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Simple Fixed Point Theorems on Linear Continua

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ABSTRACT

A simple fixed point theorem is formulated for multivalued maps with a connected graph on closed intervals of linear continua. These intervals either cover themselves or are concerned with self-maps. We discuss a question when the original map can possess a fixed point, provided the same assumptions are satisfied only for some of its iterate. We are particularly interested in a situation on noncompact connected linearly ordered spaces. Many illustrating examples are supplied.

RESUMEN

Un teorema simple de punto fijo es formulado para aplicaciones multivaluadas con gráfico conexo sobre intervalos cerrados de un linear contínuo. Estos intervalos cubrem ellos mismos o son relacionados con auto-aplicaciones. Discutimos cuando la aplicación original puede poseer un punto fijo, con tal que las mismas condiciones sean satisfechas solamente para algunos de sus iterados. Nosostros estamos particurlamente interesados en una situación sobre espacios linealmente ordenados conexos no compactos. Ejemplos son exhibidos.

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Math. Subj. Class.: *37E05, 47H04, 47H10, 58C30.*

1. Introduction (Fixed point theorems in low dimensions)

The celebrated Sharkovskii cycle coexistence theorem is, roughly speaking, based on many times repeated *Bolzano's intermediate value theorem* (cf. [Sh]) which in turn *is equivalent with a one-dimensional version of the Brouwer fixed point theorem* (cf. [B1], p. 273). Since Bernard Bolzano proved his statement already in 1817, the intermediate value theorem can be regarded in a certain sense as probably the oldest fixed point theorem at all.

More precisely, if a closed interval covers continuously itself, then there exists a subinterval which is mapped onto the image of the given closed interval (endpoints onto the endpoints), and the application of Bolzano's theorem to the restriction on this subinterval leads to the existence of a fixed point. Moreover, a related periodic point theorem can be regarded as such a fixed point theorem applied to the iterate, but the minimal period must be still guaranteed which follows from the fact that the interiors of the subintervals are mapped onto the interiors.

Although there exists an n-dimensional version of the intermediate value theorem due to H. Poincaré which was shown after more than fifty years to be equivalent with the Brouwer fixed point theorem by C. Miranda (for more details, see e.g. [B1], p. 273), a *closed square covering continuously itself need not contain a fixed point* (see Example 1 and cf. [Ka], [Kl]).

Example 1. Letting $f(x,y) := (1-x-y, 2-2x)$ and $A := [0,1]^2$, f is obviously continuous on A and $A \subset f(A) = [-1,1] \times [0,2]$. Fixed points of f must satisfy the system $1-x-y = x$, $2-2x = y$ which is equivalent with finding the intersection points of lines $y_1 = 1-2x$ and $y_2 = 2-2x$ inside the square A.

Since there are evidently no intersections inside A (see Fig. 1), f is fixed point free in A.

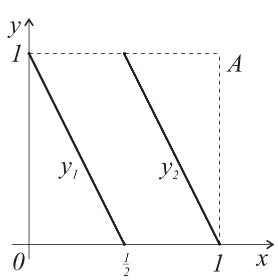

Figure 1: *Lines y_1 and y_2 from Example 1.*

The additional conditions imposed on given maps which assert fixed points are rather drastic (cf. [A2], [Sn]). That is also why *Sharkovskii's theorem holds in principle only in one dimension.*

For multivalued maps, the situation is much more delicate. B. O'Neil gave in 1947 an example (see e.g. [Mi], p. 6) of a continuous fixed point free mapping whose values are homeomorphic to S^1 which sends a closed ball in the plane into itself. J. Jezierski constructed in 1987 (see e.g. [G2], pp. 249-250) a continuous fixed point free map whose values are 1, 2 or 3 points, again on a closed ball in the plane. One can easily find only upper semicontinuous maps whose values are 1 or a fixed number $n \in \mathbb{N}$ of points which are fixed point free on closed intervals (for $n = 2$, see Example 2).

Example 2. The upper semicontinuous map (observe that its graph is closed)

$$\varphi(x) := \begin{cases} 1, & \text{for } x \in \left[0, \frac{1}{2}\right), \\ \{0,1\}, & \text{for } x = \frac{1}{2}, \\ 0, & \text{for } x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

is evidently fixed point free on the unit iterval (see Fig. 2).

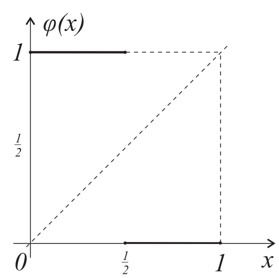

Figure 2: *Function φ from Example 2.*

In spite of these counter-examples, there exists a fixed point theory for multivalued maps with discontinuous values, of course under suitable regularity assumptions (see [Dz], [G1], [G2], [Sk], and the references therein).

