CUBO A Mathematical Journal
Vol.11, Nº01, (123–143). March 2009

Regular quaternionic functions and conformal mappings

Alessandro Perotti [1]
Department of Mathematics,
University of Trento, Via Sommarive, 14,
I–38050 Povo Trento, Italy.
email: `perotti@science.unitn.it`

ABSTRACT

In this paper we study the action of conformal mappings of the quaternionic space on a class of regular functions of one quaternionic variable. We consider functions in the kernel of the Cauchy-Riemann operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3},$$

a variant of the Cauchy–Fueter operator. This choice is motivated by the strict relation existing between this type of regularity and holomorphicity w.r.t. the whole class of complex structures on $\mathbb{H}$. For every imaginary unit $p \in \mathbb{S}^2$, let J_p be the corresponding complex structure on $\mathbb{H}$. Let $Hol_p(\Omega, \mathbb{H})$ be the space of holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$, where L_p is defined by left multiplication by p. Every element of $Hol_p(\Omega, \mathbb{H})$ is regular, but there exist regular functions that are not holomorphic for any p. These properties can be recognized by computing the *energy quadric* of a function. We show that the energy quadric is invariant w.r.t. three–dimensional rotations of $\mathbb{H}$. As an application, we obtain that every rotation of the space $\mathbb{H}$ can be generated by biregular rotations, invertible regular functions with regular inverse. Moreover, we prove that the energy quadric of a regular function can always be diagonalized by means of a three–dimensional rotation.

[1] Work partially supported by MIUR (PRIN Project "Proprietà geometriche delle varietà reali e complesse") and GNSAGA of INdAM

RESUMEN

En este artículo estudiamos la acción de aplicaciones conforme del espacio de cuaterniones sobre la clase de funciones regulares de una variable cuaternionica. Nosotros consideramos funciones en el kernel del operador de Cauchy–Riemann

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3},$$

una variante del operador de Cauchy–Fueter. Esta elección es motivada por la relación estricta existente entre este tipo de regularidad y holomorficidad w.r.t. de la clase entera de estructuras complejas sobre $\mathbb{H}$. Para todo imaginario unitario $p \in \mathbb{S}^2$, sea J_p la correspodiente estructura compleja sobre $\mathbb{H}$. Sea $Hol_p(\Omega, \mathbb{H})$ el espacio de aplicaciones holomórficas de (Ω, J_p) a $(\mathbb{H}, L_p)$, donde L_p es definido por multiplicación a la izquierda por p. Todo elemento de $Hol_p(\Omega, \mathbb{H})$ es regular, pero existen funciones regulares que no son holomórficas para cualquer p. Estas propiedades pueden ser reconocidas mediante el cálculo de la *energía cuadrica* de una función. Nosotros mostramos que la energía cuadrica es invariante w.r.t. por rotaciones tres–dimensionales de H. Como aplicación, obtenemos que toda rotación del espacio $\mathbb{H}$ puede ser generada por rotaciones bi regulares, funciones regulares invertibles con inversa regular. Además mostramos que la energía cuadrica de una función regular siempre puede ser diagonalizada por una rotación tres–dimensional.

Key words and phrases: *quaternionic regular functions, hyperholomorphic functions, conformal mappings, Möbius transformations.*

Math. Subj. Class.: *Primary 30G35; Secondary 30A30*

1 Introduction.

The aim of this paper is to analyze the action of the conformal group of the one–point compactification $\mathbb{H}^*$ of $\mathbb{H}$ on a class of regular functions of one quaternionic variable.

Let Ω be a smooth bounded domain in $\mathbb{C}^2$. Let $\mathbb{H}$ be the space of real quaternions $q = x_0 + ix_1 + jx_2 + kx_3$, where i, j, k denote the basic quaternions. We identify $\mathbb{H}$ with $\mathbb{C}^2$ by means of the mapping that associates the quaternion $q = z_1 + z_2 j$ with the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$. We consider the class $\mathcal{R}(\Omega)$ of *left–regular* (also called *hyperholomorphic*) functions $f : \Omega \to \mathbb{H}$ in the kernel of the Cauchy–Riemann operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}.$$

This differential operator is a variant of the original Cauchy–Riemann–Fueter operator (cf. for example [19] and [5, 5])

$$\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

Hyperholomorphic functions have been studied by many authors (see for instance [1, 7, 11, 12, 14, 17, 18]). Many of their properties can be easily deduced from known properties satisfied by Fueter–regular functions, since a function f is regular on Ω if and only if $f \circ \gamma$ is Fueter–regular on $\gamma(\Omega) = \gamma^{-1}(\Omega)$, where γ is the reflection of $\mathbb{C}^2$ defined by $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$. However, regular functions in the space $\mathcal{R}(\Omega)$ have some characteristics that are more intimately related to the theory of holomorphic functions of two complex variables. In particular, the space $\mathcal{R}(\Omega)$ contains the spaces of holomorphic maps with respect to any constant complex structure. This is no longer true if we adopt the original definition of Fueter regularity (see Section 2 for more details).

Let J_1, J_2 be the complex structures on the tangent bundle $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on the cotangent bundle $T^*\mathbb{H} \simeq \mathbb{H}$ and set $J_3^* = J_1^* J_2^*$. For every complex structure $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ (p a imaginary unit in the unit sphere $\mathbb{S}^2$), let

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right)$$

be the Cauchy–Riemann operator with respect to the structure J_p. Let us define $Hol_p(\Omega, \mathbf{H}) = \operatorname{Ker} \overline{\partial}_p$, the space of holomorphic maps from (Ω, J_p) to $(\mathbf{H}, L_p)$, where L_p is the complex structure defined by left multiplication by p. Then every element of $Hol_p(\Omega, \mathbf{H})$ is regular. These subspaces do not fill the whole space of regular functions (cf. [13]). This result is a consequence of a criterion of J_p–holomorphicity, based on the concept of *energy quadric* of a regular function (cf. Section 3.2 for exact definitions).

In Section 4 we come to conformal transformations. >From a theorem of Liouville, the general conformal mapping of $\mathbb{H}^*$ is the composition of a sequence of translations, dilations, rotations and inversions. It can be written as a quaternionic *Möbius transformation*, i.e. a fractional linear map of the form

$$L_A(q) = (aq + b)(cq + d)^{-1},$$

with $A \in GL(2, \mathbb{H})$. For properties of these maps, see for example [2], [5]§6.2, [11] and [19] and the references cited in those papers.

Given a function $f \in C^1(\Omega)$ and a conformal transformation L_A, let f^A be the function

$$f^A(q) = \frac{(c\gamma(q) + d)^{-1}}{|c\gamma(q) + d|^2} f(L'_{\gamma(A)}(q)),$$

where $L'_{\gamma(A)}(q) = \gamma \circ L_A \circ \gamma(q)$. In Theorem 3, we prove that f is regular on Ω if and only if f^A is regular on $\Omega' = (L'_{\gamma(A)})^{-1}(\Omega)$. Moreover, $(f^A)^B = f^{AB}$ for every $A, B \in GL(2, \mathbb{H})$. The first property can be deduced from Theorem 6 of Sudbery [19] using the reflection γ.

We are interested also in the action of conformal mappings on the energy quadric and on the holomorphicity properties of the maps. For a general conformal transformation L_A, the energy and, *a fortiori*, the energy quadric of a regular function is not conserved. However, we show that three–dimensional rotations of $\mathbb{H}$ (those which fix the real numbers) conserve the energy quadric (for translations this it is a trivial fact).

