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On the Structure of Primitive n-Sum Groups

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ABSTRACT

For a finite group G, let $\sigma(G)$ be least cardinality of a collection of proper subgroups whose set-theoretical union is all of G. We study the structure of groups G containing no normal non-trivial subgroup N such that $\sigma(G/N) = \sigma(G)$.

RESUMEN

Para un grupo G, sea $\sigma(G)$ la menor cardinalidad de la colección de subgrupos propios cujas union (de conjuntos) es todo G. Nosotros estudiamos la estructura de grupos G contiendo no trivial no normales subgrupos N tal que $\sigma(G/N) = \sigma(G)$.

Key words and phrases: *n-sum groups; minimal coverings; monolithic groups.*

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1 Introduction

If G is a non cyclic finite group, then there exists a finite collection of proper subgroups whose set-theoretical union is all of G; such a collection is called a *cover* for G. A minimal cover is one of least cardinality and the size of a minimal cover of G is denoted by $\sigma(G)$ (and for convenience we shall write $\sigma(G) = \infty$ if G is cyclic). The study of minimal covers was introduced by J.H.E. Cohn [8]; following his notation, we say that a finite group G is an n-sum group if $\sigma(G) = n$ and that a group G is a *primitive n-sum group* if $\sigma(G) = n$ and G has no normal non-trivial subgroup N such that $\sigma(G/N) = n$. We will say that G is *σ-primitive* if it is a primitive n-sum group for some integer n. Notice that if N is a normal subgroup of G, then $\sigma(G) \leq \sigma(G/N)$; indeed a cover of G/N can be lifted to a cover of G.

It is clear that if G is a non cyclic monolithic primitive group (i.e. G has a unique minimal normal subgroup and the Frattini subgroup of G is trivial) and $G/\operatorname{soc}(G)$ is cyclic, then G is a σ-primitive group.

Moreover Cohn proved that an abelian σ-primitive group is the direct product of two cyclic groups of order p, a prime number.

Tomkinson [14] showed that in a finite solvable group G, $\sigma(G) = |V| + 1$, where V is a chief factor of G with least order among chief factors of G with multiple complements. This allows to prove (see for example [5]) that a σ-primitive solvable group G is as described above, i.e. either G is abelian or G is monolithic and $G/\operatorname{soc}(G)$ is cyclic.

However there exist examples of σ-primitive groups with $G/\operatorname{soc}(G)$ non cyclic: actually with $G/\operatorname{soc}(G) \cong \operatorname{Alt}(p)$ for some prime p (see Corollary 9 and Corollary 12).

The aim of this paper is to collect information on the structure of the σ-primitive groups. In particular we prove that *if G is σ-primitive, then G contains at most one abelian minimal normal subgroup; moreover two non-abelian minimal normal subgroups of G are not G-equivalent* (we refer to an equivalence relation among the chief factors of a finite group introduced in [10] and [9], whose main properties are summarized at the beginning of Section 2). *Furthermore if G is non-abelian, then all the solvable factor groups of $G/\operatorname{soc}(G)$ are cyclic.*

No example is known of a non-abelian σ-primitive group containing two distinct minimal normal subgroups. This leads to conjecture that a non-abelian σ-primitive group is monolithic. We prove a partial result supporting this conjecture.

Theorem 1. *Let G be a σ-primitive group with no abelian minimal normal subgroups. Then either G is a primitive monolithic group and $G/\operatorname{soc}(G)$ is cyclic, or $G/\operatorname{soc}(G)$ is non-solvable and all the non-abelian composition factors of $G/\operatorname{soc}(G)$ are alternating groups of odd degree.*

A better knowledge of the σ-primitive groups is useful in dealing with several questions about the minimal covers. For example, confirming a conjecture of Tomkinson, we prove:

Theorem 2. *There is no finite group G with $\sigma(G) = 11$.*

Another application concerns the study of $\sigma(G)$ when $G = H \times K$ is a direct product of two finite groups. Cohn proved that if H and K have coprime order, then $\sigma(H \times K) = \min\{\sigma(H), \sigma(K)\}$. We prove the following more general result:

Theorem 3. *Let $G = H \times K$ be the direct product of two subgroups. If no alternating group* $\mathrm{Alt}(n)$ *with n odd is a homomorphic image of both H and K, then either $\sigma(G) = \min\{\sigma(H), \sigma(K)\}$ or $\sigma(G) = p + 1$ and the cyclic group of order p is a homomorphic image of both H and K.*

2 Preliminary results and easy remarks

For the reader's convenience, we recall the definition of an equivalence relation among the elements of the set $\mathcal{CF}(G)$ of the chief factors of G, that was introduced in [10] and studied in details in [9]. A group G is said to be primitive if it has a maximal subgroup with trivial core. The socle $\mathrm{soc}(G)$ of a primitive group G can be either an abelian minimal normal subgroup (I), or a non-abelian minimal normal subgroup (II), or the product of two non-abelian minimal normal subgroups (III); we say respectively that G is *primitive of type I, II, III* and in the first two cases we say that G is *monolithic*. Two chief factors of a finite group G are said to be G*-equivalent* if either they are G-isomorphic between them or to the two minimal normal subgroups of a primitive epimorphic image of type III of G. This means that two G-equivalent chief factors of G are either G-isomorphic between them or to two chief factors of G having a common complement (which is a maximal subgroup of G). A chief factor H/K is called Frattini if $H/K \leq \Phi(G)$. For any $A \in \mathcal{CF}(G)$ we denote by $I_G(A)$ the set of those elements of G which induces by conjugation an inner automorphism in A. Moreover we denote by $R_G(A)$ the intersection of the normal subgroups N of G contained in $I_G(A)$ and with the property that $I_G(A)/N$ is non-Frattini and G-equivalent to A. We collect here a sequence of basic properties of the subgroups $I_G(A)$ and $R_G(A)$, proved and discussed in [9]:

Proposition 4. *Let $A \in \mathcal{CF}(G)$ and let $I/R = I_G(A)/R_G(A)$. Then:*

1. *either $R = I$, in which case we set $\delta_G(A) = 0$, or $I/R = \mathrm{soc}(G/R)$ and it is a direct product of $\delta_G(A)$ minimal normal subgroups G-equivalent to A;*
2. *each chief series of G contains exactly $\delta_G(A)$ non-Frattini chief factors G-equivalent to A;*
3. *if A is abelian, then I/R has a complement in G/R;*
4. *if $\delta_G(A) \geq 2$, then any two different minimal normal subgroups of I/R have a common complement, which is a maximal subgroup;*
5. *a chief factor H/K of G is non-Frattini and G-equivalent to A if and only if $RH/RK \neq 1$ and $RH \leq I$.*

Note that if $\delta_G(A) = 1$, then $G/R_G(A)$ is a monolithic primitive group (the monolithic primitive group associated to A).

In the rest of the section we will discuss some basis results on the relation between $\sigma(G)$ and $\sigma(G/N)$ when N is a minimal normal subgroup of G. We start summarizing some known properties of σ.