Although the conjecture posed in [G1] by L. Górniewicz that the Brouwer fixed point theorem holds for Borsuk-continuous maps (for the definition, see e.g. [G2]) with compact connected values was recently answered in a negative way on B^4 by D. Miklaszewski [Mi], the same author proved that (even a weaker) *notion of Hausdorff-continuity implies for maps with ANR-values a fixed point, on any finite-dimensional ball.* In particular, for (Hausdorff-) continuous n-valued (n fixed) maps, a similar result was already achieved in 1984 by H. Schirmer on finite polyhedra (see [S1]). For n-valued maps, R. F. Brown obtained the Anosov theorem on the circle, namely that the well-defined Nielsen and Lefschetz numbers are absolutely equal (for more details, see [B2], [B3]). Let us note that E. Kudryavtseva (Moscow State University) disproved the Anosov property for 2-maps on the two-dimensional torus after its conjecturing by R. F. Brown during his talk at the conference TTFPP 2007 in Polish Będlewo.

Hence, *in one dimension* (which is sufficient for our needs here), *the Brouwer theorem holds at least for continuous maps whose values are finite unions of closed* (possibly degenerate) *intervals.* For upper semicontinuous maps with closed connected values, a special case of the well-known Kakutani theorem applies (cf. [AG], [G2]), while fixed point theorems for those with disconnected values must be formulated in a more sophisticated way (cf. [Dz], [G2], [Sk]).

If a closed interval only covers itself, then one can easily check that the same assumptions are insufficient for the existence of a fixed point. For instance, continuous 2-point maps are often fixed point free (see Example 3).

Example 3. Since $\varphi(x) = f_1(x) \cup f_2(x)$, where $f_1(x) := x - \frac{1}{2}$ and $f_2(x) := x + \frac{1}{2}$, φ is a continuous 2-point map such that $[-\frac{1}{2}, \frac{1}{2}] \subset \varphi([-\frac{1}{2}, \frac{1}{2}]) = [-1, 1]$, but there are evidently (see Fig. 3) no fixed points of φ on $[-\frac{1}{2}, \frac{1}{2}]$.

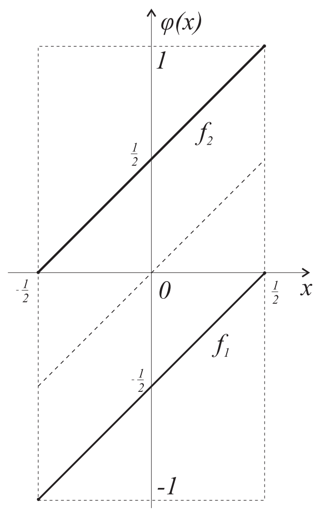

Figure 3: *Function φ from Example 3.*

Thus, some additional connectivity restrictions should be imposed (see Theorem 1 below).

The main purpose of our paper is to discuss possible improvements and consequences of Theorem 1, especially in terms of the iterates of given maps. This will be done, including the formulation of Theorem 1, in Section 3. Before we recall auxiliary definitions. Concluding remarks concern some further possibilities, higher-dimensional analogies and open problems.

2. Auxiliary definitions

In the entire text, all topological spaces will be Hausdorff and all multivalued maps will have nonempty values, i.e. by $\varphi : X \multimap Y$, we mean $\varphi : X \to 2^Y \setminus \{\emptyset\}$. By a *fixed point* of φ, we mean a point $x_0 \in X \cap Y$ such that $x_0 \in \varphi(x_0)$. By a *k-orbit* of φ, we mean a sequence $\{x_1, \ldots, x_k\}$ such that

(i) $x_{i+1} \in \varphi(x_i)$, for all $i = 1, \ldots, k-1$, $x_1 \in \varphi(x_k)$, and

(ii) the orbit is not a product orbit formed by going p-times around a shorter m-orbit, where $mp = k$.

We say that a linearly ordered set $\mathbb{L}$ with more than one point is a *linear continuum* (cf. [Mu], [S2]) if

(i) $\mathbb{L}$ has the least upper bound property,

(ii) $\mathbb{L}$ is ordered densely, i.e. if $x < y$, then there exists z so that $x < z < y$,

(iii) $\mathbb{L}$ is endowed with the order topology by which $\mathbb{L}$ becomes a topological (Hausdorff) space.

It is well-known (see e.g. [AS]) that *connected linearly ordered topological spaces* can be fully characterized by conditions (i)-(iii). The typical examples of linear continua are the real line, its intervals of all types, the long line, the unit square in the dictionary order, etc.

By *intervals* (a, b) or $[a, b]$ of $\mathbb{L}$, where $a, b \in \mathbb{L}$, we understand the sets $\{x \in \mathbb{L} : a < x < b\}$ or $\{x \in \mathbb{L} : a \leq x \leq b\}$, respectively.

Let us also recall that a multivalued map $\varphi : X \multimap Y$ is *upper* (*lower*) *semicontinuous* if $\varphi^{-1}(U) := \{x \in X : \varphi(x) \subset U\}$ is open (closed) in X, for every open (closed) subset U of Y, or equivalently, if $\varphi_+^{-1}(U) := \{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is closed (open) in X, for every closed (open) subset U of Y. The map φ is *continuous* if it is both upper and lower semicontinuous.

We call multivalued maps with compact connected values which are upper semicontinuous or lower semicontinuous or continuous as *M-maps* or *N-maps* or *S-maps*, respectively (cf. [AFP]).

It is well-known (see e.g. [AG], [G2]) that, for compact-valued maps, the notions of continuity and Hausdorff continuity (i.e. the continuity w.r.t. the Hausdorff metric) coincide and that compact maps are upper semicontinuous if and only if their graph is a closed set.

