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Semi-Classical Dispersive Estimates for the Wave and Schrödinger Equations with a Potential in Dimensions $n \geq 4$

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ABSTRACT

We expand the operators $|t|^{(n-1)/2}e^{it\sqrt{-\Delta+V}}\varphi(h\sqrt{-\Delta+V})$ and $|t|^{n/2}e^{it(-\Delta+V)}$ $\psi(h^2(-\Delta+V))$, $0 < h \ll 1$, modulo operators whose $L^1 \to L^\infty$ norm is $O_N(h^N)$, $\forall N \geq 1$, where $\varphi, \psi \in C_0^\infty((0,+\infty))$ and $V \in L^\infty(\mathbf{R}^n)$, $n \geq 4$, is a real-valued potential satisfying $V(x) = O\left(\langle x\rangle^{-\delta}\right)$, $\delta > (n+1)/2$ in the case of the wave equation and $\delta > (n+2)/2$ in the case of the Schrödinger equation. As a consequence, we give sufficent conditions in order that the wave and the Schrödinger groups satisfy dispersive estimates with a loss of ν derivatives, $0 \leq \nu \leq (n-3)/2$. Roughly speaking, we reduce this problem to estimating the $L^1 \to L^\infty$ norms of a finite number of operators with *almost* explicit kernels. These kernels are completely explicit when $4 \le n \le 7$ in the case of the wave group, and when $n = 4, 5$ in the case of the Schrödinger group.

RESUMEN

En este trabajo son expandidos los operadores $|t|^{(n-1)/2}e^{it\sqrt{-\Delta+V}}\varphi(h\sqrt{-\Delta+V})$ y $|t|^{n/2}e^{it(-\Delta+V)}\psi(h^2(-\Delta+V))$, $0 < h \ll 1$, modulo operadores cuja $L^1 \to L^\infty$ norma es $O_N(h^N)$, $\forall N \ge 1$, donde $\varphi, \psi \in C_0^\infty((0, +\infty))$ y $V \in L^\infty(\mathbf{R}^n)$, $n \ge 4$, es un potencial real satisfaziendo $V(x) = O\left(\langle x\rangle^{-\delta}\right)$, $\delta > (n+1)/2$ en el caso de la ecuación de la onda y $\delta > (n+2)/2$ en el caso de la ecuación de Schrödinger. Como consequencia presentamos condiciones suficientes a fin de que los grupos de la onda y Schrödinger cumplan estimativas dispersivas con una perdida de ν derivadas $0 \le \nu \le (n-3)/2$. Rigurosamente hablando, reduzimos este problema a estimar las normas $L^1 \to L^\infty$ de un número finito de operadores con nucleos casi explicitos. Estos nucleos son completamente explicitos cuando $4 \le n \le 7$ en el caso del grupo de la onda y cuando $n = 4, 5$ en el caso del grupo de Schrödinger.

Key words and phrases: *Potential, dispersive estimates.*

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1 Introduction and statement of results

Denote by G the self-adjoint realization of the operator $-\Delta + V$ on $L^2(\mathbf{R}^n)$, $n \ge 4$, where $V \in L^\infty(\mathbf{R}^n)$ is a real-valued potential satisfying

$$|V(x)| \le C\langle x\rangle^{-\delta}, \quad \forall x \in \mathbf{R}^n, \tag{1.1}$$

with constants $C > 0$, $\delta > (n+1)/2$. It is well known that G has no strictly positive eigenvalues and resonances. We will also denote by G_0 the self-adjoint realization of the operator $-\Delta$ on $L^2(\mathbf{R}^n)$. It is well known that the free wave group satisfies the following *semi-classical* dispersive estimate

$$\left\|e^{it\sqrt{G_0}}\varphi(h\sqrt{G_0})\right\|_{L^1\to L^\infty} \le Ch^{-(n+1)/2}|t|^{-(n-1)/2}, \quad \forall t \ne 0,\ h > 0, \tag{1.2}$$

where $\varphi \in C_0^\infty((0, +\infty))$. The natural question is to find the bigest possible class of potentials for which we have an analogue of (1.2) for the perturbed wave group. It is proved in [16] that under the assumption (1.1) only, we have such an estimate but with a significant loss in h for $0 < h \ll 1$, namely

$$\left\|e^{it\sqrt{G}}\varphi(h\sqrt{G})\right\|_{L^1\to L^\infty} \le Ch^{-n+1}|t|^{-(n-1)/2}, \quad \forall t \ne 0,\ 0 < h \le 1, \tag{1.3}$$

and this seems hard to improve without extra assumptions on the potential. This estimate is then used in [16] to obtain dispersive estimates with a loss of $(n-3)/2$ derivatives for $e^{it\sqrt{G}}\chi_a(\sqrt{G})$, $\forall a>0$, where $\chi_a\in C^\infty((-\infty,+\infty))$, $\chi_a(\lambda)=0$ for $\lambda\le a$, $\chi_a(\lambda)=1$ for $\lambda\ge 2a$.

In the present work we will expand $e^{it\sqrt{G}}\varphi(h\sqrt{G})$ modulo remainders whose $L^1\to L^\infty$ norm is upper bounded by $C_m h^{m-n+1}|t|^{-(n-1)/2}$, $0<h\le h_0\ll 1$, for every integer $m\ge 0$. In order to state the precise result we need to introduce some notations. Let $\varphi_1\in C_0^\infty((0,+\infty))$ be such that $\varphi_1=1$ on $\operatorname{supp}\varphi$, and set $\widetilde{\varphi}(\lambda)=\lambda\varphi(\lambda)$, $\widetilde{\varphi}_1(\lambda)=\lambda^{-1}\varphi_1(\lambda)$. Under (1.1) there exists a constant $h_0>0$ so that for $0<h\le h_0$, the operator

$$T(h):=\left(Id+\varphi_1(h\sqrt{G_0})-\varphi_1(h\sqrt{G})\right)^{-1}=Id+O(h^2)$$

is uniformely bounded on L^p, $1\le p\le+\infty$, as well as on weighted L^2 spaces (see Lemma 2.3 of [16] and Lemma A.1 of [11]). Set

$$U_0(t,h)=\widetilde{\varphi}_1(h\sqrt{G_0})\sin(t\sqrt{G_0}),\quad E_0^0(t,h)=e^{it\sqrt{G_0}}\varphi(h\sqrt{G_0}),$$

$$E_0(t,h)=\varphi_1(h\sqrt{G_0})\cos(t\sqrt{G_0})\varphi(h\sqrt{G})+i\widetilde{\varphi}_1(h\sqrt{G_0})\sin(t\sqrt{G_0})\widetilde{\varphi}(h\sqrt{G}).$$