Lemma 5. *Let N be a minimal normal subgroup of a group G. If $\sigma(G) < \sigma(G/N)$, then*

1. *if N has c complements which are maximal subgroups, then $c+1 \leq \sigma(G)$;*
2. *if $N = S^r$ where S is a non-abelian simple group and $l(S)$ is the minimal index of a maximal subgroup of S, then $l(S)^r + 1 \leq \sigma(G)$.*

Proof. (1) (See e.g. [14, Proof of Theorem 2.2]) Let $\mathcal{M} = \{M_i \mid i = 1, \dots, \sigma(G)\}$ be a set of maximal subgroups whose union covers G and let M be a complement of N. Clearly $M = \bigcup_{1 \leq i \leq \sigma(G)} M \cap M_i$, however $\sigma(M) = \sigma(G/N) > \sigma(G)$, hence $M = M \cap M_i$ for some i; in particular if M is a maximal subgroup of G, then $M = M_i \in \mathcal{M}$. So $\mathcal{M}$ contains all the c complements of N which are maximal; since the union of these complements does not cover N, we need at least $c+1$ subgroups in $\mathcal{M}$.

(2) Let $l_G(N)$ be the smallest index of a proper subgroup of G supplementing N. By Lemma 3.2 in [14] a minimal cover $\mathcal{M}$ of G contains at least $l_G(N)$ subgroups which supplement N. On the other hand, if all the subgroups in $\mathcal{M}$ are supplements of N, then by [8, Lemma 1] we have $l_G(N) \leq \sigma(G) - 1$. In any case we conclude $\sigma(G) \geq l_G(N) + 1 \geq l(S)^r + 1$. □

Corollary 6. *Let N be a minimal normal subgroup of a group G. If $\sigma(G) < \sigma(G/N)$, then*

1. *if N is abelian, complemented and non-central, then $|N| + 1 \leq \sigma(G)$;*
2. *if $N = S^r$ where S is a non-abelian simple group, then $5^r + 1 \leq \sigma(G)$.*

Proposition 7. *Let N be a non-solvable normal subgroup of a finite group G. Then $\sigma(G) \leq |N| - 1$.*

Proof. Consider the centralizers in G of the nontrivial elements of N: if there exists an element $g \in G$ which does not belong to $\bigcap_{1 \neq n \in N} C_G(n)$ then the subgroup $\langle g \rangle$ acts fixed point freely on N. By the classification of finite simple groups (see e.g. [15]), it follows that N is solvable, a contradiction. Hence $\sigma(G) \leq |N| - 1$. □

Corollary 8. *If N is a non-abelian minimal normal subgroup of G and $\delta_G(N) > 1$, then $\sigma(G) = \sigma(G/N)$.*

Proof. Assume by contradiction that $\sigma(G) < \sigma(G/N)$. Since $\delta_G(N) > 1$, there exists a maximal subgroup M of G, such that G/M_G is a primitive group of type III and M/M_G is a common complement of the two minimal normal subgroups of the socle $H/M_G \times NM_G/M_G$ of G/M_G. In particular M is a non-normal complement of N and it has $|N|$ conjugates, hence $|N| + 1 \leq \sigma(G)$ by Lemma 5. This contradicts Proposition 7. □

Corollary 9. *Let p a large prime not of the form $(q^k - 1)/(q - 1)$ where q is a prime power and k an integer; then $\sigma(\mathrm{Alt}(5) \wr \mathrm{Alt}(p)) < \sigma(\mathrm{Alt}(p))$.*

Proof. By Proposition 7, $\sigma(\mathrm{Alt}(5) \wr \mathrm{Alt}(p)) < |\mathrm{Alt}(5)|^p$. On the other hand, by Theorem [12, 4.4], $\sigma(\mathrm{Alt}(p)) \geq (p-2)! > 60^p$ for a large enough prime not of the form $(q^k - 1)/(q - 1)$. □

Proposition 10. *Let G be a finite group. If V is a complemented normal abelian subgroup of G and $V \cap Z(G) = 1$, then $\sigma(G) < 2|V|$. In particular, if V is a minimal normal subgroup, then $\sigma(G) \leq 1 + q + \cdots + q^n$ where $q = |\operatorname{End}_G(V)|$ and $|V| = q^n$.*

Proof. Let H be a complement of V in G; we shall prove that G is covered by the family of subgroups $\mathcal{A} = \{H^v \mid v \in V\} \cup \{C_H(v)V \mid 1 \neq v \in V\}$. Let $g = hw \in G$, where $h \in H$, $w \in V$. If $h \notin C_H(v)$ for every $v \in V \setminus \{1\}$, then $C_V(h) = 1$ and the cardinality of the set $\{h^v \mid v \in V\}$ is $|V : C_V(h)| = |V|$. Therefore $\{h^v \mid v \in V\} = \{hv \mid v \in V\}$ and $g = hw \in H^v$ for some $v \in V$. Thus $\sigma(G) \leq |\mathcal{A}| \leq |V| + (|V| - 1) < 2|V|$. In particular, if V is H-irreducible, then $\operatorname{End}_G(V) = \operatorname{End}_H(V) = \mathbb{F}$ is a finite field. We may identify H with a subgroup of $\mathrm{GL}(n, q)$, where $|\mathbb{F}| = q$ and $\dim_{\mathbb{F}} V = n$. In this case G is covered by $\mathcal{A} = \{H^v \mid v \in V\} \cup \{C_H(W)V \mid W \leq V,\ \dim_{\mathbb{F}} W = 1\}$, so $\sigma(G) \leq q^n + (1 + \cdots + q^{n-1})$. ☐

Corollary 11. *Let H be a finite group, V an H-module, $G = V \rtimes H$ the semidirect product of V by H and assume that $C_V(H) = 0$. Then*

1. *if $\mathrm{H}^1(H, V) \neq 0$, then $\sigma(G) = \sigma(H)$;*
2. *if $\sigma(H) \geq 2|V|$, then $\mathrm{H}^1(H, V) = 0$.*

Proof. Assume by contradiction that $\sigma(G) < \sigma(H)$. By Lemma 5, $c + 1 \leq \sigma(G)$ where c is the number of complements of V in G. If $\mathrm{H}^1(H, V) \neq 0$, then there are at least two conjugacy classes of complements for V in G and, since $C_V(H) = 0$, any conjugacy class consists of $|V|$ complements, hence $c \geq 2|V|$ and $\sigma(G) > 2|V|$ against Proposition 10. ☐

Corollary 12. *Let V the fully deleted module for $\mathrm{Alt}(n)$ over $\mathbb{F}_2$ and let G be the semidirect product of V by $\mathrm{Alt}(n)$.*

1. *If $n = p$ is a large odd prime not of the form $(q^k - 1)/(q - 1)$ where q is a prime power and k an integer, then $\sigma(G) < \sigma(\mathrm{Alt}(n))$.*
2. *If n is even, then $\sigma(G) = \sigma(\mathrm{Alt}(n))$*

Proof. 1) Since $|V| = 2^{p-1}$ (see e.g. [11, Prop. 5.3.5]), Proposition 10 gives that $\sigma(G) < 2|V| < 2^p$. On the other hand, by Theorem [12, 4.4], $\sigma(\mathrm{Alt}(p)) \geq (p - 2)! > 2^p$ for a large enough prime not of the form $(q^k - 1)/(q - 1)$.

2) This follows from Corollary 11 and the fact that $\mathrm{H}^1(\mathrm{Alt}(n), V) \neq 0$ whenever n is even (see e.g. [2, p. 74]). ☐

Corollary 13. *Let $V \neq W$ be non-Frattini non-central abelian minimal normal subgroups of G. Then*

1. *if $\delta_G(V) > 1$, then $\sigma(G) = \sigma(G/V)$;*

2. $\sigma(G) = \min\{\sigma(G/V), \sigma(G/W)\}$.