We say that a multivalued mapping $\varphi : \mathbb{L} \multimap \mathbb{L}$ is determined by a *connectivity relation* in $\mathbb{L}^2$ if its graph Γ_φ, restricted to every interval $I \subset \mathbb{L}$, i.e. $\Gamma_{\varphi\restriction_I}$, is connected, for every $I \subset \mathbb{L}$ (including intervals degenerated to one point).

Lemma 1. *The mapping $\varphi : \mathbb{L} \multimap \mathbb{L}$ is determined by a connectivity relation in $\mathbb{L}^2$ if and only if φ has a connected graph and connected values.*

Proof. It suffices to show that if $\varphi : \mathbb{L} \multimap \mathbb{L}$ has a connected graph and connected values, then it it is determined by a connectivity relation in $\mathbb{L}^2$, because the reverse implication follows directly from the definition.

Consider an arbitrary closed interval $I = [a, b] \subset \mathbb{L}$.

At first, we show that the set $M_b^\bullet := \{(x, y) \in \Gamma_\varphi : x \leq b\}$ is connected. We suppose that there exist nonempty sets A and B such that $M_b^\bullet = A \cup B$ and

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset. \tag{1}$$

Considering the set $M^b := \{(x, y) \in \Gamma_\varphi : x > b\}$, we have

$$\Gamma_\varphi = M_b^\bullet \cup M^b = (A \cup M^b) \cup B.$$

Since Γ_φ is connected, either $\overline{A \cup M^b} \cap B \neq \emptyset$ or $(A \cup M^b) \cap \overline{B} \neq \emptyset$. Because of formula (1) and the form of M^b, we can find $y_0 \in \varphi(b)$ such that $y_0 \in (\overline{M^b} \cap B)$. Therefore, since $\varphi(b)$ is connected, formula (1) implies that

$$\overline{M^b} \cap A = M^b \cap \overline{A} = \emptyset.$$

Thus, using formula (1) again, we obtain

$$\overline{(M^b \cup B)} \cap A = (M^b \cup B) \cap \overline{A} = \emptyset,$$

but it is a contradiction, because $\Gamma_\varphi = M^b \cup B \cup A$ is connected.

We can show in an analogous way that the set $\{(x, y) \in M_b^\bullet : x \geq a\}$, i.e. $\Gamma_{\varphi\restriction_I} := \{(x, y) \in \Gamma_\varphi : a \leq x \leq b\}$, is connected, too.

Since any interval $J \subset \mathbb{L}$ can be expressed as the union of closed intervals of $\mathbb{L}$ that have a point in common, the graph $\Gamma_{\varphi\restriction_J}$ can be expressed, in view of the above conclusions, as the union of connected sets of $\mathbb{L}^2$ that have a point in common. This is sufficient (see e.g. [Mu, p. 150]) in order the graph $\Gamma_{\varphi\restriction_J}$ to be connected which completes the proof. $\square$

The map φ is determined by a *G_δ-relation* in $\mathbb{L}^2$ if its graph Γ_φ is a G_δ-subset of $\mathbb{L}^2$, i.e. if $\Gamma_\varphi = \bigcap_{m\in\mathbb{N}} G_m$, where all $G_m \subset \mathbb{L}^2$ are open. Map φ is determined by a *connectivity G_δ-relation* if it has both the above properties.

In [ASS] resp. [AFP], we have shown that M-maps resp. N-maps in $\mathbb{R}$ are determined by connectivity G_δ-relations in $\mathbb{R}^2$.

3. Statements on linear continua

The following fixed point theorem is intuitively obvious (for $\mathbb{L} = \mathbb{R}$, cf. Lemma 2.4 in [ASS]).

Theorem 1. *Let $I = [a, b] \subset \mathbb{L}$ be a closed interval of a linear continuum $\mathbb{L}$ and $\varphi : I \multimap \mathbb{L}$ be a multivalued mapping with a connected graph. Assume that either $I \subset \varphi(I)$ or $\varphi(I) \subset I$. Then φ has a fixed point in I.*

Proof. Denote by $\Gamma_\varphi \subset I \times \mathbb{L} \subset \mathbb{L}^2$ the graph of φ and define the sets P, P_1 and P_2 as

$$P := \{(x,x) \in \mathbb{L}^2\},\ P_1 := \{(x,y) \in \mathbb{L}^2 : x < y\},\ P_2 := \{(x,y) \in \mathbb{L}^2 : y < x\}.$$

Obviously, P_1 and P_2 are nonempty disjoint open sets in $\mathbb{L}^2$ and $\mathbb{L}^2 = P \cup P_1 \cup P_2$.

Assume that $\mathrm{Fix}\ \varphi := \{x \in I : x \in \varphi(x)\} = \emptyset$, i.e. $P \cap \Gamma_\varphi = \emptyset$.

- If $I \subset \varphi(I)$, then there exist points $c, d \in [a,b]$ such that $a \in \varphi(c)$ and $b \in \varphi(d)$. Moreover, $a < c$ (otherwise, $a \in \varphi(a)$ and a is a fixed point) and $d < b$ (otherwise, $b \in \varphi(b)$ and b is a fixed point).