Furthermore, given any integer $j\ge 1$, define the operators

$$E_j(t,h)=-h\int_0^t U_0(t-\tau,h)VT(h)E_{j-1}(\tau,h)d\tau,$$

$$E_j^0(t,h)=-h\int_0^t U_0(t-\tau,h)VE_{j-1}^0(\tau,h)d\tau.$$

Theorem 1.1 *Let V satisfy (1.1). Then, there exists a constant $h_0>0$ so that for all $0<h\le h_0$, $t\ne 0$, we have the estimate*

$$\left\|e^{it\sqrt{G}}\varphi(h\sqrt{G})-T(h)\sum_{j=0}^m E_j(t,h)\right\|_{L^1\to L^\infty}\le C_m h^{m-n+1}|t|^{-(n-1)/2}, \tag{1.4}$$

for every integer $m\ge 0$ with a constant $C_m>0$ independent of t and h. Moreover, the operators E_j satisfy the estimates

$$\|E_0(t,h)\|_{L^1\to L^\infty}\le Ch^{-(n+1)/2}|t|^{-(n-1)/2}, \tag{1.5}$$

$$\|E_j(t,h)\|_{L^1\to L^\infty}\le C_j h^{j-n}|t|^{-(n-1)/2},\quad j\ge 1, \tag{1.6}$$

$$\left\|E_j(t,h)-E_j^0(t,h)\right\|_{L^1\to L^\infty}\le C_j h^{j+2-n}|t|^{-(n-1)/2},\quad j\ge 1. \tag{1.7}$$

It follows from this theorem that to improve the estimate (1.3) in h, it suffices to improve the estimate (1.6). We also have the following

Corollary 1.2 *Let V satisfy (1.1) and suppose in addition that there exists $0 \le k \le (n-3)/2$ such that the operators E_j satisfy the estimate*

$$\|E_j(t,h)\|_{L^1\to L^\infty} \le Ch^{k-n+1}|t|^{-(n-1)/2}, \tag{1.8}$$

for all integers $1 \le j < k+1$. Then, for every $a > 0$, $0 < \epsilon \ll 1$, we have the estimate

$$\left\|e^{it\sqrt{G}}(\sqrt{G})^{k-n+1-\epsilon}\chi_a(\sqrt{G})\right\|_{L^1\to L^\infty} \le C_\epsilon |t|^{-(n-1)/2}, \quad \forall t \ne 0, \tag{1.9}$$

while for every $0 \le q \le (n-3)/2 - k$, $2 \le p < \frac{2(n-1-2q-2k)}{n-3-2q-2k}$, we have

$$\left\|e^{it\sqrt{G}}(\sqrt{G})^{-\alpha((n+1)/2+q)}\chi_a(\sqrt{G})\right\|_{L^{p'}\to L^p} \le C|t|^{-\alpha(n-1)/2}, \quad \forall t \ne 0, \tag{1.10}$$

where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$. Moreover, when $4 \le n \le 7$ the estimates (1.9) and (1.10) hold true if we suppose (1.8) fulfilled with E_j replaced by E_j^0.

The estimate (1.8) with $k > 0$ seems hard to establish (even if we replace E_j by E_j^0) and the proof would probably require some regularity condition on the potential. Note that when $n = 2$ and $n = 3$ the estimates (1.9) and (1.10) (with $k = (n-3)/2$, $q = 0$) are proved in [2] under (1.1) only. In the case of $n = 2$ these estimates are proved (for a large enough) in [10] for a much larger class of potentials satisfying

$$\sup_{y\in\mathbf{R}^2}\int_{\mathbf{R}^2}\frac{|V(x)|dx}{|x-y|^{1/2}} \le C < +\infty. \tag{1.11}$$

When $n = 3$ these estimates are proved in [4] for a quite large subclass of potentials satisfying

$$\sup_{y\in\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|V(x)|dx}{|x-y|} \le C < +\infty. \tag{1.12}$$

When $n \ge 4$ optimal dispersive estimates (that is, without loss of derivatives) are proved in [1] for potentials belonging to the Schwartz class. When $n \ge 4$, as mentioned above, the estimates (1.9) and (1.10) with $k = 0$ are proved in [16] under (1.1) only. The proof of Theorem 1.1 and Corollary 1.2, which will be given in Section 2, is based very much on the analysis developed in [16].

A similar analysis as above can be carried out for the Schrödinger group as well. The free one satisfies the following dispersive estimate

$$\left\|e^{itG_0}\psi(h^2G_0)\right\|_{L^1\to L^\infty} \le C|t|^{-n/2}, \quad \forall t \ne 0,\ h > 0, \tag{1.13}$$

where $\psi \in C_0^\infty((0,+\infty))$. On the other hand, it is proved in [15] that under the assumption (1.1) with $\delta > (n+2)/2$ only, the perturbed Schrödinger group satisfies

$$\left\|e^{itG}\psi(h^2G)\right\|_{L^1\to L^\infty} \le Ch^{-(n-3)/2}|t|^{-n/2}, \quad \forall t \ne 0,\ 0 < h \le 1. \tag{1.14}$$

This estimate is used in [15] to obtain dispersive estimates with a loss of $(n-3)/2$ derivatives for $e^{itG}\chi_a(G)$, $\forall a > 0$.