Proof. 1) By a result in [3], the number c of complements of V in G is

$$c = |\operatorname{Der}(G/V, V)| = |\operatorname{End}_{G/V}(V)|^{\delta_G(V)-1}|\operatorname{Der}(G/C_G(V), V)|$$

hence $c \geq 2|V|$ whenever $\delta_G(V) > 1$. If $\sigma(G) < \sigma(G/V)$, then by Lemma 5 and Proposition 10, $2|V| < c + 1 \leq \sigma(G) < 2|V|$, a contradiction.

2) If V and W are G-equivalent, then by (1) $\sigma(G) = \sigma(G/V) = \sigma(G/W)$. So assume that V and W are not G-equivalent and, by contradiction, that $\sigma(G) < \min\{\sigma(G/V), \sigma(G/W)\}$. A complement of V in G has at least $|V|$ conjugates and it is a maximal subgroup of G, so we can find at least $|V|$ complements of V. In the same way there are at least $|W|$ distinct complements of $|W|$ in G. Moreover, since V and W are not G-equivalent, V and W cannot have a common complement. Arguing as in Lemma 5 we see that all the complements of V and W have to appear in a minimal cover of G. Therefore $\sigma(G) \geq |V| + |W| \geq \min\{2|V|, 2|W|\}$, against Proposition 10.

□

3 The structure of σ-primitive groups

We collect some known properties of σ-primitive groups and some consequences of the previous section.

Corollary 14. *Let G be a non-abelian σ-primitive group. Then:*

1. $Z(G) = 1$;
2. *the Frattini subgroup of G is trivial;*
3. *if N is a minimal normal subgroup of G, then $\delta_G(N) = 1$;*
4. *there is at most one abelian minimal normal subgroup of G;*
5. *the socle $\operatorname{soc}(G) = G_1 \times \cdots \times G_n$ is a direct product of non-G-equivalent minimal normal subgroups and at most one of them is abelian.*
6. *G is a subdirect product of the monolithic primitive groups $X_i = G/R_G(G_i)$ associated to the minimal normal subgroups G_i, $1 \leq i \leq n$.*

Proof. Part (1) is Theorem 4 in [8]. If $\Phi(G)$ is the Frattini subgroup of G and H is a proper subgroup of G, then also $H\,\Phi(G)$ is a proper subgroup of G. Hence we can assume that $\Phi(G)$ is contained in every subgroup of a minimal cover of G so that $\sigma(G) = \sigma(G/\Phi(G))$ and therefore (2) holds. Parts (3) and (4) follows from Corollaries 8 and 13. Then (3) and (4) implies (5). To prove (6) we consider the intersection $R = \bigcap_{i=1}^n R_G(G_i)$. If $R \neq 1$, then R contains a minimal normal subgroup N of G. By (2) and (5), N is non-Frattini and G-equivalent to G_i for some $1 \le i \le n$. Hence by Proposition 4 (5), $R_G(G_i)N \neq R_G(G_i)$, in contradiction with $N \le R \le R_G(G_i)$. □

Definition 15. *Let X be a primitive monolithic group and let N be its socle. For any non-empty union $\Omega = \bigcup_i \omega_i N$ of cosets of N in X with the property that $\langle \Omega \rangle = X$, define $\sigma_\Omega(X)$ to be the minimum number of supplements of N in G needed to cover Ω. Then we define*

$$\sigma^*(X) = \min \left\{ \sigma_\Omega(X) \mid \Omega = \bigcup_i \omega_i N,\ \langle \Omega \rangle = X \right\}.$$

Proposition 16. *Let G be a non-abelian σ-primitive group, $G_1, \dots, G_n$ the minimal normal subgroups, and $X_1, \dots X_n$ the monolithic primitive groups associated to G_i, $i = 1, \dots n$. Then $\sigma(G) \ge \sum_{i=1}^n \sigma^*(X_i)$.*

Proof. Let $\mathcal{M}$ be a set of $\sigma = \sigma(G)$ maximal subgroups whose union is G. Define $\mathcal{M}_{\neg G_i} = \{M \in \mathcal{M} \mid M \not\geq G_i\}$; note that

- $\mathcal{M}_{\neg G_i} \neq \varnothing$ for each $1 \le i \le n$; otherwise every maximal subgroup of $\mathcal{M}$ would contain G_i and the set $\{M/G_i \mid M \in \mathcal{M}\}$ would cover G/G_i with $\sigma(G) < \sigma(G/G_i)$ subgroups.
- $\mathcal{M}_{\neg G_i} \cap \mathcal{M}_{\neg G_j} = \varnothing$ for $i \neq j$; indeed if there exists $M \in \mathcal{M}_{\neg G_i} \cap \mathcal{M}_{\neg G_j}$, then $G_i M_G / M_G$ and $G_j M_G / M_G$ are minimal normal subgroups of the primitive group G/M_G, hence $\delta_G(G_i) \ge 2$, contrary to Corollary 14.

Therefore $\mathcal{M}$ contains the disjoint union of the non-empty sets $\mathcal{M}_{\neg G_i}$, $1 \le i \le n$, and we are reduced to prove that $|\mathcal{M}_{\neg G_i}| \ge \sigma^*(X_i)$, for every i. Let us fix an index i and let $\pi : G \mapsto X$ be the projection of G over $X = X_i$. We set $N = \operatorname{soc} X \cong G_i$, $\mathcal{M}_i = \{M \in \mathcal{M} \mid M \ge G_i\} = \mathcal{M} \setminus \mathcal{M}_{\neg G_i}$ and

$$\Omega = \left\{ \pi(g) \mid g \in G \setminus \bigcup_{M \in \mathcal{M}_i} M \right\}.$$

By minimality of the cover $\mathcal{M}$, $G \neq \bigcup_{M \in \mathcal{M}_i} M$ hence $\Omega \neq \varnothing$. Moreover, as $G_i \le M \in \mathcal{M}_i$ and $\pi(G_i) = \operatorname{soc} X = N$, we get that for every $x \in \Omega$ the coset xN is contained in Ω. If $\langle \Omega \rangle = H \neq X$, then G is covered by the set $\mathcal{M}_i \cup \{\pi^{-1}(H)\}$ and this actually is a minimal cover of G, since $|\mathcal{M}_i| + 1 \le \sigma$. But then, as $\pi^{-1}(H) \ge G_i$, we would have $\sigma(G/G_i) \le |\mathcal{M}_i| + 1 = \sigma(G)$, a contradiction. Hence $\langle \Omega \rangle = X$.