  Then
  $$d < b \Rightarrow (d,b) \in P_1 \cap \Gamma_\varphi \Rightarrow P_1 \cap \Gamma_\varphi \neq \emptyset$$
  and
  $$a < c \Rightarrow (c,a) \in P_2 \cap \Gamma_\varphi \Rightarrow P_2 \cap \Gamma_\varphi \neq \emptyset.$$

  From the above arguments, we have $\Gamma_\varphi \subset P_1 \cup P_2$, where Γ_φ is connected and $P_1 \cup P_2$ is disconnected which is a contradiction.

- If $\varphi(I) \subset I$, then $a < p$, for all $p \in \varphi(a)$ (otherwise, $a \in \varphi(a)$ and a is a fixed point) and $q < b$, for all $q \in \varphi(b)$ (otherwise, $b \in \varphi(b)$ and b is a fixed point).

  Then
  $$a < p \Rightarrow (a,p) \in P_1 \cap \Gamma_\varphi \Rightarrow P_1 \cap \Gamma_\varphi \neq \emptyset$$
  and
  $$q < b \Rightarrow (b,q) \in P_2 \cap \Gamma_\varphi \Rightarrow P_2 \cap \Gamma_\varphi \neq \emptyset.$$

  From the above arguments, we have $\Gamma_\varphi \subset P_1 \cup P_2$, where Γ_φ is connected and $P_1 \cup P_2$ is disconnected which is again a contradiction.

□

The following slight generalization of a one-dimensional version of the Brouwer theorem is well-known, because it can be easily deduced from the evident intermediate value property.

Corollary 1. *If a single-valued map $f : I \to I$, where $I \subset \mathbb{L}$ is a closed interval of a linear continuum $\mathbb{L}$, has a connected graph, then f has a fixed point in I.*

Example 4. The function $f : [-1,1] \to [-1,1]$ defined by

$$f(x) := \begin{cases} \sin\frac{1}{x}, & \text{for } x \in [-1,1] \setminus \{0\}, \\ 1, & \text{for } x = 0. \end{cases}$$

is not continuous, but has a connected graph. It admits, in fact, infinitely many fixed points in $[-1,1]$ (see Fig. 4).

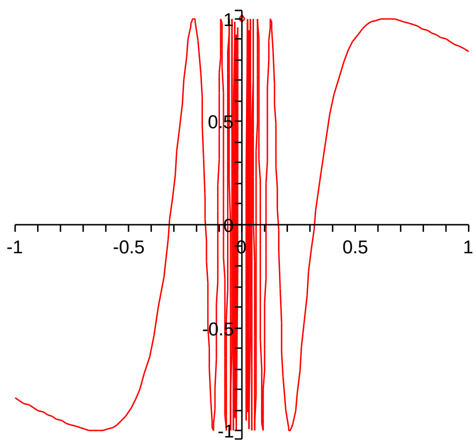

Figure 4: *Function f from Example 4.*

Lemma 2. *Let $\varphi : \mathbb{L} \multimap \mathbb{L}$ have a connected graph Γ_φ. If φ has an n-orbit, for some $n \in \mathbb{N}$, then it also has a fixed point.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be an n-orbit of φ. Assume that Fix $\varphi := \{x \in \mathbb{L} : x \in \varphi(x)\} = \emptyset$. Denote $a := \min\{x_i : i = 1, \ldots n\}$ and $b := \max\{x_i : i = 1, \ldots n\}$. There exist $k, l \in \{1, 2, \ldots n\}$ such that $x_k \in \varphi(a)$ and $x_l \in \varphi(b)$. Then $x_k > a$ and $x_l < b$ (otherwise, φ has a fixed point x_k or x_l). Hence,

$$(a, x_k) \in P_1 := \{(x, y) \in \Gamma_\varphi : x < y\} \quad \text{and} \quad (b, x_l) \in P_2 := \{(x, y) \in \Gamma_\varphi : y < x\}.$$

Since the sets P_1 and P_2 are nonempty disjoint open sets in Γ_φ and $\Gamma_\varphi = P_1 \cup P_2$ (we suppose Fix $\varphi = \emptyset$), we obtain a contradiction with the connectedness of Γ_φ. $\square$

The class of maps with a connected graph is rather large. In particular, it trivially contains maps determined by connectivity relations, and since in $\mathbb{R}$ upper and lower semicontinuous maps with closed connected values are determined by connectivity G_δ-relations (see [ASS] or [AFP]), they also have connected graph.

Since the maps satisfying the assumptions of Theorem 1 possess a fixed point, so obviously do their iterates whose graph is not necessarily connected like e.g. the a map $\varphi : [0,1] \multimap [\frac{1}{4}, \frac{3}{4}]$, where

$$\varphi(x) := \begin{cases} \{\frac{1}{4}, \frac{3}{4}\}, & \text{for } x \in [0,1)\ , \\ \\ \left[\frac{1}{4}, \frac{3}{4}\right], & \text{for } x = 1, \end{cases}$$

because $\varphi^2 : [0,1] \multimap [\frac{1}{4}, \frac{3}{4}]$, where $\varphi^2(x) = \{\frac{1}{4}, \frac{3}{4}\}$, for $x \in [0,1]$.