In this work we will also expand $e^{itG}\psi(h^2G)$ modulo remainders whose $L^1 \to L^\infty$ norm is upper bounded by $C_m h^{m-(n-2)/2-\epsilon}|t|^{-n/2}$, $0 < h \le h_0 \ll 1$, for every integer $m \ge 0$, similarly to the wave group above. To this end, choose a function $\psi_1 \in C_0^\infty((0,+\infty))$ such that $\psi_1 = 1$ on $\mathrm{supp}\,\psi$, and set

$$T(h) := \left(Id + \psi_1(h^2G_0) - \psi_1(h^2G)\right)^{-1} = Id + O(h^2),$$

$$F_0^0(t,h) = e^{itG_0}\psi(h^2G_0), \quad F_0(t,h) = \psi_1(h^2G_0)e^{itG_0}\psi(h^2G), \quad W_0(t,h) = e^{itG_0}\psi_1(h^2G_0),$$

$$F_j(t,h) = i\int_0^t W_0(t-\tau,h)VT(h)F_{j-1}(\tau,h)d\tau, \quad j \ge 1,$$

$$F_j^0(t,h) = i\int_0^t W_0(t-\tau,h)VF_{j-1}^0(\tau,h)d\tau, \quad j \ge 1.$$

Theorem 1.3 *Let V satisfy (1.1) with $\delta > (n+2)/2$. Then, there exists a constant $h_0 > 0$ so that for all $0 < h \le h_0$, $t \ne 0$, $0 < \epsilon \ll 1$, we have the estimate*

$$\left\| e^{itG}\psi(h^2G) - T(h)\sum_{j=0}^m F_j(t,h) \right\|_{L^1\to L^\infty} \le C_m h^{m-(n-2)/2-\epsilon}|t|^{-n/2}, \tag{1.15}$$

for every integer $m \ge 0$ with a constant $C_m > 0$ independent of t and h. Moreover, the operators F_j satisfy the estimates

$$\|F_0(t,h)\|_{L^1\to L^\infty} \le C|t|^{-n/2}, \tag{1.16}$$

$$\|F_j(t,h)\|_{L^1\to L^\infty} \le C_j h^{j-n/2-\epsilon}|t|^{-n/2}, \quad j \ge 1, \tag{1.17}$$

$$\left\|F_j(t,h) - F_j^0(t,h)\right\|_{L^1\to L^\infty} \le C_j h^{j+2-n/2-\epsilon}|t|^{-n/2}, \quad j \ge 1. \tag{1.18}$$

Thus, to improve the estimate (1.14) in h, it suffices to improve the estimate (1.17). We also have the following

Corollary 1.4 *Let V satisfy (1.1) with $\delta > (n+2)/2$ and suppose in addition that there exists $0 \le k \le (n-3)/2$ such that the operators F_j satisfy the estimate*

$$\|F_j(t,h)\|_{L^1\to L^\infty} \le Ch^{k-(n-3)/2}|t|^{-n/2}, \tag{1.19}$$

for all integers $1 \le j \le k + 3/2$. Then, for every $a > 0$, $0 < \epsilon \ll 1$, we have the estimate

$$\left\| e^{itG}G^{k/2-(n-3)/4-\epsilon}\chi_a(G)\right\|_{L^1\to L^\infty} \le C_\epsilon|t|^{-n/2}, \quad \forall t \ne 0, \tag{1.20}$$

while for every $0 \le q \le (n-3)/2 - k$, $2 \le p < \frac{2(n-1-2q-2k)}{n-3-2q-2k}$, we have

$$\left\| e^{itG}G^{-\alpha q/2}\chi_a(G)\right\|_{L^{p'}\to L^p} \le C|t|^{-\alpha n/2}, \quad \forall t \ne 0, \tag{1.21}$$

where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$. Moreover, if there exists an operator $\mathcal{F}_k(t)$, independent of h, such that the following estimates hold

$$\left\| \mathcal{F}_k(t)G_0^{k/2-(n-3)/4}\right\|_{L^1\to L^\infty} \le C|t|^{-n/2}, \tag{1.22}$$

$$\left\|F_1(t,h)-\mathcal{F}_k(t)\psi(h^2G_0)\right\|_{L^1\to L^\infty}\le Ch^{k-(n-3)/2+\varepsilon}|t|^{-n/2}, \tag{1.23}$$

$$\|F_j(t,h)\|_{L^1\to L^\infty}\le Ch^{k-(n-3)/2+\varepsilon}|t|^{-n/2}, \tag{1.24}$$

for $2\le j\le k+3/2$ with some $\varepsilon>0$, then we have

$$\left\|e^{itG}G^{k/2-(n-3)/4}\chi_a(G)\right\|_{L^1\to L^\infty}\le C|t|^{-n/2},\quad \forall t\neq 0. \tag{1.25}$$

Furthermore, when $n=4,5$ the estimates (1.20), (1.21) and (1.25) hold true if we suppose (1.19), (1.23) and (1.24) fulfilled with F_j replaced by F_j^0.

As in the case of the wave group above, the estimates (1.19), (1.22), (1.23) and (1.24) with $k>0$ seem hard to establish (even if we replace F_j by F_j^0) and the proof would certainly require some regularity condition on the potential. Indeed, it follows from the results in [5] that there exist compactly supported potentials $V\in C^\nu(\mathbf{R}^n)$, $\forall\,\nu<(n-3)/2$, for which these estimates with $k=(n-3)/2$ fail to hold. Therefore, it is naural to expect that they hold true for potentials $V\in C^{(n-3)/2-k}(\mathbf{R}^n)$. We also conjecture that the statements of Theorem 1.3 and Corollary 1.4 hold true for potentials satisfying (1.1) with $\delta>(n+1)/2$ as for the wave group above. Note that when $n=2$ the estimate (1.25) without loss of derivatives (that is, with $k=(n-3)/2$) is proved in [12] under (1.1) with $\delta>2$. In this case this estimate is proved (for a large enough) in [10] for potentials satisfying (1.11). When $n=3$ this estimate is proved in [6] for potentials $V\in L^{3/2-\epsilon}\cap L^{3/2+\epsilon}$, $0<\epsilon\ll 1$, and in particular for potentials satisfying (1.1) with $\delta>2$. In this case it is also proved in [13] for potentials satisfying (1.12) with $C<4\pi$. When $n\ge 4$ the optimal dispersive estimate (that is, without loss of derivatives) is proved in [9] for potentials satisfying (1.1) with $\delta>n$ as well as $\widehat{V}\in L^1$. This result has been recently extended in [11] to potentials satisfying (1.1) with $\delta>n-1$ as well as $\widehat{V}\in L^1$. When $n\ge 4$, as mentioned above, the estimates (1.21) and (1.25) with $k=0$ are proved in [15] under (1.1) with $\delta>(n+2)/2$ only. The proof of Theorem 1.3 and Corollary 1.4, which will be given in Section 3, relies very much on the analysis developed in [15].