Let $a \in \mathbb{H}$, $a \neq 0$. Let $rot_a(q) = aqa^{-1}$ be the three–dimensional rotation of $\mathbb{H}$ defined by a. In Theorem 4, we prove that the function

$$f^a = rot_{\gamma(a)} \circ f \circ rot_a$$

is regular on $\Omega^a = rot_a^{-1}(\Omega)$ if and only if f is regular on Ω. Moreover, the energy density of f^a is $\mathcal{E}(f^a) = \mathcal{E}(f) \circ rot_a$ and the matrix function $M(f)$ (for f regular $M(f)$ is the energy quadric, cf. Section 3) transforms in the following way

$$M(f^a) = Q_a(M(f) \circ rot_a)Q_a^T,$$

where $Q_a \in SO(3)$ is the orthogonal matrix associated to the rotation $rot_{\gamma(a)}$ of the space $\mathbb{R}^3 = \langle i, j, k \rangle$.

This formula has many consequences. It allows to obtain (Corollary 3) that f^a is J_p–holomorphic if and only if f is $J_{p'}$–holomorphic, with $p' = rot_{\gamma(a)}^{-1}(p)$. Moreover, we get (Corollary 4) that the energy quadric of a regular function can always be diagonalized by means of a three–dimensional rotation. Finally, we obtain a biregularity result about rotations (Proposition 2 and Corollary 5). We prove that every three-dimensional rotation is the composition of (at most) two three-dimensional biregular rotations, and that every four-dimensional rotation is the composition of two biregular rotations.

2 Notations and definitions

2.1 Fueter regular functions

We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is *(left) regular* (or *hyperholomorphic*) on Ω if

$$\mathcal{D}f = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

We will denote by $\mathcal{R}(\Omega)$ the space of regular functions on Ω.

With respect to this definition of regularity, the space $\mathcal{R}(\Omega)$ contains the identity mapping and every holomorphic mapping (f_1, f_2) on Ω (with respect to the standard complex structure) defines a regular function $f = f_1 + f_2 j$. We recall some properties of regular functions, for which we refer to the papers of Sudbery[19], Shapiro and Vasilevski[17] and Nōno[12]:

1. The complex components are both holomorphic or both non–holomorphic.

2. Every regular function is harmonic.
3. If Ω is pseudoconvex, every complex harmonic function is the complex component of a regular function on Ω.
4. The space $\mathcal{R}(\Omega)$ of regular functions on Ω is a *right* $\mathbb{H}$–module with integral representation formulas.
5. f is regular $\Leftrightarrow$ $\dfrac{\partial f_1}{\partial \bar{z}_1} = \dfrac{\partial \overline{f_2}}{\partial z_2}, \quad \dfrac{\partial f_1}{\partial \bar{z}_2} = -\dfrac{\partial \overline{f_2}}{\partial z_1}.$

We note that a function $f = f_1 + f_2 j$ is regular on Ω if and only if its Jacobian matrix has the form

$$J(f) = \left(\frac{\partial(f_1, f_2, \bar{f}_1, \bar{f}_2)}{\partial(z_1, z_2, \bar{z}_1, \bar{z}_2)}\right) = \left(\begin{array}{cc|cc} a_1 & -\bar{b}_2 & -\bar{c}_2 & -c_1 \\ a_2 & \bar{b}_1 & \bar{c}_1 & -c_2 \\ \hline -c_2 & -\bar{c}_1 & \bar{a}_1 & -b_2 \\ c_1 & -\bar{c}_2 & \bar{a}_2 & b_1 \end{array}\right)$$

at every $z \in \Omega$, where $a = \left(\dfrac{\partial f_1}{\partial z_1}, \dfrac{\partial f_2}{\partial z_1}\right)$, $b = \left(\dfrac{\partial \bar{f}_2}{\partial \bar{z}_2}, -\dfrac{\partial \bar{f}_1}{\partial \bar{z}_2}\right)$, $c = \left(\dfrac{\partial \bar{f}_2}{\partial z_1}, -\dfrac{\partial \bar{f}_1}{\partial z_1}\right) = -\left(\dfrac{\partial f_1}{\partial \bar{z}_2}, \dfrac{\partial f_2}{\partial \bar{z}_2}\right)$. We shall call a matrix of this form a *regular matrix*. Note that a regular matrix can have rank $0, 2, 3$ or 4 but not rank 1.

A definition equivalent to regularity has been given by Joyce[6] in the setting of hypercomplex manifolds. Joyce introduced the module of *q–holomorphic* functions on a hypercomplex manifold.

A hypercomplex structure on the manifold $\mathbb{H}$ is given by the complex structures J_1, J_2 on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on $T^*\mathbb{H} \simeq \mathbb{H}$. In complex coordinates

$$\begin{cases} J_1^* dz_1 = i\, dz_1, & J_1^* dz_2 = i\, dz_2 \\ J_2^* dz_1 = -d\bar{z}_2, & J_2^* dz_2 = d\bar{z}_1 \\ J_3^* dz_1 = i\, d\bar{z}_2, & J_3^* dz_2 = -i\, d\bar{z}_1 \end{cases}$$

where we make the choice $J_3^* = J_1^* J_2^*$, which is equivalent to $J_3 = -J_1 J_2$. In real coordinates, the action of the structures is the following

$$\begin{cases} J_1 dx_0 = -dx_1, & J_1 dx_2 = -dx_3, \\ J_2 dx_0 = -dx_2, & J_2 dx_1 = dx_3, \\ J_3 dx_0 = dx_3, & J_3 dx_1 = dx_2. \end{cases}$$

A function f is regular if and only if f is *q–holomorphic*, i.e.

$$df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0.$$

In complex components $f = f_1 + f_2 j$, we can rewrite the equations of regularity as

$$\overline{\partial} f_1 = J_2^*(\partial \overline{f}_2).$$

The original definition of regularity given by Fueter (cf. [19] or [5]) differs from that adopted here by a real coordinate reflection. Let γ be the transformation of $\mathbb{C}^2$ defined by $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$. Then a C^1 function f is regular on the domain Ω if and only if $f \circ \gamma$ is Fueter–regular on $\gamma(\Omega) = \gamma^{-1}(\Omega)$, i.e. it satisfies the differential equation

$$\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right)(f \circ \gamma) = 0 \quad \text{on } \gamma^{-1}(\Omega).$$

2.2 Biregular functions

A quaternionic function $f \in C^1(\Omega)$ is called *biregular* if f is invertible and f, f^{-1} are regular. If this property holds locally, f is called *locally biregular*. These functions were studied in [8], [9] and [15].

The class $\mathcal{BR}(\Omega)$ of biregular functions is closed with respect to right multiplication by a non-zero quaternion, but it is not closed with respect to composition or sum: even if $f + g$ is invertible and $f, g \in \mathcal{BR}(\Omega)$, the sum can be not biregular.

2.2.0.1 Examples

1. Every biholomorphic map (f_1, f_2) on Ω defines a biregular function $f = f_1 + f_2 j$.

2. The identity function is biregular on $\mathbb{H}$. More generally, the affine functions $f(q) = qa + b$, $a \in \mathbb{H}^*$, $b \in \mathbb{H}$, are biregular on $\mathbb{H}$.

3. $f = \bar{z}_1 + \bar{z}_2 j \in \mathcal{R}(\mathbb{H})$, $f^{-1} = f \in \mathcal{BR}(\mathbb{H})$.

