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On Two-Sided Centralizers of Rings and Algebras*

JOSO VUKMAN
Department of Mathematics and Computer Sciences,
Faculty of Natural Sciences and Mathematics, University of Maribor,
Koroška 160, 2000 Maribor, Slovenia
email: joso.vukman@uni-mb.si

AND

IRENA KOSI-ULBL
Faculty of Education, University of Maribor,
Koroška 160, 2000 Maribor, Slovenia
email: irena.kosi@uni-mb.si

ABSTRACT

In this paper we prove the following result. Let A be a semisimple H^*-algebra and let $T : A \to A$ be an additive mapping satisfying the relation $(n+1)T(x^{nm+1}) = T(x)x^{nm} + x^m T(x)x^{(n-1)m} + \cdots + x^{nm}T(x)$, for all $x \in A$ and some fixed integers $m \geq 1$, $n \geq 1$. In this case T is a two-sided centralizer.

RESUMEN

En este artículo probamos el siguiente resultado. Sea A una H^*-algebra semi-simple y $T : A \to A$ una aplicación aditiva satisfaziendo la relación $(n+1)T(x^{nm+1}) =$

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$T(x)x^{nm}+x^mT(x)x^{(n-1)m}+\cdots+x^{nm}T(x)$, para todo $x \in A$ y ciertos $m \geq 1$, y $n \geq 1$ enteros fixados. En este caso T es un centralizador "two-sided".

Key words and phrases: *Prime ring, semiprime ring, Banach space, standard operator algebra, H^*-algebra, left (right) centralizer, left (right) Jordan centralizer, two-sided centralizer.*

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Introduction

Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be $n-$torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$ .Let us recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. An additive mapping $x \longmapsto x^*$ on a ring R is called involution in case $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all pairs $x, y \in R$. A ring equipped with an involution is called a ring with involution or $^*-$ring. We denote by Q_r and C Martindale right ring of quotients and extended centroid of a semiprime ring R. For the explanation of Q_r and C we refer to [3] . An additive mapping $T : R \to R$, where R is an arbitrary ring, is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. The concept appears naturally in C^*-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would write $T : R_R \to R_R$ of a right ring module R into itself. For a semiprime ring R all such homomorphisms are of the form $T(x) = qx$, for all $x \in R$, where q is some fixed element of Q_r (see Chapter 2 in [3]). In case R has the identity element $T : R \to R$ is a left centralizer iff T is of the form $T(x) = ax$, for all $x \in R$, where a is some fixed element of R.An additive mapping $T : R \to R$ is called a left Jordan centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$.The definitions of right centralizer and right Jordan centralizer are self-explanatory. We call $T : R \to R$ a two-sided centralizer in case T is both a left and a right centralizer. In case $T : R \to R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$, for all $x \in R$ (see Theorem 2.3.2 in [3]). One of the initial papers using the concept of centralizers (also called multipliers) is due to Wendel [33] for group algebras. Helgason [9] introduced centralizers for Banach algebras. Wang [32] studied centralizers of commutative Banach algebras. Johnson [11] introduced the concept of centralizers for rings. We refer to Busby [7] for a study of so-called double centralizers in the extension of C^*-algebras. Akemann, Pedersen and Tomiyama [1] have studied centralizers of C^*-algebras. Several authors have also studied spectral properties of centralizers on Banach algebras (see [15, 16]). Johnson [12] has studied centralizers on some topological algebras. Johnson [13] has studied the continuity of centralizers on Banach algebras (see also [11]). Husain [10] has also investigated centralizers on topological algebras with particular reference to complete metrizable locally convex algebras and topological algebras with orthogonal bases. Khan, Mohammad and Thaheem [14] have studied centralizers and double centralizers on certain topological algebras. Centralizers have also appeared in a variety, among which we mention representation theory of Banach algebras, the study of Banach modules, Hopf algebras (see [18, 19]), the theory of singular integrals, interpolation theory, stohastic processes, the theory of semigroups of operators, partial differential equations and the study of approximation problems (see Larsen [16] for more details). Zalar [34] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Molnár [17] has proved that in case we have an additive mapping $T : A \to A$, where A is a semisimple H^*-algebra, satisfying the relation $T(x^3) = T(x)x^2$ $(T(x^3) = x^2T(x))$ for all $x \in A$, then T is a left (right) centralizer. Let us recall that a semisimple H^*-algebra is a semisimple Banach *-algebra A whose norm is a Hilbert space norm such that $(x, yz^*) = (xz, y) = (z, x^*y)$ is fulfilled for all $x, y, z \in A$ (see [2]). Benkovič and Eremita [4] have proved that in case there exists an additive mapping $T : R \to R$,where R is a prime ring with suitable characteristic restrictions, satisfying the relation $T(x^n) = T(x)x^{n-1}$, for all $x \in R$ and some fixed integer $n > 1$, then T is a left centralizer. Vukman and Kosi-Ulbl [26] have proved that any additive mapping T , which maps a semisimple H^*-algebra A into itself and satisfies the relation $2T(x^{n+1}) = T(x)x^n + x^nT(x)$, for all $x \in A$ and some fixed integer $n \geq 1$, is a two-sided centralizer (see also [5]). A result of Vukman and Kosi-Ulbl [27] states that in case there exists an additive mapping $T : R \to R$, where R is a $2-$torsion free semiprime $^*-$ring, satisfying the relation $T(xx^*) = T(x)x^*$ $(T(x^*x) = x^*T(x))$, for all $x \in R$,then T is a left (right) centralizer. For results concerning centralizers on prime and semiprime rings, operator algebras and H^*-algebras we refer to $[8, 20-31]$. Let X be a real or complex Banach space and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subset L(X)$ is said to be standard in case $F(X) \subset A(X)$. Let us point out that any standard algebra is prime, which is a consequence of Hahn-Banach theorem. We denote by X^* the dual space of a Banach space X and by I the identity operator on X.