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2 Semi-classical expansion of $e^{it\sqrt{G}}\varphi(h\sqrt{G})$

We keep the same notations as in the introduction. Our starting point is the following identity which can be derived easily from Duhamel's formula (see [16])

$$\left(Id+\varphi_1(h\sqrt{G_0})-\varphi_1(h\sqrt{G})\right)e^{it\sqrt{G}}\varphi(h\sqrt{G})$$

$$=E_0(t,h)-h\int_0^t U_0(t-\tau,h)Ve^{i\tau\sqrt{G}}\varphi(h\sqrt{G})d\tau. \tag{2.1}$$

We rewrite (2.1) as follows

$$e^{it\sqrt{G}}\varphi(h\sqrt{G}) = \widetilde{E}_0(t,h) + \int_0^t \widetilde{U}_0(t-\tau,h)Ve^{i\tau\sqrt{G}}\varphi(h\sqrt{G})d\tau, \tag{2.2}$$

where

$$\widetilde{E}_0(t,h) = T(h)E_0(t,h), \quad \widetilde{U}_0(t,h) = -hT(h)U_0(t,h).$$

Iterating (2.2) m times leads to the identity

$$e^{it\sqrt{G}}\varphi(h\sqrt{G}) = \sum_{j=0}^{m} \widetilde{E}_j(t,h) + R_{m+1}(t,h), \tag{2.3}$$

where the operators $\widetilde{E}_j$, $j \geq 1$, are defined by

$$\widetilde{E}_j(t,h) = \int_0^t \widetilde{U}_0(t-\tau,h)V\widetilde{E}_{j-1}(\tau,h)d\tau,$$

while the operators R_m, $m \geq 0$, are defined as follows

$$R_0(t,h) = e^{it\sqrt{G}}\varphi(h\sqrt{G}),$$

$$R_{m+1}(t,h) = \int_0^t \widetilde{U}_0(t-\tau,h)VR_m(\tau,h)d\tau.$$

It is clear from (2.3) that the estimate (1.4) follows from the following

Proposition 2.1 *Under the assumptions of Theorem 1.1, for all* $0 < h \leq h_0$, $t \neq 0$, $1/2 - \epsilon/4 \leq s \leq (n-1)/2$, $0 < \epsilon \ll 1$, *we have the estimates*

$$\|R_{m+1}(t,h)\|_{L^1 \to L^\infty} \leq C_m h^{m-n+1}|t|^{-(n-1)/2}, \tag{2.4}$$

$$\left\|\langle x\rangle^{-s-\epsilon}R_{m+1}(t,h)\right\|_{L^1\to L^2} \leq C_m h^{m-n/2+1}|t|^{-s}, \tag{2.5}$$

for every integer $m \geq 0$.

Proof. For $m = 0$ the estimate (2.4) is proved in [16] (see (4.10)). We will now derive (2.4) for $m \geq 1$ from (2.5) and the following estimate proved in [16] (see (2.4)):

$$\int_{-\infty}^{\infty} |t|^{2s} \left\|\langle x\rangle^{-1/2-s-\epsilon}e^{it\sqrt{G_0}}\varphi(h\sqrt{G_0})f\right\|_{L^2}^2 dt \leq Ch^{-n}\|f\|_{L^1}^2, \quad \forall f \in L^1, \tag{2.6}$$

for $0 \leq s \leq (n-1)/2$, $0 < \epsilon \ll 1$. By (2.5) and (2.6), we have

$$|t|^{(n-1)/2}\left|\langle R_{m+1}(t,h)f,g\rangle\right|$$

$$\leq C\int_{t/2}^{t} |\tau|^{(n-1)/2}\left\|\langle x\rangle^{-1-\epsilon}\widetilde{U}_0(t-\tau,h)^*g\right\|_{L^2}\left\|\langle x\rangle^{-(n-1)/2-\epsilon}R_m(\tau,h)f\right\|_{L^2}d\tau$$

$$+C\int_{t/2}^{t}|\tau|^{(n-1)/2}\left\|\langle x\rangle^{-n/2-\epsilon}\widetilde{U}_0(\tau,h)^*g\right\|_{L^2}\left\|\langle x\rangle^{-1/2-\epsilon}R_m(t-\tau,h)f\right\|_{L^2}d\tau$$

$$\leq Ch^{m-n/2}\|f\|_{L^1}\left(\int_{-\infty}^{\infty}\langle\tau'\rangle^{1+\epsilon/2}\left\|\langle x\rangle^{-1-\epsilon}\widetilde{U}_0(\tau',h)^*g\right\|_{L^2}^2d\tau'\right)^{1/2}$$

$$+C\left(\int_{-\infty}^{\infty}|\tau|^{n-1}\left\|\langle x\rangle^{-n/2-\epsilon}\widetilde{U}_0(\tau,h)^*g\right\|_{L^2}^2d\tau\right)^{1/2}\left(\int_{-\infty}^{\infty}\left\|\langle x\rangle^{-1/2-\epsilon}R_m(\tau',h)f\right\|_{L^2}^2d\tau'\right)^{1/2}$$

$$\leq Ch^{m+1-n}\|f\|_{L^1}\|g\|_{L^1}.$$

We will now prove (2.5) by induction in m. For $m=0$ it is proved in [16] (see (4.6)) with $s=(n-1)/2$ but the proof for general s is the same. We will show that (2.5) for R_{m+1} follows from (2.5) for R_m and the following estimate proved in [16] (see (2.1)):

$$\left\|\langle x\rangle^{-s}e^{it\sqrt{G_0}}\varphi(h\sqrt{G_0})\langle x\rangle^{-s}\right\|_{L^2\to L^2}\leq C\langle t\rangle^{-s},\quad \forall t,\ 0<h\leq 1. \tag{2.7}$$