4. The function $f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$ is regular, but the inverse map

$$f^{-1} = \frac{1}{3}\left(z_1 + z_2 + \bar{z}_1 - 2\bar{z}_2 + (z_1 + z_2 - 2\bar{z}_1 + \bar{z}_2)j\right)$$

is not regular. Note that in this case the Jacobian determinant is negative. This cannot happen for a biregular function (cf. [15]).

2.3 Holomorphic functions w.r.t. a complex structure J_p

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the orthogonal complex structure on $\mathbb{H}$ defined by a unit imaginary quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere $\mathbb{S}^2 = \{p \in \mathbb{H} \mid p^2 = -1\}$. In particular, J_1 is the standard complex structure of $\mathbb{C}^2 \simeq \mathbb{H}$.

Let $\mathbb{C}_p = \langle 1, p\rangle$ be the complex plane spanned by 1 and p and let L_p be the complex structure defined on $T^*\mathbb{C}_p \simeq \mathbb{C}_p$ by left multiplication by p. If $f = f^0 + if^1 : \Omega \to \mathbb{C}$ is a J_p*–holomorphic* function, i.e. $df^0 = J_p^*(df^1)$ or, equivalently, $df + iJ_p^*(df) = 0$, then f defines a regular function $\tilde{f} = f^0 + pf^1$ on Ω. We can identify $\tilde{f}$ with a holomorphic function

$$\tilde{f} : (\Omega, J_p) \to (\mathbb{C}_p, L_p).$$

We have $L_p = J_{\gamma(p)}$, where $\gamma(p) = p_1 i + p_2 j - p_3 k$. More generally, we can consider the space of holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$

$$Hol_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \text{ of class } C^1 \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = \operatorname{Ker} \overline{\partial}_p$$

where $\overline{\partial}_p$ is the Cauchy–Riemann operator with respect to the structure J_p

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right).$$

These functions will be called J_p*–holomorphic maps* on Ω.

For any positive orthonormal basis $\{1, p, q, pq\}$ of $\mathbb{H}$ $(p, q \in \mathbb{S}^2)$, let $f = f_1 + f_2 q$ be the decomposition of f with respect to the orthogonal sum

$$\mathbb{H} = \mathbb{C}_p \oplus (\mathbb{C}_p)q.$$

Let $f_1 = f^0 + pf^1$, $f_2 = f^2 + pf^3$, with f^0, f^1, f^2, f^3 the real components of f w.r.t. the basis $\{1, p, q, pq\}$. Then the equations of regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J_q^*(\partial_p \overline{f}_2),$$

where $\overline{f}_2 = f^2 - pf^3$ and $\partial_p = \frac{1}{2}\left(d - pJ_p^* \circ d\right)$. Therefore every $f \in Hol_p(\Omega, \mathbb{H})$ is a regular function on Ω.

Remark 1. *1. The identity map belongs to the space* $Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$ *but not to* $Hol_k(\Omega, \mathbb{H})$.

2. For every $p \in \mathbb{S}^2$, $Hol_{-p}(\Omega, \mathbb{H}) = Hol_p(\Omega, \mathbb{H})$.

3. Every $\mathbb{C}_p$*–valued regular function is a* J_p*–holomorphic function.*

4. If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_q(\Omega, \mathbb{H})$, *with* $p \neq \pm q$, *then* $f \in Hol_r(\Omega, \mathbb{H})$ *for every* $r = \frac{\alpha p + \beta q}{\|\alpha p + \beta q\|}$ $(\alpha, \beta \in \mathbb{R})$ *in the circle of* $\mathbb{S}^2$ *generated by* p *and* q.

If the almost complex structure J_p is not constant, i.e. not compatible with the hyperkähler structure of $\mathbb{H}$, we get a similar result. Note that in this case the structure is not necessarily integrable. Let $f \in C^1(\Omega)$ and assume that $p = p(z) \in \mathbb{S}^2$ varies continuously with z in Ω. We will say that p is f*-equivariant* if $f(z) = f(z')$ implies $p(z) = p(z')$ $(z, z' \in \Omega)$. This property allows to define $p^* : f(\Omega) \to \mathbb{S}^2$ such that $p^* \circ f = p$ on Ω. In [15], the following result was proved.

Proposition 1. *If* $f \in C^1(\Omega)$ *satisfies the equation*

$$\overline{\partial}_{p(z)} f = \frac{1}{2}\left[df(z) + p(z)J_{p(z)}^* \circ df(z)\right] = 0 \tag{1}$$

at every $z \in \Omega$, *then* f *is a regular function on* Ω. *If, moreover, the structure* p *is* f*-equivariant and* p^* *admits a continuous extension to an open set* $U \supseteq f(\Omega)$, *then* f *is a (pseudo)holomorphic map from* (Ω, J_p) *to* (U, L_{p^*}).

Example 1. *$f(z)=\bar z_1+z_2^2+\bar z_2 j$ is regular on $\mathbb{H}$. On $\Omega=\mathbb{H}\setminus\{z_2=0\}$ f is holomorphic w.r.t. the almost complex structure J_p, where*

$$p(z)=\frac{1}{\sqrt{|z_2|^2+|z_2|^4}}\left(|z_2|^2 i+(\operatorname{Im} z_2)j-(\operatorname{Re} z_2)k\right).$$

Note that $p(z)$ can not be continued to $\mathbb{H}$ as a continuous map. Also the inverse map $f^{-1}(z)=\bar z_1-z_2^2+\bar z_2 j$ is regular on $\mathbb{H}$. Then f is biregular on $\mathbb{H}$. But f is also (pseudo)biholomorphic on Ω: $f(\Omega)=\Omega$ and $f^{-1}:(\Omega,J_{p'})\to(\mathbb{H},L_{p'\circ f})$ is holomorphic, where

$$p'(z)=\frac{1}{\sqrt{|z_2|^2+|z_2|^4}}\left(|z_2|^2 i-(\operatorname{Im} z_2)j+(\operatorname{Re} z_2)k\right).$$

Note that $L_{p^*}=L_{p\circ f^{-1}}=J_{p'}$ at $f(z)$ and $L_{p'\circ f}=J_p$ at $z\in\Omega$.

3 A criterion for holomorphicity

3.1 Energy and regularity

In [13] it was proved that on every domain Ω there exist regular functions that are not J_p-holomorphic for any p. A similar result was obtained by Chen and Li[3] for the larger class of *q-maps* between hyperkähler manifolds.

The criterion for holomorphicity is based on an energy-minimizing property of holomorphic maps.

The *energy density* (w.r.t. the euclidean metric) of a function $f:\Omega\to\mathbb{H}$, of class $C^1(\Omega)$, is given by

$$\mathcal{E}(f)=\tfrac{1}{2}\|df\|^2=\tfrac{1}{2}\operatorname{tr}(J(f)\overline{J(f)}^T).$$

After integration on Ω, we get the *energy* of $f\in C^1(\overline{\Omega})$:

$$\mathcal{E}_\Omega(f)=\tfrac{1}{2}\int_\Omega \mathcal{E}(f)dV.$$

Using ideas from [10] and [3], it was proved in [13] that a regular function $f\in C^1(\overline{\Omega})$ minimizes energy in the homotopy class constituted by maps u with $u_{|\partial\Omega}=f_{|\partial\Omega}$ which are homotopic to f relative to $\partial\Omega$:

Now we introduce the Lichnerowicz invariants. Let $A(f)=(a_{\alpha\beta})$ be the 3×3 matrix with entries the real functions $a_{\alpha\beta}=-\langle J_\alpha,f^*L_{i_\beta}\rangle$, where $(i_1,i_2,i_3)=(i,j,k)$. For $f\in C^1(\overline{\Omega})$, we set

$$A_\Omega(f)=\int_\Omega A(f)\,dV \quad\text{and}\quad M_\Omega(f)=\tfrac{1}{2}\left((\operatorname{tr} A_\Omega(f))I_3-A_\Omega(f)\right),$$

where I_3 denotes the identity matrix.

