,(\cdot,T)\right)_\xi\right](\gamma,t)\Big|_{t=T-\xi-0}^{t=T-\xi+0}. \tag{30}$$

Using the wave basis by analogy with (24), we can represent

$$\begin{aligned} \left(u^f\,(\cdot,T)\right)_\xi &= P^\xi u^f(\cdot,T) = \lim_{N\to\infty} P_N^\xi u^f(\cdot,T)\,, \\ P_N^\xi u^f(\cdot,T) \;&=\; \sum_{k=1}^N \eta_k^N\, u_k^\xi \end{aligned} \tag{31}$$

and find the coefficients $\eta_1^N, \dots, \eta_N^N$ from the Gram system

$$\sum_{k=1}^N G_{ik}\eta_k^N \;=\; \theta_i\,, \qquad i = 1, \dots, N$$

with the r.h.s.

$$\theta_i := \left(u^f(\cdot,T), u_i^\xi\right)_{\mathcal{H}^T} = \langle \text{see (4)} \rangle = (C^T f, f_i^\xi)_{\mathcal{F}^T}\,.$$

Then, substituting

$$O^T\left(u^f\,(\cdot,T)\right)_\xi = \langle \text{see (31)} \rangle = \lim_{N\to\infty} \sum_{k=1}^N \eta_k^N\, O^T u_k^\xi = \lim_{N\to\infty} \sum_{k=1}^N \eta_k^N\, C^T f_k^\xi$$

in (30), we arrive at (29).

Thus, the external observer with the knowledge of the response operator R^{2T} can make the waves visible in the tube B^T under the part σ of the Earth surface. This is what we call a *visualization* (see [1]–[3]).

12 Comments

- As regards the numerical realization of the procedure outlined in section 10, the most problematic point is Step 2, which consists of solving system (25) for large N's. Namely, since C^T is a compact operator, the condition number of the matrix $\{G_{ik}\}$ grows as $N \to \infty$, so that (25) turns out to be an ill-posed system. The second problem is that the passage to the limit and the differentiation w.r.t. to t in (11) do not commute. Actually, the difficulties of this type are unavoidable: they reflect the well-known strong ill-posedness of multidimensional inverse problems. However, certain affirmative results in numerical testing do exist and show that elaboration of workable BC-algorithms is not a hopeless endeavor [2], [6].

- For numerical realization, an important issue is the proper choice of the system $\check{f}^\xi$. For many reasons, the controls producing the waves with sharp forward front are preferable. In the problem with *Neumann* boundary controls (2), it looks reasonable to simulate a complete system by the set $\{f^\xi_{qp}\}^{M,N}_{p=1,\,q=1}$:
$$f^\xi_{qp}(\gamma,t) = \delta'_\varepsilon\left(t - \left[(T-\xi) + p\,\frac{\xi}{M+1}\right]\right)\, f_q(\gamma)\,,$$
where $\delta'_\varepsilon(t)$ is a relevant regularization of the first derivative of the Dirac function $\delta(t)$, $\{f_q\}^\infty_{q=1}$ is an orthonormal basis in $L_2(\sigma)$, M and N are large integers. Also, note that in applications one deals as a rule with a certain prescribed set of 'standard' controls. Therefore, it is important to develop the algorithms with controls simulating the real sources.

- The amplitude formulae (11) and (29) can be easily extended to the case of a curved boundary Γ. Namely, assume that $\sigma \subset \Gamma$ and $T > 0$ are such that the field of the normal rays, which form the tube B^T, is regular. Then, the only correction required is to replace ρ_0 by $\frac{\rho_0}{J}$, where $J = J(\gamma,\tau)$ is the Jacobian of the passage from the semideodesic coordinates in B^T_σ to the Cartesian coordinates (see [3]).

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References

[1] M.I. Belishev, *Boundary control in reconstruction of manifolds and metrics (the BC-method).* Invers Problems., **13** (1997), No 5, R1–R45.

[2] M.I. Belishev, V.Yu. Gotlib, *Dynamical variant of the BC-method: theory and numerical testing.* Journal of Inverse and Ill-Posed Problems, **7**, no 3: 221–240, 1999.

[3] M.I. Belishev, *How to see waves under the Earth surface (the BC-method for geophysicists).* Ill-Posed and Inverse Problems, S.I.Kabanikhin and V.G.Romanov (Eds). VSP, 55–72, 2002.

[4] M.I. Belishev, *Recent progress in the boundary control method.* Invers Problems., **23** (2007), No 5, R1–R67.

[5] M. Ikawa. *Hyperbolic PDEs and Wave Phenomena*, Translations of Mathematical Monographs, v. 189 AMS; Providence. Rhode Island, 1997.

[6] S.I. Kabanikhin, M.A. Shishlenin, A.D. Satybaev, *Direct Methods of Solving Inverse Hyperbolic Problems.* Utrecht, The Netherlands, VSP, 2004.

[7] I. Lasiecka, J-L. Lions, R. Triggiani, *Non homogeneous boundary value problems for second order hyperbolic operators.* J. Math. Pures Appl, v. **65** (1986), no 3, 142–192.