Consider first the case $1\leq s\leq (n-1)/2$. We have

$$|t|^s\left\|\langle x\rangle^{-s-\epsilon}R_{m+1}(t,h)\right\|_{L^1\to L^2}$$

$$\leq C\int_{t/2}^{t}|\tau|^s\left\|\langle x\rangle^{-s-\epsilon}\widetilde{U}_0(t-\tau,h)\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^2}\left\|\langle x\rangle^{-s-\epsilon}R_m(\tau,h)\right\|_{L^1\to L^2}d\tau$$

$$+C\int_{t/2}^{t}|\tau|^s\left\|\langle x\rangle^{-s-\epsilon}\widetilde{U}_0(\tau,h)\langle x\rangle^{-s-\epsilon}\right\|_{L^2\to L^2}\left\|\langle x\rangle^{-1-\epsilon}R_m(t-\tau,h)\right\|_{L^1\to L^2}d\tau$$

$$\leq Ch^{m+1-n/2}\int_{-\infty}^{\infty}\langle\tau'\rangle^{-1-\epsilon}d\tau'+Ch\int_{-\infty}^{\infty}\left\|\langle x\rangle^{-1-\epsilon}R_m(\tau',h)\right\|_{L^1\to L^2}d\tau'\leq Ch^{m+1-n/2}.$$

Let now $1/2-\epsilon/4\leq s\leq 1$. We have

$$|t|^s\left\|\langle x\rangle^{-s-\epsilon}R_{m+1}(t,h)\right\|_{L^1\to L^2}$$

$$\leq C\int_{t/2}^{t}|\tau|^s\left\|\langle x\rangle^{-s-\epsilon}\widetilde{U}_0(t-\tau,h)\langle x\rangle^{-1/2-\epsilon}\right\|_{L^2\to L^2}\left\|\langle x\rangle^{-s-1/2-\epsilon}R_m(\tau,h)\right\|_{L^1\to L^2}d\tau$$

$$+C\int_{t/2}^{t}|\tau|^s\left\|\langle x\rangle^{-s-\epsilon}\widetilde{U}_0(\tau,h)\langle x\rangle^{-s-\epsilon}\right\|_{L^2\to L^2}\left\|\langle x\rangle^{-1-\epsilon}R_m(t-\tau,h)\right\|_{L^1\to L^2}d\tau$$

$$\leq Ch\left(\int_{-\infty}^{\infty}\langle\tau'\rangle^{-1-\epsilon}d\tau'\right)^{1/2}\left(\int_{-\infty}^{\infty}|\tau|^{2s}\left\|\langle x\rangle^{-s-1/2-\epsilon}R_m(\tau,h)\right\|_{L^1\to L^2}^2d\tau\right)^{1/2}$$

$$+Ch\int_{-\infty}^{\infty}\left\|\langle x\rangle^{-1-\epsilon}R_m(\tau',h)\right\|_{L^1\to L^2}d\tau'\leq Ch^{m+1-n/2}.$$

□

To prove (1.6) observe first that in the same way as in the proof of (2.5) one can show that the operator E_j satisfies the estimate

$$\left\|\langle x\rangle^{-s-\epsilon}E_j(t,h)\right\|_{L^1\to L^2}\leq C_jh^{j-n/2}|t|^{-s},\quad j\geq 1, \tag{2.8}$$

for $1/2-\epsilon/4\le s\le (n-1)/2$, $0<\epsilon\ll 1$. On the other hand, proceeding as in the proof of (2.4), one can easily see that (2.8) implies (1.6). To prove (1.7) we decompose $E_j-E_j^0$ as follows

$$E_j(t,h)-E_j^0(t,h)=-h\int_0^t U_0(t-\tau,h)V(T(h)-Id)E_{j-1}(\tau,h)d\tau$$

$$+h\int_0^t U_0(t-\tau,h)V(E_{j-1}(\tau,h)-E_{j-1}^0(\tau,h))d\tau:=\mathcal{E}_j^1(t,h)+\mathcal{E}_j^2(t,h). \tag{2.9}$$

Using (2.8), in the same way as in the proof of (1.6), one gets

$$\left\|\mathcal{E}_j^1(t,h)\right\|_{L^1\to L^\infty}\le C_j h^{j+2-n}|t|^{-(n-1)/2}. \tag{2.10}$$

On the other hand, it is easy to see from (2.9) by induction in j that we have the estimate

$$\left\|\langle x\rangle^{-s-\epsilon}(E_j(t,h)-E_j^0(t,h))\right\|_{L^1\to L^2}\le C_j h^{j+2-n/2}|t|^{-s},\quad j\ge 0, \tag{2.11}$$

for $1/2-\epsilon/4\le s\le (n-1)/2$, $0<\epsilon\ll 1$. It follows from (2.11) that the operator $\mathcal{E}_j^2$ satisfies the estimate

$$\left\|\mathcal{E}_j^2(t,h)\right\|_{L^1\to L^\infty}\le C_j h^{j+2-n}|t|^{-(n-1)/2}. \tag{2.12}$$

Now (1.7) follows from (2.9), (2.10) and (2.12).