On the other hand, if φ has still connected values (i.e. if it is determined by a connectivity relation; cf. Lemma 1), then all the iterates $\varphi^n, n \in \mathbb{N}$, of φ have the same property. Indeed. It directly follows from the definition of a connectivity relation that, for any (possibly degenerate) interval $I \subset \mathbb{L}$, $\varphi(I)$ is connected, i.e. an interval. Thus, $\varphi^2(I) = \varphi(\varphi(I))$ must be also connected, i.e. φ^2 is determined by a connectivity relation and, in particular, it has a connected graph. By induction, we get that it holds for all the iterates, as claimed.

Moreover, there exist maps with a disconnected graph or disconnected values whose some iterate determines a connectivity relation like the mapping $\varphi : [0,1] \multimap [\frac{1}{4}, \frac{3}{4}]$, where

$$\varphi(x) := \begin{cases} \frac{3}{4}, & \text{for } x \in \left[0, \frac{1}{2}\right) \setminus \{\frac{1}{4}\}, \\ \left[\frac{1}{4}, \frac{3}{4}\right], & \text{for } x = \frac{1}{4}, \\ \{\frac{1}{4}, \frac{3}{4}\}, & \text{for } x = \frac{1}{2}, \\ \left[\frac{1}{4}, \frac{3}{4}\right], & \text{for } x = \frac{3}{4}, \\ \frac{1}{4}, & \text{for } x \in \left(\frac{1}{2}, 1\right] \setminus \{\frac{3}{4}\}, \end{cases}$$

because $\varphi^2 : [0,1] \multimap [\frac{1}{4}, \frac{3}{4}]$, $\varphi^2(x) = [\frac{1}{4}, \frac{3}{4}]$ (see Fig. 5).

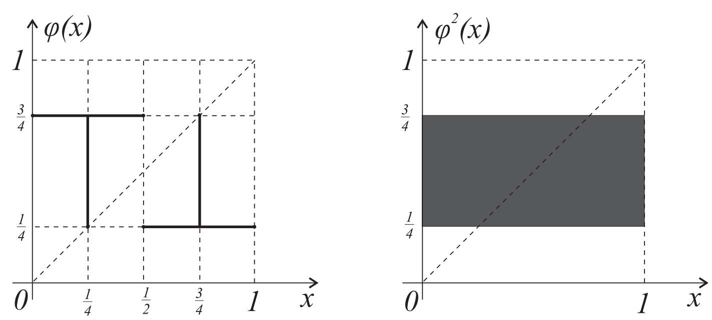

Figure 5: *Maps φ and φ^2.*

If, in Theorem 1, $\varphi = \xi^n$, for some $n \in \mathbb{N}$, then a natural question therefore arises whether or not mapping ξ itself admits a fixed point. As a partial answer, we can give the two following corollaries.

Corollary 2. *Let $\varphi : \mathbb{L} \multimap \mathbb{L}$ be a multivalued mapping with a connected graph. Assume that, for some $n \in \mathbb{N}$, the n-th iterate φ^n of φ has also a connected graph and that there exists a closed* *interval $I \subset \mathbb{L}$ of a linear continuum $\mathbb{L}$ such that either $I \subset \varphi^n(I)$ or $\varphi^n(I) \subset I$. Then φ has a fixed point.*

Proof. According to Theorem 1, φ^n has a fixed point in I. If it is not at the same time a fixed point of φ, then a nontrivial k-orbit of φ occurs, for some $k|n$. By means of Lemma 2, φ must have a fixed point. □

Corollary 3. *Let $\varphi : \mathbb{L} \multimap \mathbb{L}$ be a multivalued mapping with a connected graph and connected values (i.e. let φ determine a connectivity relation; cf. Lemma 1). Assume that, for the n-th iterate φ^n, $n \in \mathbb{N}$, of φ there exists a closed interval $I \subset \mathbb{L}$ of a linear continuum $\mathbb{L}$ such that either $I \subset \varphi^n(I)$ or $\varphi^n(I) \subset I$. Then φ has a fixed point.*

Proof. Since φ^n has, by the above arguments, a connected graph, an application of Corollary 2 completes the proof. □

The following example demonstrates that the graph connectedness in Corollaries 2 and 3 cannot be avoided.

Example 5. The mapping $\varphi : [0,1] \multimap [0,1]$ with closed connected values (observe that $\varphi([0,1]) = [0,1]$, see Fig. 6), where

$$\varphi(x) := \begin{cases} \left[\frac{1}{2}, 1\right], & \text{for } x = 0, \\ -x+1, & \text{for } x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ 0, & \text{for } x = \frac{1}{2}, \\ \left[0, \frac{1}{2}\right], & \text{for } x = 1, \end{cases}$$

has the second iterate $\varphi^2 : [0,1] \multimap [0,1]$, where

$$\varphi^2(x) = \begin{cases} \left[0, \frac{1}{2}\right], & \text{for } x = 0, \\ x, & \text{for } x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ \left[\frac{1}{2}, 1\right], & \text{for } x \in \{\frac{1}{2}, 1\}, \end{cases}$$

which is an M-mapping (see Fig. 6), but despite the fact that the set of fixed points of φ^2 is the whole interval $[0,1]$, φ itself is fixed point free.

