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The Extension of the Formula by Dupire

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ABSTRACT

We provide the extension of Dupire's PDE, as the partial integro-differential equations of market prices of call options with many maturities and strike prices for jump diffusion model.

RESUMEN

Nosotros damos la extensión de Dupire PDE, como las ecuaciones parciales integro-diferenciales de precios de mercado de opciones de llamada con muchos vencimientos y golpe de precios para modelos de difusión con saltos.

Key words and phrases: *Dupire, PDE, Jump-diffusion model.*

Math. Subj. Class.: *35K15, 60H30.*

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space. On the space $(\Omega, \mathcal{F}, P)$ we set a standard Brownian motion $W = \{W_t\}_{t\in[0,T]}$ from $W_0 = 0$ and a Poission random measure $N(dtdx)$ on $(0,T]\times\mathbf{R}$ with intensity measure $dt\nu(dx)$, where $T \in (0,\infty)$ and the measure ν on $\mathbf{R}$ satisfies

$$\int_{\mathbf{R}} (1+e^{2z}) \wedge z^2 \nu(dz) < \infty. \tag{1}$$

We consider a risk-neutral price process $\{S_t^x\}_{t\in[0,T]}$ of a risk asset satisfing

$$\begin{aligned} dS_t^x &= \sigma(t, S_t^x) S_t^x dW_t + (r-\delta) S_t^x dt; \\ S_0^x &= x \in (0,\infty), \end{aligned}$$

where $r \geq 0$ denotes the interest rate and $\delta \geq 0$ the dividend rate. The function $\sigma : [0,T]\times(0,\infty) \to [0,\infty)$ has the Lipschiz condition and is often called the volatility of the asset's price. According to the well-known discussion of option pricing model, if for each $T, K \in (0,\infty)$ we have a unique solusion $u(t,x,T,K)$ to the parabolic equation and boundary condition

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma(t,x)^2 x^2 \frac{\partial^2 u}{\partial x^2} + (r-\delta)x\frac{\partial u}{\partial x} - ru = 0, &\quad (t,x) \in [0,T) \times (0,\infty); \\ u(t,x,T,K)|_{t=T} = (x-K)^+, &\quad x \in (0,\infty), \end{aligned}$$

then a price of a call option with maturity T and strike price K is given by

$$u(t,x,T,K)|_{t=0} = e^{-rT} E[(S_T^x - K)^+].$$

Dupire[1] found that $u(t,x,T,K)$ as a function of (T,K) satisfies the following dual equation to the last parabolic equation:

$$\frac{\partial u}{\partial T} = \frac{1}{2}\sigma(T,K)^2 K^2 \frac{\partial^2 u}{\partial K^2} - (r-\delta)K\frac{\partial u}{\partial K} - \delta u, \quad (T,K) \in (t,\infty) \times (0,\infty).$$

But his approach is not enough mathematically. There are some works justifing rigorously his idea, for example, Klebaner[4] etc. Klebaner[4] gives the last equation by the Meyer-Tanaka formula. On the other hand, there are also works on option pricing model for jump-diffusion processes, for example, geometric Lévy processes by Fujiwara and Miyahara[2]. Recently, Jourdain[3] provides the extension of Dupire's work for jump-diffusion processes by stochastic flow approch.

Now, we consider the following risk-neutral evolusion $\{X_t^x\}_{t\in[0,T]}$ for the underlying risk asset's prices:

$$\begin{aligned} X_t^x &= x + \int_0^t a(u, X_u^x) X_u^x dW_u + (r-\delta)\int_0^t X_u^x du \\ &\quad + \int_{(0,t]\times\mathbf{R}} X_{u-}^x (e^z - 1)\{N(dudz) - du\nu(dz)\}, \quad t \in [0,T] \end{aligned}$$

where $a(t,x) : [0,T]\times(0,\infty) \to [0,\infty)$ satisfies the Lipschiz condition and has the second derivative with respect to x. Then $\{X_t^x\}_{t\in[0,T]}$ has the extended diffusion operator(see Yoshida[5] p.408)

$$\begin{aligned}(A_t f)(x) &= \frac{1}{2}a(t,x)^2x^2f''(x) + (r-\delta)xf'(x) \\ &\quad + \int_{\mathbf{R}} f(xe^z) - f(x) - (e^z-1)xf'(x)\nu(dz).\end{aligned}$$