Now we shall prove that $|\mathcal{M}_{\neg G_i}| \ge \sigma_\Omega(X) \ge \sigma^*(X)$. By [9, Proposition 11] the kernel $R = R_G(G_i)$ of the projection π_i of G over X has the property that if H is a proper subgroup of G such that $HG_i = G$ then $HR \neq G$. Therefore every maximal subgroup $M \in \mathcal{M}_{\neg G_i}$ contains R, $M = \pi^{-1}(\pi(M))$ and $\pi(M)$ is a a maximal subgroup of X supplementing N. Clearly, as $\bigcup_{M \in \mathcal{M}_{\neg G_i}} M$ covers $G \setminus \bigcup_{M \in \mathcal{M}_i} M$, we have that $\bigcup_{M \in \mathcal{M}_{\neg G_i}} \pi(M)$ covers Ω. Therefore $|\{\pi(M) \mid M \in \mathcal{M}_{\neg G_i}\}| = |\mathcal{M}_{\neg G_i}| \ge \sigma_\Omega(X) \ge \sigma^*(X)$. □

Remark 17. *For every primitive monolithic group X_i, $\sigma^*(X_i) \le l_{X_i}(\mathrm{soc}(X_i))$, where $l_{X_i}(\mathrm{soc}(X_i))$ is the smallest index of a proper subgroup of X_i supplementing $\mathrm{soc}(X_i)$. Indeed, if a supplement of $N_i = \mathrm{soc}(X_i)$ in X_i has non trivial intersection with a coset gN_i, then $|gN_i \cap M| = |N_i \cap M| = |gN_i|/|G : M|$, and therefore we need at least $l_{X_i}(\mathrm{soc}(X_i))$ supplements to cover gN_i. So in particular the previous proposition implies that $\sigma(G) \ge \sum_{i=1}^n l_{X_i}(N_i)$.*

Lemma 18. *Let N be a normal subgroup of a group X. If a set of subgroups covers a coset yN of N in X, then it also covers every coset $y^\alpha N$ with α prime to $|y|$.*

Proof. Let s be an integer such that $s\alpha \equiv 1 \mod |y|$. As s is prime to $|y|$, by a celebrated result of Dirichlet, there exists infinitely many primes in the arithmetic progression $\{s + |y|n \mid n \in \mathbb{N}\}$; we choose a prime $p > |X|$ in $\{s + |y|n \mid n \in \mathbb{N}\}$. Clearly, $y^p = y^s$. As p is prime to $|X|$, there exists an integer r such that $pr \equiv 1 \mod |X|$. Hence, if $yN \subseteq \cup_{i\in I} M_i$, for every $g \in y^\alpha N$ we have that $g^p \in (y^\alpha)^p N = (y^\alpha)^s N = yN \subseteq \cup_{i\in I} M_i$ and therefore also $g = (g^p)^r$ belongs to $\cup_{i\in I} M_i$. □

Corollary 19. *Let G be a non-abelian σ-primitive group, N a minimal normal subgroup and X the monolithic primitive groups associated to N. Then:*

1. *if $X = N$, then $G = N$;*
2. *if $|X/N|$ is a prime, then $G = X$.*

Proof. Note that if $X = N$, then there is only one coset of N in X hence $\Omega = N$, $\sigma^*(N) = \sigma_N(N) = \sigma(N)$. By Proposition 16, $\sigma^*(N) = \sigma(N) \le \sigma(G)$. As $N = X$ is a homomorphic image of G, we get $G = N$.

Now let $|X/N|$ be a prime. Let Ω be a non-empty union of cosets of N in X with the property that $\langle\Omega\rangle = X$; then Ω contains a coset yN which is a generator for X/N. By Lemma 18 we have that if $\bigcup_i M_i$ covers Ω, then $\bigcup_i M_i$ covers every coset of N with the exception, at most, of the subgroup N itself. Hence, $\sigma(X) \le \sigma_\Omega(X) + 1$ that is $\sigma^*(X) \ge \sigma(X) - 1$. By Proposition 16, $\sigma(G) \ge \sum_{i=1}^n \sigma^*(X_i)$. Moreover, by Remark 16, $\sigma^*(X_i) \ge 2$. Therefore, as $\sigma(G) \le \sigma(X)$, there is no room for another minimal normal subgroup in G. □

Corollary 20. *If $N = \mathrm{Alt}(n)$, $n \ne 6$, is a normal subgroup of G, then either $\sigma(G) = \sigma(G/N)$ or $G \in \{\mathrm{Sym}(n), \mathrm{Alt}(n)\}$.*

Proof. It is sufficient to consider a σ-primitive image of G and then apply Corollary 19. □

Actually, the corollary holds also for $n = 6$, thanks to the following proposition.

Proposition 21. *Let G be a σ-primitive group and let $\mathrm{O}^\infty(G)$ be the smallest normal subgroup of G such that $G/\mathrm{O}^\infty(G)$ is solvable. If G is non solvable, then $G/\mathrm{O}^\infty(G)$ is a cyclic group.*

Proof. By Corollary 14, G is a subdirect product of the monolithic primitive groups X_i associated to the minimal normal subgroups G_i, $1 \leq i \leq n$; call $N_i = \mathrm{soc}(X_i) \cong G_i$. Let $\mathcal{M}$ be a set of $\sigma = \sigma(G)$ maximal subgroups whose union is G and define $\mathcal{M}_{\neg G_i} = \{M \in \mathcal{M} \mid M \not\geq G_i\}$. Let m_i be the minimal index of a supplement of N_i in X_i: by Remark 17, $\sigma(G) \geq \sum_{i=1}^{n} m_i$.

Let $R = \mathrm{O}^{\infty}(G)$ and assume by contradiction that G/R is not cyclic. Then, by Tomkinson's result [14], $\sigma(G/R) = q + 1$ where q is the order of the smallest chief factor $A = H/K$ of G/R having more than a complement in G/R. As G is not solvable, then $\sigma(G) < \sigma(G/R) = q + 1$. Since G is the subdirect product of the X_i's, without loss of generality we can assume that A is a chief factor of $X = X_1$.

If $N = \mathrm{soc}(X)$ is an elementary abelian p-group, then, by Corollary 6 and Corollary 14 (1), $|N| + 1 \leq \sigma(G) < q + 1$. Therefore $|N| < q$ and A is a chief factor, say U/V, of an irreducible linear group $X/N \leq GL(N)$ acting on N. By Clifford Theorem, U is a completely reducible linear group hence $\mathrm{O_p}(U) = 1$. Then, by Theorem 3 in [4], $|U/U'| < |N| < q$, against $|A| = |U/V| = q$.

Assume now that $N = S$ is a simple non-abelian group. Then A is isomorphic to a chief factor of a subgroup of $\mathrm{Out}(S)$ hence $q = |A| \leq |\mathrm{Out}(S)| < m_1$ (see e.g. Lemma 2.7 [4]). But $\sigma(G) \geq \sum_{i=1}^{n} m_i \geq m_1 > q$, against $\sigma(G) < q + 1$.