Vukman [20] has proved the following result.

THEOREM A. Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be an additive mapping. Suppose that

$$2T(x^2) = T(x)x + xT(x)$$

holds for all $x \in R$. In this case T is a two-sided centralizer.

Vukman and Kosi-Ulbl [23] have proved the result below.

THEOREM B. Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be an additive mapping. Suppose that

$$3T(xyx) = T(x)yx + xT(y)x + xyT(x)$$

holds for all pairs $x, y \in R$. In this case T is of the form $T(x) = \lambda x$, for all $x \in R$ and some fixed element λ from the extended centroid C of R.

Motivated by Theorem A and Theorem B Fošner and Vukman [8] have proved the following theorem.

THEOREM C. Let R be a prime ring and let $T : R \to R$ be an additive mapping satisfying the relation

$$nT(x^{n+1}) = T(x)x^{n-1} + xT(x)x^{n-2} + ... + x^{n-1}T(x),$$

for all $x \in R$, where $n \geq 2$ is some fixed integer. If $char(R) = 0$, then T is of the form $T(x) = \lambda x$, for all $x \in R$ and some fixed element λ from the extended centroid C of R.

In the proof of Theorem C Fošner and Vukman used as the main tool the theory of functional identities (Beidar-Brešar-Chebotar theory). The theory of functional identities considers set-theoretic maps on rings that satisfy some identical relations. When threatening such relations one usually concludes that the form of the maps involved can be described, unless the ring is very special (see[6]).

It this paper we consider the following more general relation

$$(n+1)T(x^{nm+1}) = T(x)x^{nm} + x^m T(x)x^{(n-1)m} + ... + x^{nm}T(x), \tag{1}$$

where $m \geq 1$, $n \geq 1$ are some fixed integers. One can notice that the expression (1) for $n = m = 1$ is the same as hypothesis of Theorem A. Obviously, any two-sided centralizer on arbitrary ring satisfies the above relation. We proceed with the following conjecture.

CONJECTURE. Let R be a semiprime ring with suitable torsion restrictions and let $T : R \to R$ be an additive mapping satisfying the relation (1) for all $x \in R$ and some fixed integers $m \geq 1$, $n \geq 1$. In this case T is a two-sided centralizer.

It is our aim in this paper to prove the above conjecture in semisimple H^*-algebras and in semiprime rings with the identity element. Our methods differ from those used in [8].