For each maturity T and strike price K we denote

$$C(x,T,K) = e^{-rT}E[(X_T^x - K)^+] \tag{2}$$

by a call option price with an asset price x. In particular, in the case $a(\cdot,\cdot)\equiv a$ the last definition (2) is justified by Fujiwara and Miyahara[2]. If we moreover assume that $a(\cdot,\cdot)$ belongs to the class

$$\mathcal{V} = \left\{ f : [0,T]\times(0,\infty) \to \mathbf{R} | \sup_{(t,x)\in[0,T]\times(0,\infty)} \sum_{k=0}^{3} |x^k \frac{\partial^k f}{\partial x^k}(t,x)| < \infty \right\},$$

then Jourdain[3] provides the following equation of (T,K):

$$-\frac{\partial C}{\partial T} + \mathcal{A}_T C = 0, \quad (T,K)\in(0,\infty)\times(0,\infty),$$

where

$$\begin{aligned}(\mathcal{A}_T f)(K) &= \frac{1}{2}a(T,K)^2K^2f''(K) - (r-\delta)Kf'(K) - \delta f(K) \\ &\quad + \int_{\mathbf{R}} \{f(Ke^{-z}) - f(K) - (e^{-z}-1)Kf'(K)\}e^z\nu(dz).\end{aligned}$$

Here notice that the assumption $a(\cdot,\cdot)\in\mathcal{V}$ satisfies the Lipschiz condition. In this note we provide the same result of the above without $a(\cdot,\cdot)\in\mathcal{V}$ by using not only stochastic flow approch but also another one.

2 Main result

We fix $x\in(0,\infty)$ as follows. We have the following main theorem.

Theorem 2.1. *$C(x,T,K)$ as a function of (T,K) satisfies*

$$-\frac{\partial C}{\partial T} + \mathcal{A}_T C = 0, \quad (T,K)\in(0,\infty)\times(0,\infty)$$

in weak sense; that is,

$$\int_0^\infty\int_0^\infty C(x,T,K)\left\{\frac{\partial\varphi}{\partial T}(T,K) + \mathcal{A}_T^*\varphi(T,K)\right\}dTdK = 0, \ \forall\varphi\in C_0^\infty((0,\infty)^2),$$

where

$$\int_0^\infty\int_0^\infty \psi(T,K)\mathcal{A}_T^*\varphi(T,K)dTdK = \int_0^\infty\int_0^\infty \mathcal{A}_T\psi(T,K)\varphi(T,K)dTdK,$$
$$\forall\varphi,\forall\psi\in C_0^\infty((0,\infty)^2).$$

2.1 Lemmas

Lemma 2.1. *It follows that*

$$0 \le C(x,T,K) \le e^{-\delta T}x, \quad (T,K) \in (0,\infty)\times(0,\infty). \tag{3}$$

For every $\varphi(\cdot) \in C_0^2((0,\infty))$

$$e^{-rT}E[\varphi(X_T^x)] = \int_0^\infty C(x,T,K)\varphi''(K)dK, \quad T \in (0,\infty) \tag{4}$$

holds.

Remark 2.1. *It follows from* (3) *that* $C(x,T,K)$ *as a function of* (T,K) *is locally integrable on* $(0,\infty)\times(0,\infty)$*. Thus the right-hand side of* (4) *is well-defined.*

proof: By (2) we have

$$0 \le e^{rT}C(x,T,K) \le E[X_T^x].$$

Moreover, since $\{e^{-(r-\delta)t}X_t^x\}_{t\in[0,T]}$ is a nonnegative local martingale with initial value x, the right-hand side of the last inequality is

$$\le e^{(r-\delta)T}x.$$

Hence we get (3). Finally, we compute from (2) that the right-hand side of (4) is

$$\begin{aligned} &= \int_0^\infty e^{-rT}E[(X_T^x-K)^+]\varphi''(K)dK \\ &= e^{-rT}E\left[\int_0^\infty (X_T^x-K)^+\varphi''(K)dK\right] \\ &= e^{-rT}E[\varphi(X_T^x)]. \end{aligned}$$

Hence we get (4).