We recall the criterion for regularity and holomorphicity proved in [13].

Theorem 1. *1. $M_\Omega(f)$ is a relative homotopy invariant of f.*

2. f is regular on Ω if and only if $\mathcal{E}_\Omega(f) = \operatorname{tr} M_\Omega(f)$.

3. If $f \in \mathcal{R}(\Omega)$, then $M_\Omega(f)$ is symmetric and positive semidefinite.

4. If $f \in \mathcal{R}(\Omega)$, then f belongs to some space $Hol_p(\Omega, \mathbb{H})$ (for a constant structure J_p) if and only if $\det M_\Omega(f) = 0$.

5. $f \in Hol_p(\Omega, \mathbb{H})$ if and only if $X_p = (p_1, p_2, p_3)$ is a unit vector in the kernel of $M_\Omega(f)$.

>From the criterion it can be seen that almost all regular functions are not holomorphic with respect to any constant complex structure J_p.

Example 2. *$f = \bar{z}_1 + z_2 + \bar{z}_2 j$ is J_p-holomorphic, with $p = \frac{1}{\sqrt{5}}(i - 2k)$, since on the unit ball B (with normalized unit volume)*

$$\mathcal{E}_B(f) = 3 \quad \textit{and} \quad M_B(f) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix}.$$

Example 3. *$f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$ is regular, but not holomorphic:*

$$\mathcal{E}_B(f) = 6 \quad \textit{and} \quad M_B(f) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Example 4. *$f = \bar{z}_1 + \bar{z}_2 j$ is regular and has matrix*

$$M_B(f) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

of rank one. This means that $f \in Hol_j(\mathbb{H}, \mathbb{H}) \cap Hol_k(\mathbb{H}, \mathbb{H})$.

Example 5. *The identity mapping belongs to the space*

$$Hol_i(\mathbb{H}, \mathbb{H}) \cap Hol_j(\mathbb{H}, \mathbb{H}) = \bigcap_{p \in <i,j>} Hol_p(\mathbb{H}, \mathbb{H}).$$

Example 6 (Nonlinear case)**.** *$f = |z_1|^2 - |z_2|^2 + \bar{z}_1 \bar{z}_2 j$ has energy $\mathcal{E}_B(f) = 2$ on the unit ball. The matrix $M_B(f)$ is*

$$M_B(f) = \begin{bmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Therefore f is regular but not holomorphic w.r.t. any constant complex structure J_p.

3.2 The energy quadric

In [15], a pointwise version of the criterion for holomorphicity was established.

Theorem 2. *Let Ω be connected and $f \in C^1(\Omega)$. Consider the matrix of real functions on Ω*

$$M(f) = \tfrac{1}{2}\left((\operatorname{tr} A(f))I_3 - A(f)\right).$$

1. *f is regular on Ω if and only if $\mathcal{E}(f) = \operatorname{tr} M(f)$ at every point $z \in \Omega$.*
2. *If $f \in \mathcal{R}(\Omega)$, then $M(f)$ is symmetric and positive semidefinite.*
3. *If $f \in \mathcal{R}(\Omega)$, then $\det M(f) = 0$ on Ω if and only if there exists an open, dense subset $\Omega' \subseteq \Omega$ on which f satisfies equation (1) for some function $p(z) : \Omega' \to \mathbb{S}^2$. Moreover, if $\det M(f) = 0$ and $p(z)$ is f-equivariant, $p^* \circ f = p$ and p^* extends continuously to an open set $U \supseteq f(\Omega)$, then f is a (pseudo)holomorphic map from (Ω', J_p) to (U, L_{p^*}).*

Let

$$a = \left(\frac{\partial f_1}{\partial z_1}, \frac{\partial f_2}{\partial z_1}\right),\ b = \left(\frac{\partial \bar f_2}{\partial \bar z_2}, -\frac{\partial \bar f_1}{\partial \bar z_2}\right),\ c = \left(\frac{\partial \bar f_2}{\partial z_1}, -\frac{\partial \bar f_1}{\partial z_1}\right),\ d = -\left(\frac{\partial f_1}{\partial \bar z_2}, \frac{\partial f_2}{\partial \bar z_2}\right).$$

Then the energy density is given by $\mathcal{E}(f) = |a|^2+|b|^2+|c|^2+|d|^2$. A lengthy but straightforward computation gives the following expression for the matrix $M(f)$:

$$M(f) = \begin{bmatrix} |c|^2 + |d|^2 & \operatorname{Im}(\langle d, a\rangle - \langle c, b\rangle) & \operatorname{Re}(\langle d, a\rangle + \langle c, b\rangle) \\ \operatorname{Im}(\langle c, a\rangle - \langle d, b\rangle) & \frac{1}{2}|a-b|^2 + \frac{1}{2}|c-d|^2 & -\operatorname{Im}(\langle a, b\rangle + \langle c, d\rangle) \\ \operatorname{Re}(\langle c, a\rangle + \langle d, b\rangle) & -\operatorname{Im}(\langle a, b\rangle - \langle c, d\rangle) & \frac{1}{2}|a+b|^2 + \frac{1}{2}|c-d|^2 \end{bmatrix}.$$

Then $\mathcal{E}(f) = \operatorname{tr} M(f)$ if and only if $c = d$, i.e. f is regular. In this case the matrix $M(f)$ becomes

$$M(f) = \begin{bmatrix} 2|c|^2 & \operatorname{Im}\langle c, a-b\rangle & \operatorname{Re}\langle c, a+b\rangle \\ \operatorname{Im}\langle c, a-b\rangle & \frac{1}{2}|a-b|^2 & -\operatorname{Im}\langle a, b\rangle \\ \operatorname{Re}\langle c, a+b\rangle & -\operatorname{Im}\langle a, b\rangle & \frac{1}{2}|a+b|^2 \end{bmatrix}.$$

Definition 1. *For a regular function f on Ω, the family of positive semi-definite quadrics with matrices $\{M(f)(z) \mid z \in \Omega\}$ will be called the energy quadric of f.*

Remark 2. *If f is invertible, then every $p(z)$ is f-equivariant. If p is a constant complex structure, then p is f-equivariant for every f.*

Remark 3. *If f is (real) affine, $M(f)$ is a constant matrix. If f is not affine, $\det M(f) = 0$ on Ω does not imply that $\det M_\Omega(f) = 0$, but Theorems 1 and 2 imply that the converse is true.*

Example 7. *The function $f(z) = \bar{z}_1 + z_2^2 + \bar{z}_2 j$ is regular (also biregular, cf. Example 1) on $\mathbb{H}$. We have*

$$\mathcal{E}(f) = 2 + 4|z_2|^2, \quad M(f) = 2\begin{bmatrix} 1 & -\operatorname{Im} z_2 & \operatorname{Re} z_2 \\ -\operatorname{Im} z_2 & |z_2|^2 & 0 \\ \operatorname{Re} z_2 & 0 & |z_2|^2 \end{bmatrix}.$$

Then the energy quadric of f is singular on $\mathbb{H}$. On the domain $\Omega' = \mathbb{H}\setminus\{z_2 = 0\}$, where $M(f)$ has maximum rank 2, the kernel of $M(f)$ is spanned by the vector $X = (|z_2|^2, \operatorname{Im} z_2, -\operatorname{Re} z_2)$. Then f is J_p-holomorphic on Ω', with

$$p(z) = \frac{1}{\sqrt{|z_2|^2+|z_2|^4}}\left(|z_2|^2 i + (\operatorname{Im} z_2) j - (\operatorname{Re} z_2) k\right).$$

On the unit ball B, $\mathcal{E}_B(f) = \frac{10}{3}$ and the matrix

$$M_B(f) = \int_B M(f) dV = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

is non-singular. Therefore, f is not J_q-holomorphic for any constant complex structure J_q.