THEOREM 1. Let A be a semisimple H^*-algebra. Suppose $T : A \to A$ is an additive mapping satisfying the relation (1) for all $x \in A$ and some fixed integers $m \geq 1$, $n \geq 1$. In this case T is a two-sided centralizer.

For the proof of Theorem 1 we need the theorem below which is of independent interest.

THEOREM 2. Let X be a Banach space over the real or complex field $\mathcal{F}$, let $A(X) \subset L(X)$ be a standard operator algebra. Suppose $T : A(X) \to L(X)$ is an additive mapping satisfying the relation

$$(n+1)T(A^{nm+1}) = T(A)A^{nm} + A^m T(A)A^{(n-1)m} + ... + A^{nm}T(A),$$

for all $A \in A(X)$ and some fixed integers $m \geq 1$, $n \geq 1$. In this case T is of the form $T(A) = \lambda A$, for all $A \in A(X)$ and some fixed $\lambda \in \mathcal{F}$.In particular, T is continuous.

Proof. We have the relation

$$(n+1)T(A^{nm+1}) = T(A)A^{nm} + A^m T(A)A^{(n-1)m} + ... + A^{nm}T(A). \tag{2}$$

Let us first consider the restriction of T on $F(X)$. Let A be from $F(X)$ and let $P \in F(X)$, be a projection with $AP = PA = A$. From the above relation one obtains $T(P) = PT(P)P$, which gives

$$T(P)P = PT(P). \tag{3}$$

Putting $A + P$ for A in the relation (2), we obtain

$$\begin{gathered}(n+1)\sum_{i=0}^{nm+1}\binom{nm+1}{i}T\left(A^{nm+1-i}P^i\right) = (T(A)+B)\left(\sum_{i=0}^{nm}\binom{nm}{i}A^{nm-i}P^i\right) + \\ \left(\sum_{i=0}^{m}\binom{m}{i}A^{m-i}P^i\right)(T(A)+B)\left(\sum_{i=0}^{(n-1)m}\binom{(n-1)m}{i}A^{(n-1)m-i}P^i\right) + ... + \\ \left(\sum_{i=0}^{nm}\binom{nm}{i}A^{nm-i}P^i\right)(T(A)+B),\end{gathered} \tag{4}$$

where B stands for $T(P)$. Using (2) and rearranging the equation (4) in sense of collecting together terms involving equal number of factors of P we obtain

$$\sum_{i=1}^{nm} f_i(A,P) = 0, \tag{5}$$

where $f_i(A,P)$ stands for the expression of terms involving i factors of P. Replacing A by $A+2P$, $A+3P$, $\ldots$, $A+nmP$ in turn in the equation (1), and expressing the resulting system of nm homogeneous equations of variables $f_i(A,P)$, $i = 1,2,...,nm$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{nm} \\ \vdots & \vdots & \vdots & \vdots \\ nm & (nm)^2 & \cdots & (nm)^{nm} \end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only the trivial solution.

In particular,

$$f_{nm-1}(A,P) = (n+1)\binom{nm+1}{nm-1}T\left(A^2\right) - \binom{nm}{nm-1}T(A)A - \binom{nm}{nm-2}BA^2 -$$
$$\binom{m}{m-2}\binom{(n-1)m}{(n-1)m}A^2B - \binom{m}{m-1}\binom{(n-1)m}{(n-1)m}AT(A)P - \binom{m}{m-1}\binom{(n-1)m}{(n-1)m-1}ABA -$$
$$\binom{m}{m}\binom{(n-1)m}{(n-1)m-1}PT(A)A - \binom{m}{m}\binom{(n-1)m}{(n-1)m-2}BA^2 - \cdots -$$
$$\binom{m}{m}\binom{(n-1)m}{(n-1)m-2}A^2B - \binom{m}{m}\binom{(n-1)m}{(n-1)m-1}AT(A)P - \binom{m}{m-1}\binom{(n-1)m}{(n-1)m-1}ABA -$$
$$\binom{m}{m-1}\binom{(n-1)m}{(n-1)m}PT(A)A - \binom{m}{m-2}\binom{(n-1)m}{(n-1)m}BA^2 -$$
$$\binom{nm}{nm-2}A^2B - \binom{nm}{nm-1}AT(A) = 0$$