We are left with the case $N = S^r$ where S is a simple non-abelian group. Then X/N is isomorphic to a subgroup of $\mathrm{Out}(S) \wr \mathrm{Sym}(r)$. If A is isomorphic to a chief factor of a transitive subgroup of $\mathrm{Sym}(r)$, then Theorem 2 in [4] gives that $q = |A| \leq 2^r < (n_1)^r \leq m_1$, where n_1 is the minimal index of a subgroup of S. But this contradicts $m_1 \leq \sigma(G) \leq q$. Therefore A has to be a chief factor of a subgroup of $\mathrm{Out}(S)^r$. Then $q = |A| \leq |\mathrm{Out}(S)|^r \leq n_1^r \leq m_1$ gives the final contradiction. $\square$

Lemma 22. *Let G be a non-solvable transitive permutation group of degree n. Then either $\sigma(G) \leq 4^n$ or every non-abelian composition factor of G is isomorphic to an alternating group of odd degree.*

Proof. Let G be a non-solvable transitive permutation group of degree n. We can embed G into a wreath product of its primitive components, let say $G \leq K_1 \wr K_2 \wr \cdots \wr K_t$ where K_i is a primitive permutation group of degree n_i and $n_1 n_2 \cdots n_t = n$ (see for example [7]). Let K_j be a non-solvable component and assume that K_j is not an alternating or symmetric group of odd degree; then G has an homomorphic image $\overline{G}$ which is embedded in a wreath product $K \wr H$ where $K = K_j$ is a permutation group of degree $a = n_j$ and H has degree b with $ab \leq n$. If K does not contain $\mathrm{Alt}(a)$ then $|K| \leq 4^a$ [13] and $\overline{G}$ has a non-solvable normal subgroup of order at most 4^{ab}. By Proposition 7 this implies that $\sigma(G) \leq \sigma(\overline{G}) \leq 4^{ab} \leq 4^n$. So assume that K contains $\mathrm{Alt}(a)$ where a is even. We identify $\overline{G}$ with its image in $K \wr H$: $\overline{G}$ is a transitive group of degree ab, with a system of imprimitivity $\mathcal{B}$ with blocks of size a and K is the permutation group induced on a block by its stabilizer. Let $\mathcal{M}_1$ be the set of subgroups $\overline{G} \cap M$ where M is a maximal intransitive subgroup of $\mathrm{Sym}(ab)$ and let $\mathcal{M}_2$ be the set of subgroups $\overline{G} \cap (M \wr H)$ where $M \cong \mathrm{Sym}(a/2) \wr \mathrm{Sym}(2)$ is a maximal imprimitive subgroup of $\mathrm{Sym}(a)$; if $T \in M_2$ and $B \in \mathcal{B}$, then the permutation group induced on B by the stabilizer T_B is isomorphic to the imprimitive proper subgroup $\mathrm{Sym}(a/2) \wr \mathrm{Sym}(2)$ of K, hence T is a proper subgroup of $\overline{G}$. Now let $x \in \overline{G}$: if x is not a cycle of length ab then there exists $T \in \mathcal{M}_1$ containing x; otherwise there exists $T \in \mathcal{M}_2$ containing x. Hence the set $\mathcal{M}_1 \cup \mathcal{M}_2$ covers $\overline{G}$ with

$$\sum_{i=1}^{ab/2} \binom{ab}{i} + \frac{1}{2}\binom{a}{a/2} \leq 2^{ab} \leq 2^n$$

proper subgroups. Therefore $\sigma(G) \leq 2^n$. □

Proposition 23. *Let G be a σ-primitive group with a non-abelian minimal normal subgroup N. If $G/NC_G(N)$ is not cyclic, then all the non-abelian composition factors of $G/NC_G(N)$ are alternating groups of odd degree.*

Proof. Let $N = S^r$, where S is a non-abelian simple group. By Corollary 6, $5^r + 1 \leq \sigma(G)$. Denote by X the monolithic primitive group associated to the G-group N; then X is a subgroup of $\mathrm{Aut}(S) \wr \mathrm{Sym}(r)$. Let K be the image of X in $\mathrm{Sym}(r)$. If K is solvable, then, by Schreier Conjecture, $X/\operatorname{soc}(X) \cong G/NC_G(N)$ is solvable. By Proposition 21 it follows that $G/NC_G(N)$ is cyclic.

Thus, if $G/NC_G(N)$ is not cyclic, then K is non-solvable. Since $5^r + 1 \leq \sigma(G) \leq \sigma(K)$, the previous lemma implies that every non-abelian composition factor of K is an alternating group of odd degree. Then, by Schreier Conjecture, the same holds for $G/NC_G(N)$. □

Theorem 24. *Let G be a σ-primitive group with no abelian minimal normal subgroups. Then either G is a primitive monolithic group and $G/\operatorname{soc}(G)$ is cyclic, or $G/\operatorname{soc}(G)$ is non-solvable and all the non-abelian composition factors of $G/\operatorname{soc}(G)$ are alternating groups of odd degree.*

Proof. By Corollary 14, G is a subdirect product of the monolithic primitive groups X_i associated to the minimal normal subgroups G_i, $1 \leq i \leq n$. By Proposition 23 and Proposition 21, for every i, $G/G_iC_G(G_i) \cong X_i/\operatorname{soc}(X_i)$ is either cyclic or non-solvable and all of its non-abelian composition factors are alternating groups of odd degree. Therefore either $G/\operatorname{soc}(G)$ is solvable (hence cyclic by Proposition 21) or non-solvable and all of its non-abelian composition factors are alternating groups of odd degree.

We are left to prove that if $G/\operatorname{soc}(G)$ is cyclic then $n = 1$. Assume by contradiction that $n \geq 2$.

Let u_i be the number of distinct prime divisors of the order of the cyclic groups $X_i/\operatorname{soc}(X_i)$ and assume that $u_1 \leq \cdots \leq u_n$.

Step 1. *Let m_i be the minimal index of a supplement of $\operatorname{soc}(X_i)$ in X_i; then $m_i \geq u_i$*

If $\operatorname{soc}(X_i) = S$ is a simple group, then X_i/S is isomorphic to a subgroup of $\mathrm{Out}(S)$, and thus $u_i \leq 2^{u_i} \leq |\,\mathrm{Out}(S)| \leq m_i$ (see e.g. Lemma 2.7 [4]).

If $\operatorname{soc}(X_i) = S^r$ where $r \neq 1$, then $X_i/\operatorname{soc}(X_i)$ is isomorphic to a subgroup Y of $\mathrm{Out}(S) \wr \mathrm{Sym}(r)$. Let K be the intersection of Y with the base subgroup $(\mathrm{Out}(S))^r$ of the wreath product

$\mathrm{Out}(S) \wr \mathrm{Sym}(r)$ and let a be the number of distinct prime divisors of $|K|$; since $|K|$ divides $|\,\mathrm{Out}(S)|^r$, we get that $2^a \leq |\,\mathrm{Out}(S)| \leq n_i$ where n_i is the minimal index of a subgroup of S. Now $b = u_i - a$ is smaller or equal than the number of distinct prime divisors of the order of Y/K which is isomorphic to a non trivial subgroup of $\mathrm{Sym}(r)$, hence $1 \leq b < r$ and thus $u_i = a + b \leq (2^a)^b \leq (2^a)^r \leq (n_i)^r \leq m_i$ whenever $a > 0$. If $a = 0$, then $X_i/\operatorname{soc}(X_i)$ is isomorphic to a subgroup of $\mathrm{Sym}(r)$ and thus $u_i < r \leq (n_i)^r \leq m_i$. This proves the first step.

Let π be the projection of G over $X = X_1$ and call $N = \operatorname{soc} X$. Note that there exist precisely u_1 maximal subgroups of the cyclic group X/N; let $H_1, \ldots, H_{u_1}$ be the maximal subgroups of G such that their images in X/N give all the maximal subgroups of X/N.