Before we moreover introduce lemmas, for every $\varphi(\cdot,\cdot) \in C_0^\infty((0,\infty)^2)$ we set a family $\{\Phi_h\}_{h>0}$ of all functions

$$\Phi_h(T,x) = \frac{1}{h}\{E[\varphi(T,X_{T+h}^x)] - E[\varphi(T,X_T^x)]\}, \quad (T,K)\in(0,\infty)\times(0,\infty).$$

Lemma 2.2.

$$\lim_{h\downarrow 0}\int_0^\infty \Phi_h(T,x)dT = -\int_0^\infty\int_0^\infty e^{rT}C(x,T,K)\frac{\partial^3\varphi}{\partial T\partial K^2}(T,K)dTdK.$$

proof: First, we set $\tilde{C}(x,T,K) = e^{rT}C(x,T,K)$. By using (4), we have

$$\int_0^\infty \Phi_h(T,x)dT = \int_0^\infty\left\{\int_0^\infty \frac{\tilde{C}(x,T+h,K)-\tilde{C}(x,T,K)}{h}\frac{\partial^2\varphi}{\partial K^2}(T,K)dK\right\}dT,$$

where $h>0$. Moreover we compute that the right-hand side of the last equality is

$$\begin{aligned} &= \int_0^\infty \left\{ \int_0^\infty \frac{\tilde{C}(x,T+h,K)-\tilde{C}(x,T,K)}{h} \frac{\partial^2 \varphi}{\partial K^2}(T,K)dT \right\} dK \\ &= \int_0^\infty \left\{ \int_0^\infty \tilde{C}(x,T,K)\frac{1}{h}\left(\frac{\partial^2 \varphi}{\partial K^2}(T-h,K) - \frac{\partial^2 \varphi}{\partial K^2}(T,K)\right) dT \right\} dK, \end{aligned}$$

and so we have

$$\int_0^\infty \Phi_h(T,x)dT = \int_0^\infty \int_0^\infty \tilde{C}(x,T,K)\frac{1}{h}\left(\frac{\partial^2 \varphi}{\partial K^2}(T-h,K) - \frac{\partial^2 \varphi}{\partial K^2}(T,K)\right) dTdK.$$

Then, by using the dominated convergence theorem, $\varphi(\cdot,\cdot) \in C_0^\infty((0,\infty)^2)$ and (3) imply that the right-hand side of the last equality converges to

$$-\int_0^\infty \int_0^\infty \tilde{C}(x,T,K)\frac{\partial^3 \varphi}{\partial T \partial K^2}(T,K)dTdK$$

as $h \downarrow 0$. Hence we get the desired result.

We denote by the following operator depended on time $t \in [0,\infty)$:

$$\begin{aligned} (\tilde{\mathcal{A}}_t f)(x) &= \frac{1}{2}a(t,x)^2 x^2 f''(x) + \{\frac{\partial}{\partial x}(a(t,x)^2x^2) + (r-\delta)x\} f'(x) \\ &\quad + \left\{ \frac{1}{2}\frac{\partial^2}{\partial x^2}(a(t,x)^2x^2) + (r-2\delta) \right\} f(x) \\ &\quad + \int_{\mathbf{R}} e^{2z} f(xe^z) - (2e^z-1)f(x) - (e^z-1)xf'(x)\nu(dz). \end{aligned}$$

Lemma 2.3.

$$\lim_{h\downarrow 0} \int_0^\infty \Phi_h(T,x)dT = \int_0^\infty \int_0^\infty e^{rT}C(x,T,K)\left(\tilde{\mathcal{A}}_T \frac{\partial^2 \varphi}{\partial K^2}(T,K)\right) dTdK.$$

proof: First, we divide $A.$ into two parts as follows:

$$\begin{aligned} (A.f)(x) &= \Bigg\{ \frac{1}{2}a(\cdot,x)^2x^2f''(x) + (r-\delta)xf'(x) \\ &\quad + \int_{|z|<1} f(xe^z) - f(x) - zxf'(x)\nu(dz) \\ &\quad - \int_{|z|<1} (e^z-1-z)xf'(x)\nu(dz) - \int_{|z|\geq 1} f(x) + (e^z-1)xf'(x)\nu(dz) \Bigg\} \\ &\quad + \int_{|z|\geq 1} f(xe^z)\nu(dz) \\ &= (A^0_. f)(x) + \int_{|z|\geq 1} f(xe^z)\nu(dz). \end{aligned}$$