and

$$f_{nm}(A,P) = (n+1)\binom{nm+1}{nm}T(A) - \binom{nm}{nm}T(A)P - \binom{nm}{nm-1}BA -$$
$$\binom{m}{m-1}\binom{(n-1)m}{(n-1)m}AB - \binom{m}{m}\binom{(n-1)m}{(n-1)m}PT(A)P - \binom{m}{m}\binom{(n-1)m}{(n-1)m-1}BA - \cdots -$$
$$\binom{m}{m}\binom{(n-1)m}{(n-1)m-1}AB - \binom{m}{m}\binom{(n-1)m}{(n-1)m}PT(A)P - \binom{m}{m-1}\binom{(n-1)m}{(n-1)m}BA -$$
$$\binom{nm}{nm-1}AB - \binom{nm}{nm}PT(A) = 0.$$

The above equations reduce to

$$6(n+1)(nm+1)T\left(A^2\right) = 12(T(A)A + AT(A)) + 6(n-1)(AT(A)P + PT(A)A) +$$
$$(n+1)((2n+1)m-3)\left(A^2B + BA^2\right) + 2m(n-1)(n+1)ABA, \tag{6}$$

and

$$2(n+1)(nm+1)T(A) = 2(T(A)P + PT(A)) +$$
$$n(n+1)m(AB+BA) + 2(n-1)PT(A)P. \tag{7}$$

Right multiplication of the relation (7) by P gives

$$2(n+1)(nm+1)T(A)P = 2(T(A)P + PT(A)) +$$
$$n(n+1)m(AB+BA) + 2(n-1)PT(A)P. \tag{8}$$

Similarly one obtains

$$2(n+1)(nm+1)PT(A) = 2(T(A)P + PT(A)) +$$

$$n\,(n+1)\,m\,(AB+BA)+2\,(n-1)\,PT\,(A)\,P. \tag{9}$$

Combining (8) with (9) we arrive at

$$T(A)P = PT(A),$$

which reduces the relation (6) to

$$\begin{gathered} 6\,(mn+1)\,T\left(A^2\right) = 6\,(T(A)A + AT(A)) + \\ ((2n+1)\,m-3)\left(A^2B+BA^2\right)+2m\,(n-1)\,ABA, \end{gathered} \tag{10}$$

and the relation (7) to

$$2\,(mn+1)\,T\,(A) = 2T\,(A)\,P + mn\,(AB+BA)\,. \tag{11}$$

Right multiplication of the above relation by P and combining the relation so obtained with (11) gives

$$T(A) = T(A)P.$$

According to the above relation the relation (11) reduces to

$$2T\,(A) = AB+BA. \tag{12}$$

From the above relation one obtains

$$2T\left(A^2\right) = A^2B+BA^2. \tag{13}$$

Right and then left multiplication of the relation (12) by A gives

$$2T\,(A)\,A = ABA+BA^2 \tag{14}$$

and

$$2AT\,(A) = A^2B+ABA, \tag{15}$$

respectively. Using the relations (13), (14) and (15) in the relation (10) gives after some calculation

$$A\,(m,n)\,BA^2 + A\,(m,n)\,A^2B - 2A\,(m,n)\,ABA = 0,$$

where $A\,(m,n)$ stands for $mn-m+3$. The above relation reduces to

$$A^2B+BA^2-2ABA=0. \tag{16}$$

Applying the relations (13) and (16) in the relation (10) one obtains

$$2T\left(A^2\right) = T(A)A + AT(A). \tag{17}$$

From the relation (12) one can conclude that T maps $F(X)$ into itself. We have therefore an additive mapping $T : F(X) \rightarrow F(X)$ satisfying the relation (17) for all $A \in F(X)$. Since $F(X)$

is prime one can apply Theorem A and conclude that T is a two-sided centralizer of $F(X)$. We intend to prove that there exists an operator $C \in L(X)$, such that