Proof of Corollary 1.2. Following [16] we set

$$\Phi(t,h)=e^{it\sqrt{G}}\varphi(h\sqrt{G})-e^{it\sqrt{G_0}}\varphi(h\sqrt{G_0}).$$

It follows from (1.4) and (1.8) that the operator Φ satisfies the estimate

$$\|\Phi(t,h)\|_{L^1\to L^\infty}\le Ch^{k-n+1}|t|^{-(n-1)/2}. \tag{2.13}$$

On the other hand, we have (see Theorem 3.1 of [16])

$$\|\Phi(t,h)\|_{L^2\to L^2}\le Ch,\quad \forall t. \tag{2.14}$$

By interpolation between (2.13) and (2.14) we conclude

$$\|\Phi(t,h)\|_{L^{p'}\to L^p}\le Ch^{1-\alpha(n-k)}|t|^{-\alpha(n-1)/2}, \tag{2.15}$$

for every $2\le p\le +\infty$, where $1/p+1/p'=1$, $\alpha=1-2/p$. Now we will make use of the identity

$$\sigma^{-\alpha((n+1)/2+q)}\chi_a(\sigma)=\int_0^1\varphi(\theta\sigma)\theta^{\alpha((n+1)/2+q)-1}d\theta,$$

where $\varphi(\sigma)=\sigma^{1-\alpha((n+1)/2+q)}\chi_a'(\sigma)\in C_0^\infty((0,+\infty))$. By (2.15) we get

$$\left\|e^{it\sqrt{G}}(\sqrt{G})^{-\alpha((n+1)/2+q)}\chi_a(\sqrt{G})-e^{it\sqrt{G_0}}(\sqrt{G_0})^{-\alpha((n+1)/2+q)}\chi_a(\sqrt{G_0})\right\|_{L^{p'}\to L^p}$$

$$\leq \int_0^1 \|\Phi(t,\theta)\|_{L^{p'}\to L^p}\,\theta^{\alpha((n+1)/2+q)-1}d\theta$$

$$\leq C|t|^{-\alpha(n-1)/2}\int_0^1 \theta^{-\alpha((n-1)/2-k-q)}d\theta \leq C|t|^{-\alpha(n-1)/2}, \tag{2.16}$$

provided $\alpha((n-1)/2-k-q)<1$, that is, for $2\leq p<\frac{2(n-1-2q-2k)}{n-3-2q-2k}$. Now (1.10) follows from (2.16) and the fact that it holds for the free operator. Similarly, by (2.13) we get

$$\left\|e^{it\sqrt{G}}(\sqrt{G})^{k-n+1-\epsilon}\chi_a(\sqrt{G})-e^{it\sqrt{G_0}}(\sqrt{G_0})^{k-n+1-\epsilon}\chi_a(\sqrt{G_0})\right\|_{L^1\to L^\infty}$$

$$\leq \int_0^1 \|\Phi(t,\theta)\|_{L^1\to L^\infty}\,\theta^{n-k-2+\epsilon}d\theta$$

$$\leq C|t|^{-(n-1)/2}\int_0^1 \theta^{-1+\epsilon}d\theta \leq C_\epsilon|t|^{-(n-1)/2}. \tag{2.17}$$

Now (1.9) follows from (2.17) and the fact that it holds for the free operator. □

3 Semi-classical expansion of $e^{itG}\psi(h^2G)$

We keep the same notations as in the introduction. We will make use of the following identity which can be derived easily from Duhamel's formula (see [15])

$$\left(Id+\psi_1(h^2G_0)-\psi_1(h^2G)\right)e^{itG}\psi(h^2G)$$

$$=F_0(t,h)+i\int_0^t W_0(t-\tau,h)Ve^{i\tau G}\psi(h^2G)d\tau. \tag{3.1}$$

We rewrite (3.1) as follows

$$e^{itG}\psi(h^2G)=\widetilde{F}_0(t,h)+\int_0^t \widetilde{W}_0(t-\tau,h)Ve^{i\tau G}\psi(h^2G)d\tau, \tag{3.2}$$

where

$$\widetilde{F}_0(t,h)=T(h)F_0(t,h),\quad \widetilde{W}_0(t,h)=iT(h)W_0(t,h).$$

Iterating (3.2) m times leads to the identity

$$e^{itG}\psi(h^2G)=\sum_{j=0}^m \widetilde{F}_j(t,h)+\mathcal{R}_{m+1}(t,h), \tag{3.3}$$

where the operators $\widetilde{F}_j$, $j\geq 1$, are defined by

$$\widetilde{F}_j(t,h)=\int_0^t \widetilde{W}_0(t-\tau,h)V\widetilde{F}_{j-1}(\tau,h)d\tau,$$

while the operators $\mathcal{R}_m$, $m\geq 0$, are defined as follows

$$\mathcal{R}_0(t,h)=e^{itG}\psi(h^2G),$$

$$\mathcal{R}_{m+1}(t,h)=\int_0^t \widetilde{W}_0(t-\tau,h)V\mathcal{R}_m(\tau,h)d\tau.$$

It is clear from (3.3) that the estimate (1.15) follows from the following

Proposition 3.1 *Under the assumptions of Theorem 1.2, for all* $0<h\le h_0$, $t\neq 0$, $1/2-\epsilon/4\le s\le (n-1)/2$, $0<\epsilon\ll 1$, *we have the estimates*

$$\|\mathcal{R}_{m+1}(t,h)\|_{L^1\to L^\infty}\le C_m h^{m-(n-2)/2-\epsilon}|t|^{-n/2}, \tag{3.4}$$

$$\left\|\langle x\rangle^{-1/2-s-\epsilon}\mathcal{R}_{m+1}(t,h)\right\|_{L^1\to L^2}\le C_m h^{m+s-(n-3)/2-\epsilon/6}|t|^{-s-1/2}, \tag{3.5}$$

for every integer $m\ge 0$.