Since $\varphi(\cdot,\cdot) \in C_0^\infty((0,\infty)^2)$ we can choose subintervals $I_1 = [\alpha_1, \beta_1]$ and $I_2 = [\alpha_2, \beta_2]$ of $(0,\infty)$ such that *supp* $\varphi \subset I_1 \times I_2$. We pick $\delta > 0$ and set $\tilde{I}_1 = \{x | \alpha_1 \le x \le \beta_1 + \delta\}$ and $\tilde{I}_2 = \{x | \alpha_2 e^{-1} \le x \le \beta_2 e\}$. We denote by $\| f \|_{C(\Gamma)} = \sup_{x\in\Gamma} |f(x)|$, where Γ is a compact subset of $(0,\infty)^2$ and $f \in C(\Gamma) = \{f$ is a real-valued continuous function on $\Gamma\}$. Then observe that $A_T^0 \varphi(T,\cdot)$ belongs to $C^2((0,\infty))$, since $\varphi(\cdot,\cdot) \in C_0^\infty((0,\infty)^2)$ and $a(T,\cdot)$ has the second derivative, and

$$
\begin{aligned}
|A_u^0 \varphi(T,K)| \le\ & \frac{1}{2} \| a^2 \|_{C(\tilde{I}_1 \times I_2)} \| K^2 \frac{\partial^2 \varphi}{\partial K^2} \|_{C(I_1\times I_2)} 1_{I_1\times I_2}(T,K) \\
& + |r-\delta| \, \| K \frac{\partial \varphi}{\partial K} \|_{C(I_1\times I_2)} 1_{I_1 \times I_2}(T,K) \\
& + \int_{|z|<1} z^2 \, \nu(dz) \, \| K^2 \frac{\partial^2 \varphi}{\partial K^2} + K \frac{\partial \varphi}{\partial K} \|_{C(I_1 \times \tilde{I}_2)} 1_{I_1 \times \tilde{I}_2}(T,K) \\
& + \int_{|z|<1} e^z - 1 - z \, \nu(dz) \, \| K \frac{\partial \varphi}{\partial K} \|_{C(I_1 \times I_2)} 1_{I_1\times I_2}(T,K) \\
& + \nu(|z| \ge 1) \, \| \varphi \|_{C(I_1\times I_2)} 1_{I_1\times I_2}(T,K) \\
& + \int_{|z|\ge 1} |e^z - 1| \, \nu(dz) \, \| K \frac{\partial \varphi}{\partial K} \|_{C(I_1\times I_2)} 1_{I_1 \times I_2}(T,K) \\
\le\ & \Bigg\{ \frac{1}{2} \| a^2 \|_{C(\tilde{I}_1\times I_2)} \| K^2 \frac{\partial^2 \varphi}{\partial K^2} \|_{C(I_1 \times I_2)} \\
& + (|r-\delta| + \int_{|z|<1} e^z - 1 - z \, \nu(dz) + \int_{|z|\ge 1} |e^z - 1| \, \nu(dz)) \| K \frac{\partial \varphi}{\partial K} \|_{C(I_1\times I_2)} \\
& + \int_{|z|<1} z^2 \, \nu(dz) \| K^2 \frac{\partial^2 \varphi}{\partial K^2} + K \frac{\partial \varphi}{\partial K} \|_{C(I_1 \times \tilde{I}_2)} + \nu(|z| \ge 1) \| \varphi \|_{C(I_1 \times I_2)} \Bigg\} \\
& \times 1_{I_1\times \tilde{I}_2}(T,K) \\
=\ & C_1 \times 1_{I_1 \times \tilde{I}_2}(T,K), \quad \forall u \in \tilde{I}_1,
\end{aligned}
$$

where $C_1 < \infty$ holds since $\varphi(\cdot,\cdot) \in C_0^\infty((0,\infty)^2)$, (1), and $a(\cdot,\cdot)$ is continuous. Moreover, it is easy that we have

$$\left| \int_{|z|\ge 1} \varphi(T, Ke^z) \nu(dz) \right| \le C_2 1_{I_1}(T),$$

where C_2 is a positive constant not depending on T and K. Therefore the inequality of the observation and the last inequality imply

$$|A_u \varphi(T,K)| \le (C_1 + C_2) 1_{I_1}(T), \quad \forall u \in \tilde{I}_1.$$

Here, fix T and by using Appendix 3.2 it follows from $\varphi(\cdot,\cdot) \in C_0^\infty((0,\infty)^2)$ that

$$
\begin{aligned}
\Phi_h(T,x) &= \frac{1}{h} \int_T^{T+h} E[(A_u \varphi(T,\cdot))(X_u^x)] du \\
&= \frac{1}{h} \int_T^{T+h} E[A_u \varphi(T, X_u^x)] du,
\end{aligned}
$$

for all $0 < h < \delta$. Then the last two inequality and equality imply

$$\begin{aligned} \lim_{h\downarrow 0}\Phi_h(T,x) &= E[A_T\varphi(T,X_T^x)]; \\ |\Phi_h(T,x)| &\leq (C_1+C_2)1_{I_1}(T), \quad 0<\forall h<\delta. \end{aligned}$$