Let $\mathcal{M}$ be a set of $\sigma = \sigma(G)$ maximal subgroups whose union is G and define $\mathcal{A}$ to be the set of maximal subgroups of $\mathcal{M}$ containing G_1, $\mathcal{B} = \mathcal{M} \setminus \mathcal{A}$ and

$$\Omega = \left\{ \pi_1(g) \mid g \in G \setminus \bigcup_{M \in \mathcal{A}} M \right\}.$$

Step 2. *Assume that Ω contains a coset yN such that $\langle yN \rangle = X/N$.*

By Lemma 18, if Ω is covered by $\sigma_\Omega(X)$ maximal subgroups, then the same subgroups cover every coset $y^\alpha N$ with α prime to $|y|$. All the other elements of X are covered by the u_1 maximal subgroups $\pi(H_1), \ldots, \pi(H_{u_1})$, since the images of these elements are not generators of X/N. Then $\sigma(X) \leq \sigma_\Omega(X) + u_1$. On the other hand, by Proposition 16, $\sigma_\Omega(X) + \sum_{i \neq 1} \sigma^*(X_i) \leq \sigma(G) < \sigma(X)$, hence $\sum_{i \neq 1} \sigma^*(X_i) < u_1$. Remark 17 and Step 1 give that $\sum_{i \neq 1} u_i \leq \sum_{i \neq 1} m_i < u_1$, and this contradicts the minimality of u_1.

Step 3. *Assume that Ω does not contain a coset yN such that $\langle yN \rangle = X/N$.*

Then Ω is covered by the images in X of the subgroups $H_1, \ldots, H_{u_1}$ and thus, by definition of Ω, G is covered by the subgroups in $\mathcal{A}$ and $H_1, \ldots, H_{u_1}$. It follows that $|\mathcal{B}| + |\mathcal{A}| = \sigma(G) \leq u_1 + |\mathcal{A}|$, hence, by Step 1, $|\mathcal{B}| \leq u_1 \leq m_1$, against Lemma 3.2 in [14]. This final contradiction implies that G has to be a primitive monolithic group and proves the proposition. □

4 There is no group for which $\sigma(G) = 11$

In this section we will show that $\sigma(G)$ can never be equal to 11. The first trivial observation is that $\sigma(G) \neq 11$ whenever G is solvable, since in this case by Tomkinson's result $\sigma(G) = q + 1$, for a prime power q.

Assume by contradiction that there exists a primitive 11-sum group G. By Corollary 14, $\operatorname{soc}(G)$ is the direct product of n non G-equivalent minimal normal subgroups $G_1, \ldots, G_n$, where at most one of them is abelian.

Lemma 25. *Suppose that G is a primitive 11-sum group. Then G has no abelian minimal normal subgroups.*

Proof. Assume by contradiction that G_1 is abelian. By Corollary 14, G_1 is a complemented non-central factor of G, hence, by Corollary 6, $|G_1| + 1 \leq \sigma(G) = 11$. Moreover, by Proposition 10, $11 = \sigma(G) < 2|G_1|$. Hence $|G_1|$ can only be $7, 2^3$ or 3^2. Actually, if $|G_1| = 7$, then the bound in Proposition 10 gives $\sigma(G) \leq 1 + 7$, against $\sigma(G) = 11$.

Note that, by Proposition 16, $\sigma(G) = 11 \geq \sum_{i=1}^{n} \sigma^*(X_i)$ where X_i are the monolithic groups associated to the G_i's; since G_1 is the only abelian subgroup and $\sigma^*(X_i) \geq 5$ if G_i is non-abelian, then G_1 is the unique minimal normal subgroup of G and $G \leq G_1 \rtimes \mathrm{Aut}(G_1)$.

If $|G_1| = 9$, then $G \leq \mathbb{F}_3^2 \rtimes \mathrm{GL}(2,3)$; hence G is solvable, a contradiction.

Thus $|G_1| = 8$ and $G = \mathbb{F}_2^3 \rtimes \mathrm{GL}(3,2)$, since every proper subgroup of $\mathrm{GL}(3,2)$ is solvable. Let $\mathcal{M} = \{M_1, \cdots, M_{11}\}$ be a set of 11 maximal subgroups covering G. In [6] it is proved that $\sigma(\mathrm{GL}(3,2)) = 15$ and, in particular, that one needs at least 7 subgroups to cover the seven point stabilizers of $\mathrm{GL}(3,2)$. It follows that all the 8 complement of G_1 in G occur in $\mathcal{M}$, let say they are $M_1, \ldots, M_8$. As in the proof of Proposition 10, for every point stabilizer $g \in \mathrm{GL}(3,2)$ there exists an element $v_g \in G_1$ such that gv_g does not belong to any complement of G_1 in G. Hence the remaining subgroups M_9, M_{10}, M_{11} of $\mathcal{M}$ have to cover all the elements gv_v where g is a point stabilizer. Since M_9, M_{10} and M_{11} contain G_1, this would imply that we can cover the seven point stabilizers of $\mathrm{GL}(3,2)$ with only three subgroups, a contradiction. $\square$

Theorem 26. *There is no group G with $\sigma(G) = 11$.*

Proof. Suppose that G is a primitive 11-sum group and let $G_1, \ldots, G_n$ be its minimal normal subgroups. By the previous lemma every G_i is non-abelian. If $G_i = \mathrm{Alt}(5)$ for some i, then, by Corollary 20, $G = \mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$. Otherwise, $\sigma^*(X_i) \geq l_{X_i}(G_i) > 5$ for every i and Proposition 16 implies that there is at most one minimal normal subgroup in G. By the same argument, if $G_1 = S^r$, where S is a simple non-abelian group, since $l_{X_1}(G_1) \geq 5^r$ and, by Lemma 5, $5^r + 1 \leq \sigma(G) = 11$, we have that $G_1 = S$ and $l_{X_1}(G_1) + 1 \leq 11$. Therefore G is an almost-simple group with socle S and $l_G(S) \leq 10$, in particular

$$S \in \{\mathrm{Alt}(n) \mid 5 \leq n \leq 10\} \cup \{\mathrm{Sym}(n) \mid 5 \leq n \leq 10\} \cup \{\mathrm{PSL}(2,q) \mid 7 \leq q \leq 8\}.$$

Thanks to the works of Maroti [12] and Bryce et al. [6], we can exclude most of these cases: indeed $\sigma(\mathrm{Alt}(n)) \geq 2^{n-2}$ if $n \neq 7, 9$, $\sigma(\mathrm{Alt}(5)) = 10$, $\sigma(\mathrm{Alt}(9)) \geq 80$, $\sigma(\mathrm{Sym}(n) = 2^{n-1}$ if n is odd and $n \neq 9$, $\sigma(\mathrm{Sym}(9)) \geq 172$, $\sigma(\mathrm{PSL}(2,7)) = 15$, $\sigma(\mathrm{PGL}(2,7)) = 29$, $\sigma(\mathrm{PSL}(2,8)) = 36$. Moreover, $\sigma(\mathrm{Aut}(\mathrm{Alt}(6))) \leq \sigma(C_2 \times C_2) = 3$ and $\sigma(\mathrm{Sym}(6)) = 13$ (see e.g. [1]). The remaining cases are $G = \mathrm{Alt}(7), \mathrm{Sym}(8), \mathrm{Sym}(10), \mathrm{M}_{10}, \mathrm{PGL}(2,9)$ and $\mathrm{Aut}(\mathrm{PSL}(2,8))$.