Proof. For $m=0$ these estimates are proved in Section 4 of [15]. We will now derive (3.4) for $m\ge 1$ from (3.5) and the following estimate proved in [15] (see (2.1)):

$$\left\|e^{itG_0}\psi(h^2G_0)\langle x\rangle^{-1/2-s-\epsilon}\right\|_{L^2\to L^\infty}\le Ch^{s-(n-1)/2}|t|^{-s-1/2},\quad t\neq 0,\ 0<h\le 1, \tag{3.6}$$

for $0\le s\le (n-1)/2$, $0<\epsilon\ll 1$. We have

$$\begin{aligned}
&|t|^{n/2}\,\|\mathcal{R}_{m+1}(t,h)\|_{L^1\to L^\infty}\\
&\le C\int_{t/2}^t |\tau|^{n/2}\left\|\widetilde{W}_0(t-\tau,h)\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-n/2-\epsilon}\mathcal{R}_m(\tau,h)\right\|_{L^1\to L^2}d\tau\\
&+C\int_{t/2}^t |\tau|^{n/2}\left\|\widetilde{W}_0(\tau,h)\langle x\rangle^{-n/2-\epsilon}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-1-\epsilon}\mathcal{R}_m(t-\tau,h)\right\|_{L^1\to L^2}d\tau\\
&\le Ch^{m-\epsilon/6}\int_{-\infty}^{\infty}\left\|\widetilde{W}_0(\tau',h)\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^\infty}d\tau'\\
&+C\int_{-\infty}^{\infty}\left\|\langle x\rangle^{-1-\epsilon}\mathcal{R}_m(\tau',h)\right\|_{L^1\to L^2}d\tau'\le Ch^{m-(n-2)/2-\epsilon/3}.
\end{aligned}$$

We will now show that (3.5) for $\mathcal{R}_{m+1}$ follows from (3.5) for $\mathcal{R}_m$ and the following estimate proved in [15] (see (2.2)):

$$\left\|\langle x\rangle^{-s}e^{itG_0}\psi(h^2G_0)\langle x\rangle^{-s}\right\|_{L^2\to L^2}\le C\langle t/h\rangle^{-s},\quad \forall t,\ 0<h\le 1. \tag{3.7}$$

We have

$$\begin{aligned}
&|t|^{s+1/2}\left\|\langle x\rangle^{-1/2-s-\epsilon}\mathcal{R}_{m+1}(t,h)\right\|_{L^1\to L^2}\\
&\le C\int_{t/2}^t |\tau|^{s+1/2}\left\|\langle x\rangle^{-1/2-s-\epsilon}\widetilde{W}_0(t-\tau,h)\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^2}\left\|\langle x\rangle^{-1/2-s-\epsilon}\mathcal{R}_m(\tau,h)\right\|_{L^1\to L^2}d\tau\\
&+C\int_{t/2}^t |\tau|^{s+1/2}\left\|\langle x\rangle^{-1/2-s-\epsilon}\widetilde{W}_0(\tau,h)\langle x\rangle^{-1/2-s-\epsilon}\right\|_{L^2\to L^2}\left\|\langle x\rangle^{-1-\epsilon}\mathcal{R}_m(t-\tau,h)\right\|_{L^1\to L^2}d\tau
\end{aligned}$$

$$\leq Ch^{m+s-(n-1)/2-\epsilon/6}\int_{-\infty}^{\infty}\langle\tau'/h\rangle^{-1-\epsilon/2}d\tau'$$

$$+Ch^{s+1/2}\int_{-\infty}^{\infty}\left\|\langle x\rangle^{-1-\epsilon}\mathcal{R}_m(\tau',h)\right\|_{L^1\to L^2}d\tau'\leq Ch^{m+s-(n-3)/2-\epsilon/6}.$$

□

To prove (1.17) observe that in the same way as in the proof of (3.5) one can show that the operator F_j satisfies the estimate

$$\left\|\langle x\rangle^{-1/2-s-\epsilon}F_j(t,h)\right\|_{L^1\to L^2}\leq C_jh^{j+s-(n-1)/2-\epsilon/6}|t|^{-s-1/2},\quad j\geq 1, \tag{3.8}$$

for $1/2-\epsilon/4\leq s\leq (n-1)/2$, $0<\epsilon\ll 1$. On the other hand, proceeding as in the proof of (3.4), one can easily see that (3.8) implies (1.17). To prove (1.18) we decompose $F_j-F_j^0$ as follows

$$F_j(t,h)-F_j^0(t,h)=i\int_0^t W_0(t-\tau,h)V(T(h)-Id)F_{j-1}(\tau,h)d\tau$$

$$-i\int_0^t W_0(t-\tau,h)V(F_{j-1}(\tau,h)-F_{j-1}^0(\tau,h))d\tau:=N_j^1(t,h)+N_j^2(t,h). \tag{3.9}$$

Using (3.8), in the same way as in the proof of (1.17), one gets

$$\left\|N_j^1(t,h)\right\|_{L^1\to L^\infty}\leq C_jh^{j+2-n/2-\epsilon}|t|^{-n/2}. \tag{3.10}$$

On the other hand, it is easy to see from (3.9) by induction in j that we have the estimate

$$\left\|\langle x\rangle^{-1/2-s-\epsilon}(F_j(t,h)-F_j^0(t,h))\right\|_{L^1\to L^2}\leq C_jh^{j+2+s-(n-1)/2-\epsilon/6}|t|^{-s-1/2},\quad j\geq 0, \tag{3.11}$$

for $1/2-\epsilon/4\leq s\leq (n-1)/2$, $0<\epsilon\ll 1$. It follows from (3.11) that the operator N_j^2 satisfies the estimate

$$\left\|N_j^2(t,h)\right\|_{L^1\to L^\infty}\leq C_jh^{j+2-n/2-\epsilon}|t|^{-n/2}. \tag{3.12}$$

Now (1.18) follows from (3.9), (3.10) and (3.12).