According to the dominated convergence theorem, the last two results imply

$$\lim_{h\downarrow 0}\int_0^\infty \Phi_h(T,x)dT = \int_0^\infty E[A_T\varphi(T,X_T^x)]dT. \tag{5}$$

On the other hand, by using (4) we have from the above observation

$$e^{-rT}E[A_T^0\varphi(T,X_T^x)] = \int_0^\infty C(x,T,K)\frac{\partial^2}{\partial K^2}(A_T^0\varphi(T,\cdot))(K)dK$$

Moreover we have

$$\begin{aligned} e^{-rT}E[\int_{|z|\geq 1}\varphi(T,X_T^x e^z)\nu(dz)] &= \int_{|z|\geq 1} e^{-rT}E[\varphi(T,X_T^x e^z)]\nu(dz) \\ &= \int_{|z|\geq 1}\int_0^\infty C(x,T,K)\frac{\partial^2}{\partial K^2}(\varphi(T,Ke^z))dK\nu(dz) \\ &= \int_0^\infty C(x,T,K)\int_{|z|\geq 1} e^{2z}\frac{\partial^2\varphi}{\partial K^2}(T,Ke^z)\nu(dz)dK, \end{aligned}$$

where the second line of the last equality holds by (4). Therefore the last two equalities imply

$$\begin{aligned} e^{-rT}E[A_T\varphi(T,X_T^x)] &= \int_0^\infty C(x,T,K)\{\frac{\partial^2}{\partial K^2}(A_T^0\varphi(T,K)) \\ &\qquad + \int_{|z|\geq 1} e^{2z}\frac{\partial^2\varphi}{\partial K^2}(T,Ke^z)\nu(dz)\}dK, \end{aligned}$$

and so by computing the right-hand side of the last equality we have

$$e^{-rT}E[A_T\varphi(T,X_T^x)] = \int_0^\infty C(x,T,K)(\tilde{\mathcal{A}}_T\frac{\partial^2\varphi}{\partial K^2}(T,K))dK.$$

Hence (5) and the last equality imply the desired result.

2.2 Proof of Theorem 2.1

First, pick any $\psi(T,K) \in C_0^\infty((0,\infty)^2)$. According to Lemma 2.2 and 2.3, for all $\varphi(T,K) \in C_0^\infty((0,\infty)^2)$ such that $e^{rT}\frac{\partial^2\varphi}{\partial K^2} = \psi$, we have

$$\int_0^\infty\int_0^\infty e^{rT}C(x,T,K)\left\{\frac{\partial^3\varphi}{\partial T\partial K^2}(T,K)+\tilde{\mathcal{A}}_T\frac{\partial^2\varphi}{\partial K^2}(T,K)\right\}dTdK = 0,$$

and so

$$\int_0^\infty\int_0^\infty C(x,T,K)\left\{\frac{\partial\psi}{\partial T}(T,K)+\tilde{\mathcal{A}}_T\psi(T,K)\right\}dTdK = 0$$

holds. On the other hand, we can compute the integral by parts

$$\int_0^\infty \int_0^\infty \psi(T,K)\tilde{\mathcal{A}}_T\varphi(T,K)dTdK = \int_0^\infty \int_0^\infty \mathcal{A}_T\psi(T,K)\varphi(T,K)dTdK, \quad \forall\varphi, \forall\psi \in C_0^\infty((0,\infty)^2).$$

Hence the last two equalities imply the desired conclusion.

3 Appendix

Appendix 3.1. *Let $\mathbf{X} \subset \mathbf{R}^d$, where d is a positive integer, be a domain and $C^k(\mathbf{X})$, where $k = 0,1,2,\cdots,\infty$, be a class of all real-valued functions on $\mathbf{X}$ which have continuous partial derivatives of order $\leq k$ if $k < \infty$; of order $< \infty$ if $k = \infty$. Let $C_0^k(\mathbf{X})$ be a class of all functions which belong to $C^k(\mathbf{X})$ and compact supports.*

Appendix 3.2. *(Dynkin's formula)*
For every $f \in C_0^2((0,\infty))$,

$$E[f(X_t^x)] = f(x) + E\left[\int_0^t (A_u f)(X_u^x)du\right], \ (t,x) \in [0,\infty) \times (0,\infty)$$

holds.

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