• $G \neq \mathrm{Alt}(7)$. Assume by contradiction $\sigma(\mathrm{Alt}(7)) = 11$. There are seven maximal subgroups of $\mathrm{Alt}(7)$ isomorphic to $\mathrm{Alt}(6)$; since $\sigma(\mathrm{Alt}(6)) = 16 > 11$, each of them has to appear in a minimal cover of G. Moreover, there are two conjugacy classes with 15 maximal subgroups isomorphic to

PSL(3, 2) and since $\sigma(\mathrm{PSL}(3,2)) = \sigma(\mathrm{PSL}(2,7)) = 15 > 11$ we have that $\sigma(\mathrm{Alt}(7))$ is at least $7 + 15 + 15$.

• $G \neq \mathrm{Sym}(8)$. If $\sigma(\mathrm{Sym}(8)) \leq 11$ then, since $\sigma(\mathrm{Sym}(7)) = 2^6$ and $\sigma(\mathrm{Alt}(8)) \geq 2^6$, arguing as in the previous case we get that a minimal cover $\mathcal{M}$ of Sym(8) contains the 8 point stabilizers and Alt(8). Let $g_1 = (1,2,3,4,5,6,7,8)$, $g_2 = (1,2,3,7,4,5,6,8)$ and $g_3 = (1,2,3,5,4,6,7,8)$; any couple of them generate Sym(8) so that we need at least 3 more subgroups in $\mathcal{M}$, and thus $\sigma(\mathrm{Sym}(8)) > 11$.

• $G \neq \mathrm{Sym}(10)$. If $\sigma(\mathrm{Sym}(10)) \leq 11$, then, as $\sigma(\mathrm{Sym}(9)) = 2^8$ and $\sigma(\mathrm{Alt}(10)) \geq 2^8$, a minimal cover $\mathcal{M}$ of Sym(10) contains 10 point stabilizers and Alt(10). But these subgroups do not cover the 10-cycles. Thus $\sigma(\mathrm{Sym}(10)) > 11$.

• $G \neq \mathrm{M}_{10}$. In M_{10} there are 180 elements of order 8. The only maximal subgroups containing elements of order 8 are the Sylow 2-subgroups and each of them contains 4 of these elements; thus we need at least $180/4 = 45$ subgroups to cover the elements of order 8.

• $G \neq \mathrm{PGL}(2,9)$. In PGL(2, 9) there are 144 elements of order 10. The only maximal subgroups containing elements of order 10 are the normalizers of the Sylow 5-subgroups and each of them contains 4 of these elements; thus we need at least $144/4 = 36$ subgroups to cover the elements of order 10.

• $G \neq \mathrm{Aut}(\mathrm{PSL}(2,8))$. In $\mathrm{Aut}(\mathrm{PSL}(2,8)) \setminus \mathrm{PSL}(2,8)$ there are 336 elements of order 9. The only maximal subgroups containing elements of this kind are the normalizers of the Sylow 3-subgroups; each of them contains 12 of these elements thus we need at least $336/12 = 28$ subgroups to cover Aut(PSL(2, 8)). □

5 Direct products

Proposition 27. *Let $G = H_1 \times H_2$ be the direct product of two subgroups. Let N_i be the smallest normal subgroup of H_i such that H_i/N_i is a direct product of simple groups. If H_1/N_1 and H_2/N_2 have at most one non-abelian simple group S in common and the multiplicity of S in H_1/N_1 is at most one, then either $\sigma(G) = \min\{\sigma(H_1), \sigma(H_2)\}$, or the cyclic group C_p is an epimorphic image of both H_1 and H_2 and $\sigma(G) = p + 1$.*

Proof. Let G be a counterexample with minimal order. We first prove that G is a σ-primitive group. As $\Phi(G) = \Phi(H_1) \times \Phi(H_2)$, we have $\Phi(G) = 1$. Let N be a minimal normal subgroup of G and assume by contradiction that $\sigma(G) = \sigma(G/N)$. If $N \leq H_1$, then, by minimality of $|G|$, we have that either $\sigma(G/N) = \sigma(H_1/N \times H_2) = \min\{\sigma(H_1/N), \sigma(H_2)\} \geq \min\{\sigma(H_1), \sigma(H_2)\} \geq \sigma(G)$, and so $\sigma(G) = \min\{\sigma(H_1), \sigma(H_2)\}$, or C_p is a common factor of H_1/NN_1 and H_2/N_2, and $\sigma(G/N) = p+1$; in this case $\sigma(G) = \sigma(G/N) = p + 1$. Now assume that N is not contained in H_1 or H_2. Then N is a central minimal normal subgroup of G, $N = C_p \cong N_1N/N_1 \cong N_2N/N_2$ and G has a factor group isomorphic to $C_p \times C_p$; therefore $\sigma(G) \leq p + 1$. On the other hand, $\overline{N} = NH_2 \cap H_1 \cong N$ is a central minimal normal subgroup of G contained in H_1; by the previous case, $\sigma(G) < \sigma(G/\overline{N})$. Since $\delta_G(\overline{N}) \geq 2$, $\overline{N}$ has at least $|\overline{N}| = p$ complements; hence, by Lemma 5, $\sigma(G) \geq p+1$ and therefore $\sigma(G) = p+1$. Thus a counterexample G with minimal order is a σ-primitive group.

If G is solvable, then either $G \cong C_p^2$ and $\sigma(G) = p+1$ or G is monolithic: the second possibility cannot occur as G is the direct product of two non trivial normal subgroups. So from now on we may assume that G is non solvable, and in particular, by Proposition 21, that H_1/N_1 and H_2/N_2 have no common abelian factor.

Now observe that if M is a maximal subgroup of G and M does not contain H_1 and H_2, then G/M_G is a primitive group with nontrivial normal subgroups H_1M_G/M_G and H_2M_G/M_G. If $H_1M_G/M_G = H_2M_G/M_G$, then $G/M_G = H_1M_G/M_G = H_2M_G/M_G$ is a central factor of G/M_G and H_1/N_1 and H_2/N_2 have a common abelian factor, a contradiction. Thus $H_1M_G/M_G \neq H_2M_G/M_G$, and since $G/M_G = H_1M_G/M_G \times H_2M_G/M_G$ is a primitive group, H_1M_G/M_G and H_2M_G/M_G are isomorphic simple groups. Therefore, if H_1/N_1 and H_2/N_2 have no simple groups in common, then every maximal subgroup M of G contains either H_1 or H_2, and we obtain the result arguing as in Lemma 4 of [8].

So, we assume that H_1/N_1 and H_2/N_2 have precisely one non-abelian simple group S in common and the multiplicity of S in H_1/N_1 is one: let $K_i \geq N_i$ be the normal subgroups of H_i such that $H_1/K_1 = S$ and $H_2/K_2 = S^n$, being n the multiplicity of S in H_2/N_2 , and set $K = K_1 \times K_2$.