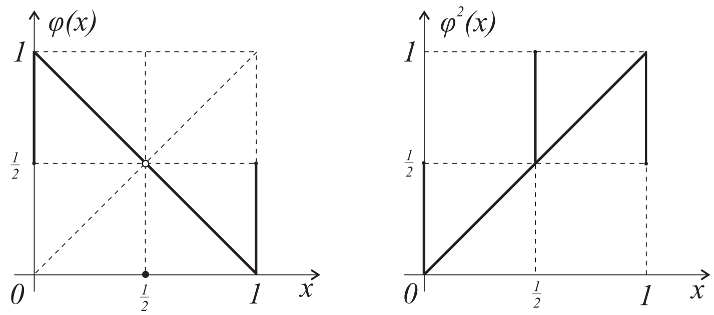

Figure 6: *Maps φ and φ^2 from Example 5.*

As a simple example of an application of Corollary 3, let us consider a continuous (single-valued) function $f : (0,\infty) \to (0,\infty)$, where $f(x) := \frac{1}{x}$, whose second iterate $f^2 : (0,\infty) \to (0,\infty)$ is $f^2(x) = x$. One can readily check that, for $I = [\frac{1}{4}, \frac{1}{2}]$, we have $f([\frac{1}{4}, \frac{1}{2}]) = [2, 4]$, i.e. $f(I) \not\subset I$ and $I \not\subset f(I)$, but $f^2([\frac{1}{4}, \frac{1}{2}]) = [\frac{1}{4}, \frac{1}{2}]$. Thus, according to Corollary 3, f has a fixed point. Observe that the only fixed point of f, $x = 1 \notin [\frac{1}{4}, \frac{1}{2}]$. On the other hand, e.g. for the interval $[\frac{1}{2}, 2]$, we already have $f([\frac{1}{2}, 2]) = [\frac{1}{2}, 2]$, and it is sufficient to apply Theorem 1, according to which f has a fixed point in $[\frac{1}{2}, 2]$.

For M-maps, Theorem 1 can be improved in the form of the following lemma.

Lemma 3 (cf. [AP], Lemma 2.2)**.** *Let $\varphi : \mathbb{L} \multimap \mathbb{L}$ be an M-map. Assume that $I_k \subset \mathbb{L}, k = 0, 1, \dots, n-1$, are closed intervals such that $I_{k+1} \subset \varphi(I_k)$, for $k = 0, 1, \dots, n-1$, and $I_n = I_0$, which we write as $I_0 \to I_1 \to \cdots \to I_n = I_0$. Then the n-th iterate φ^n of φ (i.e. the n-fold composition of φ with itself) has a fixed point x_0 (i.e. $x_0 \in \varphi^n(x_0)$) with $x_{k+1} \in \varphi(x_k)$, $x_n = x_0$, where $x_k \in I_k$, for $k = 0, 1, \dots, n-1$.*

We will finally show how Lemma 3 can be employed for restricting the problem of coexistence of periodic orbits from noncompact linearly ordered spaces to closed intervals.

Theorem 2. *Let an M-mapping $\varphi : \mathbb{L} \multimap \mathbb{L}$ have an n-orbit $\{x_1, \dots, x_n\}$, and let*

$$a := \min\{x_1, \dots, x_n\}, \qquad b := \max\{x_1, \dots, x_n\}.$$

Then there exist a closed inteval I, $[a, b] \subset I \subset \mathbb{L}$, and an M-mapping $\hat{\varphi} : I \multimap I$ such that $\hat{\varphi}(x) = \varphi(x)$ for every $x \in (a, b)$ and, for every $k \in \mathbb{N}$, the existence of a k-orbit of $\hat{\varphi}$ implies the existence of a k-orbit of φ.

Proof. If $\varphi([a, b]) = [a, b]$, it suffices to put $\hat{\varphi} = \varphi$ and $I = [a, b]$. On the contrary, let $[d, c] := \varphi([a, b])$. Now, the proof splits into the following cases.

I. $\varphi((b, c]) \not\subset [d, c]$. Setting $s := \inf\{x \in (b, c] : \varphi(x) \not\subset [d, c]\}$, then due to the upper semicontinuity of φ just one of the following possibilities occurs:

1. $c \in \varphi(s)$
   If $\varphi([d,a]) \subset [d,c]$, we put $I = [d,c]$ and define

$$\hat{\varphi}(x) = \begin{cases} \varphi(x), & \text{for every } x \in [d,s), \\ \varphi(x) \cap [d,c], & \text{for } x = s, \\ c, & \text{for every } x \in (s,c]. \end{cases}$$

   The points forming a k-orbit, $k \in \mathbb{N} \setminus \{1\}$, of $\hat{\varphi}$ form the same orbit of φ, because the point c can only form a 1-orbit of $\hat{\varphi}$ in addition to φ. On the other hand, the existence of 1-orbit of φ on $[a,b]$ follows from Theorem 1.
   If $\varphi([d,a]) \not\subset [d,c]$, then we consider $t := \sup\{x \in [d,a) : \varphi(x) \not\subset [d,c]\}$.
   If $c \in \varphi(t)$, we put $I = [d,c]$ and define

$$\hat{\varphi}(x) = \begin{cases} c, & \text{for every } x \in [d,t) \cup (s,c] \\ \varphi(x) \cap [d,c], & \text{for } x = t, s, \\ \varphi(x), & \text{for every } x \in (t,s). \end{cases}$$