Proof of Corollary 1.4. Following [15] we set

$$\Psi(t,h)=e^{itG}\psi(h^2G)-e^{itG_0}\varphi(h^2G_0).$$

It follows from (1.15) and (1.19) that the operator Ψ satisfies the estimate

$$\|\Psi(t,h)\|_{L^1\to L^\infty}\leq Ch^{k-(n-3)/2}|t|^{-n/2}. \tag{3.13}$$

On the other hand, we have (see Theorem 3.1 of [15])

$$\|\Psi(t,h)\|_{L^2\to L^2}\leq Ch,\quad\forall t. \tag{3.14}$$

By interpolation between (3.13) and (3.14) we conclude

$$\|\Psi(t,h)\|_{L^{p'}\to L^p} \leq Ch^{1-\alpha((n-1)/2-k)}|t|^{-\alpha n/2}, \tag{3.15}$$

for every $2 \leq p \leq +\infty$, where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$. Now we will make use of the identity

$$\sigma^{-\alpha q/2}\chi_a(\sigma) = \int_0^1 \psi(\theta\sigma)\theta^{\alpha q/2-1}d\theta,$$

where $\psi(\sigma) = \sigma^{1-\alpha q/2}\chi_a'(\sigma) \in C_0^\infty((0,+\infty))$. By (3.15) we get

$$\begin{aligned}
&\left\|e^{itG}G^{-\alpha q/2}\chi_a(G) - e^{itG_0}G_0^{-\alpha q/2}\chi_a(G_0)\right\|_{L^{p'}\to L^p} \\
&\leq \int_0^1 \left\|\Psi(t,\sqrt{\theta})\right\|_{L^{p'}\to L^p}\theta^{\alpha q/2-1}d\theta \\
&\leq C|t|^{-\alpha n/2}\int_0^1 \theta^{-1/2-\alpha((n-1)/2-k-q)/2}d\theta \leq C|t|^{-\alpha n/2},
\end{aligned} \tag{3.16}$$

provided $1/2 + \alpha((n-1)/2 - k - q)/2 < 1$, that is, for $2 \leq p < \frac{2(n-1-2q-2k)}{n-3-2q-2k}$. Now (1.21) follows from (3.16) and the fact that it holds for the free operator. Similarly, by (3.13) we get

$$\begin{aligned}
&\left\|e^{itG}G^{k/2-(n-3)/4-\epsilon}\chi_a(G) - e^{itG_0}G_0^{k/2-(n-3)/4-\epsilon}\chi_a(G_0)\right\|_{L^1\to L^\infty} \\
&\leq \int_0^1 \left\|\Psi(t,\sqrt{\theta})\right\|_{L^1\to L^\infty}\theta^{-k/2+(n-3)/4-1+\epsilon}d\theta \\
&\leq C|t|^{-n/2}\int_0^1 \theta^{-1+\epsilon}d\theta \leq C_\epsilon|t|^{-n/2}.
\end{aligned} \tag{3.17}$$

Now (1.20) follows from (3.17) and the fact that it holds for the free operator. By (1.15), (1.19), (1.23) and (1.24), we have

$$\left\|\Psi(t,h) - \mathcal{F}_k(t)\psi(h^2G_0)\right\|_{L^1\to L^\infty} \leq Ch^{k-(n-3)/2+\varepsilon}|t|^{-n/2}. \tag{3.18}$$

Proceeding as above with a suitably chosen function ψ, we obtain from (3.18)

$$\begin{aligned}
&\left\|e^{itG}G^{k/2-(n-3)/4}\chi_a(G) - e^{itG_0}G_0^{k/2-(n-3)/4}\chi_a(G_0) - \mathcal{F}_k(t)G_0^{k/2-(n-3)/4}\right\|_{L^1\to L^\infty} \\
&\leq \int_0^1 \left\|\Psi(t,\sqrt{\theta}) - \mathcal{F}_k(t)\psi(\theta G_0)\right\|_{L^1\to L^\infty}\theta^{-k/2+(n-3)/4-1}d\theta \\
&\leq C|t|^{-n/2}\int_0^1 \theta^{-1+\varepsilon/2}d\theta \leq C|t|^{-n/2}.
\end{aligned} \tag{3.19}$$

Now (1.25) follows from (3.19), (1.22) and the fact that it holds for the free operator. □

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References

[1] M. Beals, *Optimal L^∞ decay estimates for solutions to the wave equation with a potential*, Commun. Partial Diff. Equations **19** (1994), 1319–1369.

[2] F. Cardoso, C. Cuevas and G. Vodev, *Dispersive estimates of solutions to the wave equation with a potential in dimensions two and three*, Serdica Math. J. **31** (2005), 263–278.

[3] F. Cardoso, C. Cuevas and G. Vodev, *Weighted dispersive estimates for solutions of the Schrödinger equation*, Serdica Math. J., **34** (2008), 39–54.

[4] P. D'ancona and V. Pierfelice, *On the wave equation with a large rough potential*, J. Funct. Analysis **227** (2005), 30–77.

[5] V. Georgiev and N. Visciglia, *Decay estimates for the wave equation with potential*, Commun. Partial Diff. Equations **28** (2003), 1325–1369.

[6] M. Goldberg, *Dispersive bounds for the three dimensional Schrödinger equation with almost critical potentials*, GAFA **16** (2006), 517–536.

[7] M. Goldberg and W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*, Commun. Math. Phys. **251** (2004), 157–178.

[8] M. Goldberg and M. Visan, *A conterexample to dispersive estimates for Schrödinger operators in higher dimensions*, Commun. Math. Phys. **266** (2006), 211–238.

[9] J.-L. Journé, A. Soffer and C. Sogge, *Decay estimates for Schrödinger operators*, Commun. Pure Appl. Math. **44** (1991), 573–604.

[10] S. Moulin, *High frequency dispersive estimates in dimension two*, submitted.

[11] S. Moulin and G. Vodev, *Low frequency dispersive estimates for the Schrödinger group in higher dimensions*, Asymptot. Anal., **55** (2007), 49–71.

[12] W. Schlag, *Dispersive estimates for Schrödinger operators in two dimensions*, Commun. Math. Phys. **257** (2005), 87–117.

[13] I. Rodnianski and W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. **155** (2004), 451–513.

[14] G. Vodev, *Dispersive estimates of solutions to the Schrödinger equation*, Ann. H. Poincaré **6** (2005), 1179–1196.

[15] G. Vodev, *Dispersive estimates of solutions to the Schrödinger equation in dimensions $n \geq 4$*, Asymptot. Anal. **49** (2006), 61–86.

[16] G. Vodev, *Dispersive estimates of solutions to the wave equation with a potential in dimensions $n \geq 4$*, Commun. Partial Diff. Equations **31** (2006), 1709–1733.