Example 8. *The function $f = |z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2 j$ introduced in Example 6 has energy density $3|z|^2$ and energy quadric with matrix*

$$M(f) = \begin{bmatrix} 2|z|^2 & 0 & 0 \\ 0 & \frac{1}{2}|z|^2 & 0 \\ 0 & 0 & \frac{1}{2}|z|^2 \end{bmatrix}.$$

Therefore f is regular but not holomorphic w.r.t. any almost complex structure J_p. Note that $\det M(f) = \frac{1}{2}|z|^6$ vanishes only at the origin.

In [15], it was shown that if $f \in \mathcal{BR}(\Omega)$ is a biregular function, then there exists an open, dense subset $\Omega' \subseteq \Omega$, and an almost complex structure $p(z)$ on Ω', such that

$$f : (\Omega', J_p) \to (f(\Omega'), L_{p^*})$$

is a holomorphic map, with holomorphic inverse $f^{-1} : (f(\Omega'), J_{p'}) \to (\Omega', L_{p'\circ f})$. Here $p = p_1 i + p_2 j + p_3 k : \Omega' \to \mathbb{S}^2$, $p^* = p \circ f^{-1}$ and $p' = p_1 i + p_2 j - p_3 k$. In particular, any such map f preserves orientation.

4 Regular functions and conformal mappings

In this section we are going to analyze the action of the conformal group of $\mathbb{H}$ on regular functions. Some of the results we describe can be deduced from [19] Theorem 6 using the reflection $\gamma(z_1, z_2) =$

$(z_1, \bar{z}_2)$ introduced in §2.1, but here we want to investigate also the action on the energy quadric and the holomorphicity properties of the maps.

We recall some definitions and properties of conformal and orientation preserving mappings of the one–point compactification $\widehat{\mathbb{H}}$ of $\mathbb{H}$, for which we refer to [2], [5]§6.2, [11] and [19] and to the references cited in those papers.

The *Dieudonné determinant* of a quaternionic matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the real non–negative number

$$\det_{\mathbb{H}}(A) = \sqrt{|a|^2|d|^2 + |b|^2|c|^2 - 2Re(c\bar{a}b\bar{d})}.$$

It satisfies Binet property $\det_{\mathbb{H}}(AB) = \det_{\mathbb{H}}(A)\det_{\mathbb{H}}(B)$ and a 2×2 matrix A is (left and right) invertible if and only if $\det_{\mathbb{H}} A \neq 0$. Then we can consider the general linear group

$$GL(2, \mathbb{H}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ quaternionic matrix of order 2} \mid \det_{\mathbb{H}} A \neq 0 \right\}.$$

A theorem of Liouville tells that the general conformal transformation of $\mathbb{H}^*$ is a quaternionic *Möbius transformation*, i.e. a fractional linear map of the form

$$L_A(q) = (aq + b)(cq + d)^{-1},$$

for $A \in GL(2, \mathbb{H})$. The matrix A is determined by L_A up to a real scalar multiple. For every pair of matrices $A, B \in GL(2, \mathbb{H})$, $L_A \circ L_B = L_{AB}$. We have also the alternative representation of conformal mappings

$$L'_A(q) = (qc + d)^{-1}(qa + b), \quad \det_{\mathbb{H}} \bar{A} \neq 0.$$

Theorem 3. *Given a function $f \in C^1(\Omega)$ and a conformal transformation $L_A(q) = (aq + b)(cq + d)^{-1}$, let f^A be the function*

$$f^A(q) = \frac{(c\gamma(q) + d)^{-1}}{|c\gamma(q) + d|^2} f(L'_{\gamma(A)}(q)),$$

where $\gamma(A) = \begin{bmatrix} \gamma(a) & \gamma(b) \\ \gamma(c) & \gamma(d) \end{bmatrix}$. Then f is regular on Ω if and only if f^A is regular on $\Omega' = (L'_{\gamma(A)})^{-1}(\Omega)$. Moreover, $(f^A)^B = f^{AB}$ for every $A, B \in GL(2, \mathbb{H})$.

Proof. We deduce the first statement from the result of Sudbery (cf. [19] Theorem 6), since $f \in \mathcal{R}(\Omega)$ iff $F = f \circ \gamma$ is Fueter–regular on $\gamma(\Omega)$. This last condition is equivalent to the Fueter–regularity of the transformed function

$$F^A(p) = \frac{(cp + d)^{-1}}{|cp + d|^2} F(L_A(p))$$

on $(L_A)^{-1}(\gamma(\Omega))$. Note that this function differs from the one given by Sudbery by a real constant factor. We then obtain that f is regular iff $F^A \circ \gamma$ is regular. We have

$$F^A \circ \gamma(q) = \frac{(c\gamma(q)+d)^{-1}}{|c\gamma(q)+d|^2} f \circ \gamma \circ L_A \circ \gamma(q) = f^A(q),$$

since $\gamma \circ L_A \circ \gamma(q) = L'_{\gamma(A)}(q)$. Now we come to the last statement of the theorem. Let $B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ and $C = AB = \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}$. Then

$$\begin{aligned} (f^A)^B(q) = {} & \frac{(c'\gamma(q)+d')^{-1}}{|c'\gamma(q)+d'|^2} f^A(L'_{\gamma(B)}(q)) \\ & \frac{(c'\gamma(q)+d')^{-1}}{|c'\gamma(q)+d'|^2} \frac{(c\gamma(L'_{\gamma(B)}(q))+d)^{-1}}{|c\gamma(L'_{\gamma(B)}(q))+d|^2} f((L'_{\gamma(A)} \circ L'_{\gamma(B)})(q)) \end{aligned}$$

Let $q' = \gamma(q)$. The last statement of the theorem follows from the equalities

$$L'_{\gamma(A)} \circ L'_{\gamma(B)} = (\gamma \circ L_A \circ \gamma) \circ (\gamma \circ L_B \circ \gamma) = \gamma \circ L_{AB} \circ \gamma = L'_{\gamma(AB)}$$

and

$$\begin{aligned} \overline{(c'q'+d')}\ \overline{(c\gamma(L'_{\gamma(B)}(q))+d)} = {} & (\overline{q'c'}+\overline{d'})\,((\overline{q'c'}+\overline{d'})^{-1}(\overline{q'a'}+\overline{b'})\bar{c}+\bar{d}) \\ & (\overline{q'a'}+\overline{b'})\bar{c}+(\overline{q'c'}+\overline{d'})\bar{d} \\ & = \overline{q'}(\overline{a'}\bar{c}+\overline{c'}\bar{d})+\overline{b'}\bar{c}+\overline{d'}\bar{d} \\ & \overline{c''q'+d''} \end{aligned}$$

□

Remark 4. *If t is a non–zero real number, $f^{tA} = t^{-3} f^A$. Then f^A depends only for a real scalar multiple on the matrix chosen to represent the conformal transformation L_A. We can also restrict the choice of the matrix to the subgroup $SL(2,\mathbb{H}) = \{A \in GL(2,\mathbb{H}) \mid \det_{\mathbb{H}}(A) = 1\}$. In this case, the same conformal transformation gives rise to two functions, f^A and $f^{-A} = -f^A$.*

Every conformal transformation is the composition of a sequence of translations, dilations, rotations and inversions. In order to illustrate the preceding theorem, we now apply it to these basic cases.