   If $c \notin \varphi(t)$, then $d \in \varphi(t)$. Indeed, supposing $d \notin \varphi(t)$, either (if $t < a$) the upper semicontinuity of φ leads to a contradiction with the definition of t or (if $t = a$) we obtain a contradiction with the fact that $\varphi(a)$ is a connected interval and $\varphi(a) \cap [a,b] \neq \emptyset$. We put $I = [d,c]$ and define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d,t), \\ \varphi(x) \cap [d,c], & \text{for } x = t, s, \\ \varphi(x), & \text{for every } x \in (t,s), \\ c, & \text{for every } x \in (s,c]. \end{cases}$$

   The points forming a k-orbit, $k \in \mathbb{N} \setminus \{1\}$, of $\hat{\varphi}$ form the same orbit of φ, because the points c or d can only form a 1-orbit of $\hat{\varphi}$ in addition to φ. Again, Theorem 1 implies the existence of a fixed point on $[a,b]$.

2. $c \notin \varphi(s)$ and $d \in \varphi(s)$.
   Setting $e := \min\{y \in \varphi(x) : x \in [b,c]\}$ and $r := \min\{x \in [b,c] : e \in \varphi(x)\}$, there are the following possibilities depending on function values of φ on $[e,a]$:

   A) $\varphi([e,a]) \subset [e,c]$.
      If $\varphi((s,c]) \leq c$ (i.e., $y \leq c$, for every $y \in \varphi(x)$, where $x \in (s,c]$), it suffices to put $I = [e,c]$ and $\hat{\varphi} = \varphi$. Otherwise, setting $q := \inf\{x \in (s,c] : \varphi(x) > c\}$, we put $I = [e,c]$ and define

$$\hat{\varphi}(x) = \begin{cases} \varphi(x), & \text{for every } x \in [e,q), \\ \varphi(x) \cap [e,c], & \text{for } x = q, \\ c, & \text{for every } x \in (q,c]. \end{cases}$$

B) $\varphi([e,a]) \not\subset [e,c]$.
We consider $u := \sup\{x \in [e,a) : \varphi(x) \not\subset [e,c]\}$, and put $I = [e,c]$. The definition of $\hat{\varphi}$ depends on the relation of e and $\varphi(u)$, and on the relation of $\varphi((s,r))$ and c. If $e \in \varphi(u)$ and $\varphi((s,r)) \le c$, then we define

$$\hat{\varphi}(x) = \begin{cases} e, & \text{for every } x \in [e,u) \cup (r,c], \\ \varphi(x) \cap [e,c], & \text{for } x = u, r, \\ \varphi(x), & \text{for every } x \in (u,r). \end{cases}$$

If $e \in \varphi(u)$ and $m := \inf\{x \in (s,r) : \exists y \in \varphi(x), y > c\} \in (s,r)$, then we define

$$\hat{\varphi}(x) = \begin{cases} e, & \text{for every } x \in [e,u), \\ \varphi(x) \cap [e,c], & \text{for } x = u, m, \\ \varphi(x), & \text{for every } x \in (u,m), \\ c, & \text{for every } x \in (m,c], \end{cases}$$

and the same arguments as those at the end of part I., 1. conclude this case.
If $e \notin \varphi(u)$, then $c \in \varphi(u)$. If, moreover, $\varphi((s,r]) \le c$, we define

$$\hat{\varphi}(x) = \begin{cases} c, & \text{for every } x \in [e,u), \\ \varphi(x) \cap [e,c], & \text{for } x = u, r, \\ \varphi(x), & \text{for every } x \in (u,r), \\ e, & \text{for every } x \in (r,c]. \end{cases}$$

The points forming a k-orbit of mapping $\hat{\varphi}$, for $k \in \mathbb{N} \setminus \{1,2\}$, form the same k-orbit of φ, because the points c, e can only form a 2-orbit $\{e,c\}$ of $\hat{\varphi}$ in addition to φ. The existence of a 2-orbit of φ is also guaranteed, because it holds

$$[e,a] \to [b,c] \to [e,a].$$

Finally, if $e \notin \varphi(u)$, $c \in \varphi(u)$ and $m \in (s,r)$, we define

$$\hat{\varphi}(x) = \begin{cases} c, & \text{for every } x \in [e,u) \cup (m,c], \\ \varphi(x) \cap [e,c], & \text{for } x = u, m, \\ \varphi(x), & \text{for every } x \in (u,m). \end{cases}$$

II. $\varphi((b,c]) \subset [d,c]$.
We will discuss functional values of φ on $[d,a]$.

1. $\varphi([d,a]) \subset [d,c]$.
   It suffices to put $I = [d,c]$ and $\hat{\varphi} = \varphi$.
2. $\varphi([d,a]) \not\subset [d,c]$.
   We consider $v := \sup\{x \in [d,a) : \varphi(x) \not\subset [d,c]\}$.

If $d \in \varphi(v)$, we put $I = [d, c]$ and define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d, v), \\ \varphi(x) \cap [d, c], & \text{for } x = v, \\ \varphi(x), & \text{for every } x \in (v, c]. \end{cases}$$

If $d \notin \varphi(v)$, we set $f := \max\{y \in \varphi(x) : x \in [d, a]\}$ and $p := \max\{x \in [d, a] : f \in \varphi(x)\}$.