Let $\mathcal{M}$ be a minimal cover of G given by $\sigma(G)$ maximal subgroups of G. We set:

$$\begin{aligned}
\mathcal{M}_1 &= \{L \in \mathcal{M} \mid L \geq H_1\} = \{H_1 \times M \mid M \text{ a maximal subgroup of } H_2\},\\
\mathcal{M}_2 &= \{L \in \mathcal{M} \mid L \geq H_2\} = \{M \times H_2 \mid M \text{ a maximal subgroup of } H_1\},\\
\mathcal{M}_3 &= \mathcal{M} \setminus (\mathcal{M}_1 \cup \mathcal{M}_2).
\end{aligned}$$

Then we define the two sets

$$\Omega_1 = H_1 \setminus \bigcup_{M\times H_2 \in \mathcal{M}_2} M, \quad \Omega_2 = H_2 \setminus \bigcup_{H_1 \times M \in \mathcal{M}_1} M,$$

and their images under the projection π_{K_i} of H_i over H_i/K_i

$$\overline{\Omega}_i = \{\pi_{K_i}(w) \mid w \in \Omega_i\}.$$

As H_1/K_1 is not cyclic, we can cover $\overline{\Omega}_1$ with $|\overline{\Omega}_1|$ subgroups. Hence we can cover $H_1 = \{\bigcup_{M\times H_2 \in \mathcal{M}_2} M\} \cup \Omega_1$ with the images of the maximal subgroups in $\mathcal{M}_2$ plus $|\overline{\Omega}_1|$ maximal subgroups, and thus $\sigma(H_1) \leq |\mathcal{M}_2| + |\overline{\Omega}_1|$. On the other hand, $|\mathcal{M}_2| + |\mathcal{M}_3| \leq \sigma(G) < \sigma(H_1)$, and we obtain that

$$|\overline{\Omega}_1| > |\mathcal{M}_3|.$$

Now observe that the elements of the set $\Omega_1 \times \Omega_2$ can not belong to any of the subgroup of $\mathcal{M}_1$ or $\mathcal{M}_2$, thus the set $\Omega_1 \times \Omega_2$ has to be covered by the subgroups of $\mathcal{M}_3$. If $M \in \mathcal{M}_3$, then G/M_G is a primitive group and $G/M_G = H_1M_G/M_G \times H_2M_G/M_G = S \times S$; in particular $M \geq K$ and M/K is a maximal subgroup of diagonal type of G/K. This means that there exists an automorphism α of S and an index $i \in \{1, \dots n\}$, such that the set $(M/K) \cap (\overline{\Omega}_1 \times \overline{\Omega}_2)$ is given by elements of the type $(x, y_1, y_2, \dots, y_n)$ where $x \in \overline{\Omega}_1$, $(y_1, y_2, \dots, y_n) \in \overline{\Omega}_2$ and $y_i = x^\alpha$. For every $y \in S$ we denote by s_y the number of vectors $(y_1, y_2, \dots, y_n)$ such that $(y_1, y_2, \dots, y_n) \in \overline{\Omega}_2$ and $y_i = y$: note that

$$\sum_{y \in S} s_y = |\overline{\Omega}_2| = |\overline{\Omega}_1 \times \overline{\Omega}_2|/|\overline{\Omega}_1|.$$

On the other hand

$$|(M/K) \cap (\overline{\Omega}_1 \times \overline{\Omega}_2)| \leq \sum_{y \in S} s_y = |\overline{\Omega}_1 \times \overline{\Omega}_2|/|\overline{\Omega}_1| < |\overline{\Omega}_1 \times \overline{\Omega}_2|/|\mathcal{M}_3|,$$

since $|\overline{\Omega}_1| > |\mathcal{M}_3|$. This implies that we can not cover $\Omega_1 \times \Omega_2$ with the $|\mathcal{M}_3|$ subgroups of $\mathcal{M}_3$, a contradiction. □

Theorem 28. *Let $G = H_1 \times H_2$ be the direct product of two subgroups. If no alternating group $\mathrm{Alt}(n)$ with n odd is a homomorphic image of both H_1 and H_2, then either $\sigma(G) = \min\{\sigma(H_1), \sigma(H_2)\}$ or $\sigma(G) = p + 1$ and $S = C_p$ is a homomorphic image of both H_1 and H_2.*

Proof. Let G be a counterexample with minimal order. Let N_i be the minimal normal subgroup of H_i such that H_i/N_i is a direct product of simple groups. As in the proof of Proposition 27, it is easy to see that G is a σ-primitive group, H_1/N_1 and H_2/N_2 have at least one simple group S in common and S is non-abelian.

By Corollary 14, G has at most one abelian minimal normal subgroup, so we can assume that every minimal normal subgroup of H_1 is non-abelian.

Let K be a normal subgroup of G with $G/K \cong S$. Note that $\delta_G(G/K) \geq 2$, indeed $\delta_G(G/K)$ coincides with the multiplicity of S in $G/(N_1 \times N_2)$. Hence, by Corollary 14 (3), no minimal normal subgroup of G is G-equivalent to G/K. This implies in particular that S is an epimorphic image of $H_1/\operatorname{soc}(H_1)$, and consequently S is an homomorphic image of X/N where X is a monolithic primitive group associated to a minimal normal subgroup N of H_1. By the remark above N is non-abelian, so $N = T^r$ with T a non-abelian simple group. Since X is a subgroup of $\mathrm{Aut}(T) \wr \mathrm{Sym}(r)$ and S is non-abelian, S is an homomorphic image of a transitive group Y of degree r. Then Y satisfies the assumption of Lemma 22 and, since S is not an alternating group of odd degree, we get $\sigma(Y) \leq 4^r$. Since, by Corollary 6, $5^r + 1 \leq \sigma(G) \leq \sigma(Y)$, we get a contradiction. □

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References

[1] A. Abdollahi, F. Ashraf and S.M. Shaker, *The symmetric group of degree six can be covered by 13 and no fewer proper subgroups*, Bull. Malays. Math. Sci. Soc., **30**(2) (2007), no. 1, 57–58.

[2] M. Aschbacher, *Finite group theory*, Second edition, Cambridge Studies in Advanced Mathematics 10, Cambridge University Press, Cambridge, 2000.

[3] M. Aschbacher and R. Guralnick, *Some applications of the first cohomology group*, J. Algebra, **90**(1984), 446–460.

[4] M. Aschbacher and R. Guralnick, *On abelian quotients of primitive groups*, Proc. Amer. Math. Soc., **107**(1989), 89–95.

[5] R.A. Bryce and L. Serena, *A note on minimal coverings of groups by subgroups*, Special issue on group theory, J. Aust. Math. Soc., **71**(2001), no. 2, 159–168

[6] R.A. Bryce, V. Fedri and L. Serena, *Subgroup coverings of some linear groups*, Bull. Austral. Math. Soc., **60**(1999), 227–238.

[7] P.J. Cameron, *Permutation groups*, London Mathematical Society Student Texts 45. Cambridge University Press, 1999.

[8] J.H.E. Cohn, *On n-sum groups*, Math. Scand., **75**(1) (1994), 44–58.

[9] E. Detomi and A. Lucchini, *Crowns and factorization of the probabilistic zeta function of a finite group*, J. Algebra, **265**(2003), 651–668.

[10] P. Jiménez-Seral and J. Lafuente, *On complemented nonabelian chief factors of a finite group*, Israel J. Math., **106**(1998), 177–188.

[11] P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, London Mathematical Society Lecture Note Series, 129 Cambridge University Press, 1990.

[12] A. Maróti, *Covering the symmetric groups with proper subgroups*, J. Combin. Theory Ser. A, **110**(1) (2005), 97–111.

[13] C.E. Praeger and J. Saxl, *On the orders of primitive permutation groups*, Bull. London Math. Soc., **12**(1980), 303–307.

[14] M.J. Tomkinson, *Groups as the union of proper subgroups*, Math. Scand., **81**(2) (1997), 191–198.

[15] Y.M. Wang, *Finite groups admitting a fixed-point-free automorphism group*, Northeast. Math. J., **9**(4) (1993), 516–520.