Example 9. *The inversion $q \mapsto q^{-1}$ corresponds to the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (up to a real scalar multiple) and transforms a regular $f \in \mathcal{R}(\Omega)$ into*

$$f^{inv}(q) = \frac{\gamma(q)^{-1}}{|q|^2} f(q^{-1}),$$

regular on $\Omega' = \{q \in \mathbb{H} \mid q^{-1} \in \Omega\}$.

Example 10. *In particular, the inverted function of the constant function $f = \frac{1}{2\pi^2}$ is the Cauchy–Fueter kernel for the module of regular functions*

$$G(q) = G(z_1 + z_2 j) = \frac{1}{2\pi^2} \frac{\bar{z}_1 - \bar{z}_2 j}{|z|^4}.$$

Example 11. *A translation $q \mapsto q + b$ corresponds to the matrix $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. The transformed function is*

$$f^A(q) = f(L'_{\gamma(A)}(q)) = f(q + \gamma(b)).$$

Example 12. *A dilation $q \mapsto aq$, $a \neq 0$ real, has matrix $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$. A function f transforms into*

$$f^A(q) = f(qa).$$

Example 13. *Given two unit quaternions $a, d \in \mathbb{H}$, the diagonal matrix $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ induces the four–dimensional rotation $q \mapsto aqd^{-1}$. Given a regular function f on Ω, the function*

$$f^A(q) = d^{-1} f(\gamma(d)^{-1} q \gamma(a))$$

is regular on $\Omega' = \gamma(d)\Omega\gamma(a)^{-1}$.

Example 14. *The quaternionic Cayley transformation $\psi(q) = (q+1)(1-q)^{-1}$ maps diffeomorphically the unit ball B to the right half–space $\mathbb{H}^+ = \{q \in \mathbb{H} \mid Re(q) > 0\}$ (see [2] for geometric properties of ψ). It transforms regular functions f on $\mathbb{H}^+$ into*

$$f^{\psi}(q) = 2^{3/2} \frac{(1-\gamma(q))^{-1}}{|1-\gamma(q)|^2} f(\psi(q)),$$

regular on B. The inverse mapping $\psi^{-1}(q) = (q-1)(1+q)^{-1}$ transforms $f \in \mathcal{R}(B)$ into

$$f^{\psi^{-1}}(q) = 2^{3/2} \frac{(1+\gamma(q))^{-1}}{|1+\gamma(q)|^2} f(\psi^{-1}(q)) \in \mathcal{R}(\mathbb{H}^+).$$

The factor $2^{3/2}$ in the formulas has been chosen to get $(f^{\psi})^{\psi^{-1}} = f$.

If we take the identity map, which is regular on $\mathbb{H}$, as f, from Theorem 3 we get the following:

Corollary 1. *For every conformal transformation $L_A(q) = (aq+b)(cq+d)^{-1}$, the function*

$$\frac{(c\gamma(q)+d)^{-1}}{|c\gamma(q)+d|^2} L'_{\gamma(A)}(q),$$

is regular on $\{q \in \mathbb{H} \mid c\gamma(q) + d \neq 0\}$.

4.1 The quadric energy of rotated regular functions

A unit quaternion d defines the *three–dimensional rotation* $q \mapsto rot_d(q) := dqd^{-1}$, which gives rise to the function (cf. Example 13)

$$f^A(q) = d^{-1} f(\gamma(d)^{-1} q \gamma(d)),$$

where A is the scalar matrix $A = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$. Taking $d = \gamma(a)^{-1}$ and multiplying by $\gamma(a)^{-1}$ on the right, we obtain the function $f^a = rot_{\gamma(a)} \circ f \circ rot_a$. From Theorem 3 we immediately get:

Corollary 2. *Let $f \in C^1(\Omega)$ and let $a \in \mathbb{H}$, $a \neq 0$. Let $rot_a(q) = aqa^{-1}$ be the three–dimensional rotation of $\mathbb{H}$ defined by a. Then the function*

$$f^a = rot_{\gamma(a)} \circ f \circ rot_a$$

is regular on $\Omega^a = rot_a^{-1}(\Omega) = a^{-1}\Omega a$ if and only if f is regular on Ω.

Remark 5. *The rotated function f^a has the following properties:*

1. $(f^a)^b = f^{ab}$ *and* $(f+g)^a = f^a + g^a$.
2. $(f^a)^{a^{-1}} = f$.
3. $f^{-a} = f^a$.
4. *If* $b \in \mathbb{H}$, *then* $(fb)^a = f^a \, rot_{\gamma(a)}(b)$.

Now we analyze the action of rotations on the energy quadric. We obtain in this way a new proof of the preceding result and we get new holomorphicity properties of rotated regular functions.

Theorem 4. *Let $f \in C^1(\Omega)$ and let $a \in \mathbb{H}$, $a \neq 0$. Let $f^a = rot_{\gamma(a)} \circ f \circ rot_a$. Then the energy density of f^a is $\mathcal{E}(f^a) = \mathcal{E}(f) \circ rot_a$ and the matrix function $M(f)$ defined in Section 3 transforms in the following way*

$$M(f^a) = Q_a (M(f) \circ rot_a) Q_a^T,$$

where Q_a is the orthogonal matrix in $SO(3)$ associated to the rotation $rot_{\gamma(a)}$ of the space $\langle i, j, k \rangle$.

Before coming to the theorem, we prove a simple result about holomorphicity of rotations.

Lemma 1. *For every $p \in \mathbb{S}^2$, the three-dimensional rotation $rot_a(q) = aqa^{-1}$ is a holomorphic map from $(\mathbb{H}, J_{\gamma(p)})$ to $(\mathbb{H}, L_{rot_a(p)})$.*

Proof. Let $\mathcal{B} = \{p, p', pp'\}$ be a positive orthonormal base of $\mathbb{R}^3 = \langle i, j, k \rangle$. Let $X_p = (p_1, p_2, p_3)$, $X_{p'} = (p'_1, p'_2, p'_3)$, $X_r = (r_1, r_2, r_3)$, with $r = pp' = r_1 i + r_2 j + r_3 k$. Given the transition matrix A

with columns $X_p, X_{p'}, X_r$, the coordinates x'_α $(\alpha = 1,2,3)$ of $q = x_0 + x_1 i + x_2 j + x_3 k$ w.r.t. $\mathcal{B}$ are given by the product $(x'_1\ x'_2\ x'_3)^T = A^T(x_1\ x_2\ x_3)^T$. Then

$$x'_1 = \sum_\alpha p_\alpha x_\alpha, \quad x'_2 = \sum_\alpha p'_\alpha x_\alpha, \quad x'_3 = \sum_\alpha r_\alpha x_\alpha.$$