There are two possibilities w.r.t. functional values of φ on $(c, f]$:

A) $\varphi((c, f]) \subset [d, f]$.

If $\varphi([d, v)) \geq d$, it suffices to put $I = [d, f]$ and $\hat{\varphi} = \varphi$. Otherwise, setting $n := \sup\{x \in [d, v) : \varphi(x) < d\}$, we put $I = [d, f]$ and define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d, n), \\ \varphi(x) \cap [d, f], & \text{for } x = n, \\ \varphi(x), & \text{for every } x \in (n, f]. \end{cases}$$

B) $\varphi((c, f]) \not\subset [d, f]$.

We consider $w := \inf\{x \in (c, f] : \varphi(x) \not\subset [d, f]\}$, and put $I = [d, f]$. The definition of $\hat{\varphi}$ depends on the relation of f and $\varphi(w)$ and on the relation of $\varphi((p, v))$ and d.

If $f \in \varphi(w)$ and $\varphi((p, v)) \geq d$, then we define

$$\hat{\varphi}(x) = \begin{cases} f, & \text{for every } x \in [d, p) \cup (w, f], \\ \varphi(x) \cap [d, f], & \text{for } x = p, w, \\ \varphi(x), & \text{for every } x \in (p, w). \end{cases}$$

If $f \in \varphi(w)$ and $n := \sup\{x \in (p, v) : \exists y \in \varphi(x), y < d\} \in (p, v)$, then we define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d, n), \\ \varphi(x) \cap [d, f], & \text{for } x = n, w, \\ \varphi(x), & \text{for every } x \in (n, w), \\ f, & \text{for every } x \in (w, f]. \end{cases}$$

We can use the same ideas as in the previous cases to conclude this situation.

If $f \notin \varphi(w)$, then $d \in \varphi(w)$. If, moreover, $\varphi((p, v)) \geq d$, we define

$$\hat{\varphi}(x) = \begin{cases} f, & \text{for every } x \in [d, p), \\ \varphi(x) \cap [d, f], & \text{for } x = p, w, \\ \varphi(x), & \text{for every } x \in (p, w), \\ d, & \text{for every } x \in (w, f]. \end{cases}$$

and the analogous arguments as before conclude this case, jointly with the fact that by Lemma 3 φ has a 2-orbit, because

$$[d, a] \longrightarrow [b, f] \longrightarrow [d, a].$$

Finally, if $f \notin \varphi(w)$, $d \in \varphi(w)$ and $n \in (p, v)$, we define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d, n) \cup (w, f], \\ \varphi(x) \cap [d, f], & \text{for } x = n, w, \\ \varphi(x), & \text{for every } x \in (n, w). \end{cases}$$

□

4. Concluding remarks

If Lemma 3 could be generalized for maps determined by connectivity G_δ-relations on linear continua, then a Sharkovskii-type theorem might be formulated for these maps by means of the appropriately modified statements like Theorem 2. On the real line, this was already done in [ASS].

The Sharkovskii-type theorems establish an order relationship among the periods that the mapping can possess by means of a new (Sharkovskii's) ordering of positive integers. We already pointed out that this Sharkovskii phenomenon is in principle one-dimensional. Nevertheless, there exists a two-dimensional analogy in the sense that an order relationship can be replaced by forcing relations on braid types (see [Ha], [M1], [M2]). More precisely, one braid type is larger than the second if whenever a homeomorphism has a periodic orbit of the first type, then it also has a periodic orbit of the second type. The theory of braid types on surface dynamics was developed by several authors, but the standard reference for us is here the paper [Bo] by P. Boyland.

Higher than two-dimensional analogies of Sharkovskii's theorem require special structure of maps. For triangular maps, it was achieved (in a single-valued case) by P. Kloeden [Kl] and further extended (in a multivalued case) in [AFP], [AP], [APS]. Since we were able to do it in [APS] on a Cartesian product of linear continua $\mathbb{L}_1 \times \cdots \times \mathbb{L}_N$, where $\mathbb{L}_N$ was only a closed interval, a natural question arises whether Theorem 2 can be extended to triangular M-maps on $\mathbb{L}_1 \times \cdots \times \mathbb{L}_N$, where $\mathbb{L}_N$ is not necessarily a closed interval.

A combination of Theorem 1 and Corollary 1 in [A3] leads directly to the following random fixed point theorem (for definitions and more details, see [A3]).

Theorem 3. *Let $\Phi : \Omega \times \mathbb{L} \multimap \mathbb{L}$ be a random operator, where Ω is a complete measurable space and $\mathbb{L}$ is a complete separable metric linear continuum. Assume that, for each $\omega \in \Omega$, there exists a closed interval $I_\omega \subset \mathbb{L}$ such that $\Phi(\omega, \cdot) : I_\omega \multimap \mathbb{L}$ has a connected graph and either $I_\omega \subset \Phi(\omega, I_\omega)$ or $I_\omega \supset \Phi(\omega, I_\omega)$. Then Φ has a random fixed point, i.e. a measurable function $x : \omega \to \mathbb{L}$ such that $x(\omega) \in \Phi(\omega, x(\omega))$, for a.a. $\omega \in \Omega$.*

For more sophisticated fixed point theorems, where closed subsets are covered by their images or just intersect their images, see e.g. [A1] and the references therein.

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