$$T(A) = CA, \quad \text{for all} \quad A \in F(X). \tag{18}$$

For any fixed $x \in X$ and $f \in X^*$ we denote by $x \otimes f$ an operator from $F(X)$ defined by $(x \otimes f)y = f(y)x$, for all $y \in X$. For any $A \in L(X)$ we have $A(x \otimes f) = ((Ax) \otimes f)$.Let us choose f and y such that $f(y) = 1$ and define $Cx = T(x \otimes f)y$. Obviously, C is linear. Using the fact that T is left centralizer on $F(X)$ we obtain

$$(CA)x = C(Ax) = T((Ax) \otimes f)y = T(A(x \otimes f))y = T(A)(x \otimes f)y = T(A)x, \quad x \in X.$$

We have therefore $T(A) = CA$ for any $A \in F(X)$. Since T right centralizer on $F(X)$ we obtain $C(AP) = T(AP) = AT(P) = ACP$, where $A \in F(X)$ and P is arbitrary one-dimensional projection. We have therefore $[A, C]\, P = 0$. Since P is arbitrary one-dimensional projection it follows that $[A, C] = 0$ for any $A \in F(X)$. Using closed graph theorem one can easily prove that C is continuous. Since C commutes with all operators from $F(X)$ one can conclude that $Cx = \lambda x$ holds for any $x \in X$ and some $\lambda \in \mathcal{F}$, which gives together with the relation (17) that T is of the form

$$T(A) = \lambda A \tag{19}$$

for any $A \in F(X)$ and some $\lambda \in F$. It remains to prove that the above relation holds on $A(X)$ as well. Let us introduce $T_1 : A(X) \to L(X)$ by $T_1(A) = \lambda A$ and consider $T_0 = T - T_1$. The mapping T_0 is, obviously, additive and satisfies the relation (2). Besides, T_0 vanishes on $F(X)$. It is our aim to prove that T_0 vanishes on $A(X)$ as well. Let $A \in A(X)$, let P be an one-dimensional projection and $S = A + PAP - (AP + PA)$. Note that S can be written in the form $S = (I - P)A(I - P)$, where I denotes the identity operator on X, Since, obviously, $S - A \in F(X)$, we have $T_0(S) = T_0(A)$. Besides, $SP = PS = 0$. We have therefore the relation

$$(n+1)T_0(A^{nm+1}) = T_0(A)A^{nm} + A^m T_0(A)A^{(n-1)m} + ... + A^{nm} T_0(A), \tag{20}$$

for all $A \in A(X)$. Applying the above relation we obtain

$$\begin{gathered}
T_0(S)S^{nm} + S^m T_0(S)S^{(n-1)m} + ... + S^{nm} T_0(S) = (n+1)T_0(S^{nm+1}) = \\
(n+1)T_0(S^{nm} + P) = (n+1)T_0((S+P)^{nm+1}) = \\
T_0(S+P)(S+P)^{nm} + (S+P)^m T_0(S+P)(S+P)^{(n-1)m} + ...+ \\
(S+P)^{(n-1)m} T_0(S)(S+P)^m + (S+P)^{nm} T_0(S+P) = T_0(S)S^{nm}+ \\
S^m T_0(S)S^{(n-1)m} + ... + S^{nm} T_0(S) + T_0(S)P + S^m T_0(S)P+ \\
PT_0(S)S^{(n-1)m} + ... + S^{(n-1)m} T_0(S)P+ \\
PT_0(S)S^m + PT_0(S) + (n-1)PT_0(S)P.
\end{gathered}$$

We have therefore

$$T_0(A)P + S^m T_0(A)P + PT_0(A)S^{(n-1)m} + ... + S^{(n-1)m}T_0(A)P+$$
$$PT_0(A)S^m + PT_0(A) + (n-1)PT_0(A)P = 0. \tag{21}$$

Multiplying the above relation from both sides by P we obtain

$$PT_0(A)P = 0. \tag{22}$$

Now right multiplication of the relation (21) by P gives because of (22)

$$T_0(A)P + S^m T_0(A)P + ... + S^{(n-1)m}T_0(A)P = 0. \tag{23}$$

Replacing A by $2A$, $3A$, $\dots$, nA in turn in the equation (23), and expressing the resulting system of n homogeneous equations of variables $T_0(A)P$, $S^{im}T_0(A)P$, $i = 1, 2, ..., n-1$, we see that the coefficient matrix of the system is a matrix of the form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2^m & \cdots & 2^{(n-1)m} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & n^m & \cdots & n^{(n-1)m} \end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only the trivial solution. We have therefore $T_0(A)P = 0$. Since P is an arbitrary one-dimensional projection, one can conclude that $T_0(A) = 0$, for any $A \in A(X)$, which completes the proof of the theorem.