>From this we get that the functions $g_1 = x_0 + x'_1 rot_a(p)$ and $g_2 = x'_2 + x'_3 rot_a(p)$ are holomorphic from $(\mathbb{H}, J_{\gamma(p)})$ to $(\mathbb{H}, L_{rot_a(p)})$, since

$$J_{\gamma(p)}(dx_0) = (p_1J_1 + p_2J_2 - p_3J_3)(dx_0) = -\sum_\alpha p_\alpha dx_\alpha = -dx'_1$$

and

$$\begin{aligned} J_{\gamma(p)}(dx'_2) &= \sum_\alpha p'_\alpha (p_1J_1 + p_2J_2 - p_3J_3)(dx_\alpha) \\ &= \sum_\alpha p_\alpha p'_\alpha dx_0 - (p_2p'_3 - p_3p'_2)dx_1 - (p_3p'_1 - p_1p'_3)dx_2 - (p_1p'_2 - p_2p'_1)dx_3 \\ &= -r_1dx_1 - r_2dx_2 - r_3dx_3 = -dx'_3. \end{aligned}$$

The lemma now follows from the equality

$$\begin{aligned} rot_a(q) &= a(x_0 + x'_1 p + x'_2 p' + x'_3 r)a^{-1} \\ &= (x_0 + x'_1 rot_a(p)) + (x'_2 + x'_3 rot_a(p)) rot_a(p') = g_1 + g_2\, rot_a(p') \end{aligned}$$

□

If in the preceding lemma p is replaced by $\gamma(p)$, we get that the map $rot_a(q)$ is holomorphic also from $(\mathbb{H}, J_p)$ to $(\mathbb{H}, L_{rot_a(\gamma(p))}) = (\mathbb{H}, J_{p'})$, where $p' = \gamma(rot_a(\gamma(p))) = \gamma(a)^{-1}p\gamma(a) = rot^{-1}_{\gamma(a)}(p)$. Replacing a with $\gamma(a)$ we also get that $rot_{\gamma(a)}$ is holomorphic from $(\mathbb{H}, L_{p'}) = (\mathbb{H}, J_{rot_a(\gamma(p))})$ to $(\mathbb{H}, L_{rot_{\gamma(a)}(p')}) = (\mathbb{H}, L_p)$. Then we can draw a commutative diagram with holomorphic vertical maps

$$\begin{tikzcd} (\mathbb{H}, J_{p'}) \arrow[r, "f"] & (\mathbb{H}, L_{p'}) \arrow[d, "rot_{\gamma(a)}"] \\ (\mathbb{H}, J_p) \arrow[u, "rot_a"] \arrow[r, "f^a"'] & (\mathbb{H}, L_p) \end{tikzcd} \tag{2}$$

Proof of Theorem 4. Let J be the real Jacobian matrix of $f \circ rot_a$. Then the real Jacobian matrix of f^a is the product Q_aJ. It follows that $\mathcal{E}(f^a) = \frac{1}{2}\operatorname{tr}(Q_aJJ^TQ_a^T) = \frac{1}{2}\operatorname{tr}(JJ^T) = \mathcal{E}(f \circ rot_a)$. A similar computation gives $\mathcal{E}(f \circ rot_a) = \mathcal{E}(f) \circ rot_a$.

For the second statement of the theorem, it is sufficient to prove the equality

$$A(f^a) = Q_a(A(f) \circ rot_a)Q_a^T, \tag{3}$$

for the matrix functions $A(f)$ and $A(f^a)$ defined in Section 3, since then the matrices $A(f^a)$ and $A(f) \circ rot_a$ have the same trace and therefore

$$\begin{aligned} Q_a(M(f) \circ rot_a)Q_a^T &= \frac{1}{2}\left(\operatorname{tr} A(f) \circ rot_a\right) I_3 - \frac{1}{2}A(f^a) \\ &= \frac{1}{2}\left(\operatorname{tr} A(f^a) I_3 - A(f^a)\right) = M(f^a). \end{aligned}$$

It remains to prove (3). Let $p = p_1 i + p_2 j + p_3 k \in \mathbb{S}^2$ and $p' = rot_{\gamma(a)}^{-1}(p)$. Let us define the *p–holomorphic energy* of f

$$\mathcal{I}_p(f) = \frac{1}{2}\|df + L_p \circ df \circ J_p\|^2 = \frac{1}{2}\|df + p\, df \circ J_p\|^2 = 2\|\overline{\partial}_p f\|^2.$$

Then we obtain, as in [3],

$$\mathcal{E}(f) + \langle J_p, f^* L_p\rangle = \frac{1}{4}\mathcal{I}_p(f).$$

If $X = (p_1, p_2, p_3)$, then

$$XA(f^a)X^T = \sum_{\alpha,\beta} p_\alpha p_\beta a_{\alpha\beta} = -\langle \sum_\alpha p_\alpha J_\alpha, (f^a)^* \sum_\beta p_\beta L_{i_\beta}\rangle$$

$$= -\langle J_p, (f^a)^* L_p\rangle = \mathcal{E}(f^a) - \frac{1}{4}\mathcal{I}_p(f^a).$$

Now let $X' = (p_1', p_2', p_3') = XQ_a$. A similar computation gives

$$XQ_a A(f \circ rot_a) Q_a^T X^T = X' A(f \circ rot_a) X'^T = \mathcal{E}(f) \circ rot_a - \frac{1}{4}\mathcal{I}_{p'}(f) \circ rot_a.$$

>From the first statement of the theorem and the arbitrariness of X, equation (3) is equivalent to the equality, for any $p \in \mathbb{S}^2$, of the holomorphic energies

$$\mathcal{I}_{p'}(f) \circ rot_a = \mathcal{I}_p(f^a). \tag{4}$$

>From Lemma 1 (cf. diagram (2)) and rotational invariance of the norm we get

$$\begin{aligned} 2\mathcal{I}_p(f^a) &= \|df^a + L_p \circ df^a \circ J_p\|^2 \\ &= \|rot_{\gamma(a)} \circ df \circ drot_a + L_p \circ rot_{\gamma(a)} \circ df \circ drot_a \circ J_p\|^2 \\ &= \|rot_{\gamma(a)} \circ df \circ drot_a + rot_{\gamma(a)} \circ L_{p'} \circ df \circ J_{p'} \circ drot_a\|^2 \\ &= \|df + L_{p'} \circ df \circ J_{p'}\|^2 \circ rot_a = 2\mathcal{I}_{p'}(f) \circ rot_a. \end{aligned}$$

Then the equality (4) is true and the theorem is proved. □

Corollary 3. *Let $f \in C^1(\Omega)$ and let $a \in \mathbb{H}$, $a \neq 0$. Let $f^a = rot_{\gamma(a)} \circ f \circ rot_a$. Let $Q_a \in SO(3)$ be associated to the rotation $rot_{\gamma(a)}$ of the space $\langle i, j, k\rangle$. Then*

1. *f is regular on Ω if and only if f^a is regular on $\Omega^a = rot_a^{-1}(\Omega) = a^{-1}\Omega a$.*

2. f^a is J_p–holomorphic if and only if f is $J_{p'}$–holomorphic, with $p' = rot^{-1}_{\gamma(a)}(p)$.

3. If $f \in C^1(\overline{\Omega})$, then (cf. Theorem 1)

$$M_{\Omega^a}(f^a) = Q_a M_\Omega(f) Q_a^T.$$

Proof. 1) From Theorem 4 we get that $\operatorname{tr} M(f^a) = \operatorname{tr} M(f) \circ rot_a$ and $\mathcal{E}(f^a) = \mathcal{E}(f) \circ rot_a$. The first statement follows from Theorem 2, which tells that f is regular iff $\mathcal{E}(f) = \operatorname{tr} M(f)$.