It should be mentioned that in the proof of Theorem 2 we used some ideas similar to those used by Molnár in [17]. Let us point out that in Theorem 2 we obtain as a result the continuity of T under purely algebraic assumptions concerning T, which means that Theorem 2 might be of some interest from the automatic continuity point of view.

Proof of Theorem 1. The proof goes through using the same arguments as in the proof of Theorem in [17] with the exception that one has to use Theorem 2 instead of Lemma in [17].

We are ready for our last result.

THEOREM 3. Let $n \geq 1$, $m \geq 1$ be integers and let R be a 2, m, n, $n+1$ and $((n-1)\,m+3)-$torsion free semiprime ring with the identity element. Suppose that we have an additive mapping $T : R \to R$ satisfying the relation (1) for all $x \in R$. In this case T is of the form $T(x) = ax$, for all $x \in R$ and some fixed element $a \in Z(R)$.

Proof. We have the relation (1). Using similar approach as in the proof of Theorem 2, with the exception that we use the identity element e instead of a projection, we obtain from the above relation

$$6(nm+1)T\left(x^2\right) = 6\left(T(x)x + xT(x)\right) +$$
$$\left((2n+1)\,m-3\right)\left(x^2a + ax^2\right) + 2\,(n-1)\,mxax, \quad x \in R \tag{24}$$

and

$$2T\,(x) = xa + ax, \quad x \in R, \tag{25}$$

where a stands for $T\,(e)$. In the procedure mentioned above we used the fact that R is m, n and $n+1$-torsion free.

The substitution x^2 for x in (25)gives

$$2T\left(x^2\right) = x^2a + ax^2, \quad x \in R. \tag{26}$$

Multiplying the relation (25) first from the right side then from the left side by x we obtain

$$2T\,(x)\,x = xax + ax^2, \quad x \in R \tag{27}$$

and

$$2xT\,(x) = x^2a + xax, \quad x \in R. \tag{28}$$

Using (26), (27) and (28) in the relation (24)and applying the fact that R is $(n-1)m+3-$torsion free we obtain after some calculation

$$x^2a + ax^2 - 2xax = 0, \quad x \in R,$$

which can be written in the form

$$\left[\left[a,x\right],x\right] = 0, \quad x \in R. \tag{29}$$

Putting $x+y$ for x in the above relation we obtain

$$\left[\left[a,x\right],y\right] + \left[\left[a,y\right],x\right] = 0, \quad x,y \in R. \tag{30}$$

The substitution xy for y in relation (30) gives because of (29) and (30)

$$0 = \left[\left[a,x\right],xy\right] + \left[\left[a,xy\right],x\right] =$$
$$= \left[\left[a,x\right],x\right]y + x\left[\left[a,x\right],y\right] + \left[\left[a,x\right]y + x\left[a,y\right],x\right] =$$
$$= x\left[\left[a,x\right],y\right] + \left[\left[a,x\right],x\right]y + \left[a,x\right]\left[y,x\right] + x\left[\left[a,y\right],x\right] = \left[a,x\right]\left[y,x\right], \quad x,y \in R.$$

Thus we have

$$\left[a,x\right]\left[y,x\right] = 0, \quad x,y \in R.$$

The substitution ya for y in the above relation gives $[a,x]\,y\,[a,x]=0$, for all pairs $x,y\in R$. Let us point out that so far we have not used the assumption that R is semiprime. Since R is semiprime, it follows from the last relation that $[a,x]=0$, for all $x\in R$. In other words, $a\in Z(R)$, which reduces the relation (25) to $T(x)=ax$, $x\in R$, since R is $2-$torsion free. The proof of the theorem is complete.

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