2) It is an immediate consequence of equality (4), since a function is J_p–holomorphic iff its p–holomorphic energy vanishes.

3) It follows easily by integration of $M(f^a)$ on Ω^a. □

Corollary 4. *For every $f \in \mathcal{R}(\Omega)$, there exists $a \in \mathbb{H}$, $a \neq 0$, such that the matrices $M(f^a)$ and $M_{\Omega^a}(f^a)$ are diagonal, with non–negative entries.*

Proof. It follows immediately from Theorems 4 and 2, since when f is regular $M(f)$ is symmetric and positive semidefinite. □

Remark 6. *For a general conformal transformation L_A, the energy and, a* fortiori, *the energy quadric of a regular function is not conserved. For example, the constant function* 1 *has zero energy, while $\mathcal{E}(2\pi^2 G) \neq 0$ and $1^{inv} = 2\pi^2 G$ (cf. Example 10).*

The same happens for J_p–holomorphicity. For example, the identity function is in the spaces $Hol_i(\mathbb{H})$ and $Hol_j(\mathbb{H})$, while

$$id^{inv}(q) = \frac{\gamma(q)^{-1} q^{-1}}{|q|^2} \in \mathcal{R}(\mathbb{H} \setminus \{0\})$$

is not holomorphic w.r.t. any structure J_p. This can be seen by computing the energy quadric $M(id^{inv})$. Since $\det M(id^{inv}) = 32/|q|^{30}$ *is always non–zero, it follows from Theorem 2 that id^{inv} is not J_p–holomorphic, for any p (even non–constant). The rank of id^{inv} is three, because its image is contained in the space $\langle 1, i, j \rangle$, and the function can not have rank less than three, otherwise its quadric energy would have zero determinant (cf. [15] Theorem 7).*

A simpler example is given again by the function 1^{inv}, since the energy quadric of the kernel G is $M(G) = 2/|q|^8 I_3$.

4.2 Biregular rotations

If in Theorem 4 and its corollaries we take as f the identity map we get the following:

Proposition 2. *For every $a \in \mathbb{H}$, $a \neq 0$, the three–dimensional rotation $rot_{\gamma(a)a}$ is a biregular function on $\mathbb{H}$, with energy quadric $M(rot_{\gamma(a)a})$ of rank 1. This means that $rot_{\gamma(a)a}$ is holomorphic w.r.t. a circle of structures $p \in \mathbb{S}^2$. More precisely, $rot_{\gamma(a)a} \in Hol_p(\mathbb{H})$ for every $p \in \langle rot_{\gamma(a)}(i), rot_{\gamma(a)}(j) \rangle \cap \mathbb{S}^2$.*

Proof. We have $rot_{\gamma(a)a} = id^a$ (cf. Theorem 4). Then

$$M(rot_{\gamma(a)a}) = Q_a M(id) Q_a^T = Q_a \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} Q_a^T$$

has rank 1. Its kernel gives the structures with respect to which the rotation is holomorphic. From Corollary 3(2), these structures are generated by $rot_{\gamma(a)}(i)$ and $rot_{\gamma(a)}(j)$, since $id \in Hol_i(\mathbb{H}) \cap Hol_j(\mathbb{H})$.

Biregularity follows from $(\gamma(a)a)^{-1} = a^{-1}\gamma(a^{-1})$, which implies the equality $(id^a)^{-1} = id^{\gamma(a^{-1})} \in \mathcal{R}(\mathbb{H})$. □

Remark 7. *Not every rotation is a regular function, since the quaternion $\gamma(a)a$ is a reduced quaternion, with fourth component zero. These quaternion numbers correspond to rotations of $\mathbb{R}^3 = \langle i, j, k\rangle$ with axis orthogonal to the k axis. However, every quaternion is the product of two reduced quaternions and the map $a \mapsto \gamma(a)a$ is surjective from $\mathbb{H}$ to the space $\mathbb{H}_r$ of reduced quaternions.*

The surjectivity of $a \mapsto \gamma(a)a$ can be seen explicitly, or can be deduced from a property of the regular function id^{inv} (cf. Remark 6). Its restriction to the unit sphere S^3 is the map $q \mapsto \gamma(\bar{q})\bar{q} \in S^3 \cap \mathbb{H}_r$. It is surjective since id^{inv} has rank three.

Corollary 5.
1. *The left–multiplication map $l_{a'}(q) = a'q$ is biregular for every reduced quaternion $a' = \gamma(a)a \neq 0$.*
2. *Every three-dimensional rotation is the composition of two three-dimensional biregular rotations.*
3. *Every four-dimensional rotation is the composition of two biregular rotations.*

Proof. 1) $l_{a'}(q) = \gamma(a)aq = rot_{\gamma(a)a}(q)(a^{-1}\gamma(a)^{-1})$ has the same regularity and holomorphicity properties of $rot_{\gamma(a)a}$, since $\mathcal{R}(\Omega)$ is a right $\mathbb{H}$–module for every Ω and $\overline{\partial}_p(fb) = (\overline{\partial}_p f)b$ for every f and every $b \in \mathbb{H}$.

2) It follows from what has been said in the above remark: if $c = a'b'$, with $a' = \gamma(a)a$, $b' = \gamma(b)b \in \mathbb{H}_r$, then $rot_c = rot_{a'} \circ rot_{b'} = rot_{\gamma(a)a} \circ rot_{\gamma(b)b}$.

3) A four–dimensional rotation $rot_{c,d}(q) = cqd^{-1}$, with $|cd^{-1}| = 1$, can be decomposed as

$$rot_{c,d}(q) = cqc^{-1}\,(cd^{-1}) = rot_c(q)\,(cd^{-1}) = (rot_{a'} \circ rot_{b'})(q)\,(cd^{-1}),$$

where $c = a'b'$ as before. Let $f(q) = rot_{a'}(q)\,(cd^{-1}) \in \mathcal{BR}(\mathbb{H})$. Then $rot_{c,d} = f \circ rot_{b'}$. □

The pair of biregular functions in the corollary can be chosen in the same space $Hol_p(\mathbb{H})$. This comes from Proposition 2, because the two great circles of complex structures in $\mathbb{S}^2$ coincide or intersect in two antipodal points defining a space $Hol_p(\mathbb{H})$. Note that this space is not closed under composition, unless $J_p = L_p$, which happens only when $p = \gamma(p)$ is a reduced quaternion.

Received: July 2008. Revised: August 2008.

References

[1] R. Abreu-Blaya, J. Bory-Reyes, M. Shapiro, *On the notion of the Bochner-Martinelli integral for domains with rectifiable boundary.* Complex Anal. Oper. Theory **1** (2007), no. 2, 143–168.

[2] C. Bisi, G. Gentili, *Möbius transformations and the Poincaré distance in the quaternionic setting*, 2008 (arXiv:0805.0357v2).

[3] J. Chen and J. Li, *Quaternionic maps between hyperkähler manifolds*, J. Differential Geom. **55** (2000), 355–384.

[4] K. Gürlebeck, K. Habetha and W. Sprössig, *Holomorphic functions in the plane and n-dimensional space.* Translated from the 2006 German original *Funktionentheorie in Ebene und Raum*, Birkhäuser Verlag, Basel, 2008.

[5] K. Gürlebeck and W. Sprössig, *Quaternionic Analysis and Elliptic Boundary Value Problems.* Birkhäuser, Basel, 1990.

[6] D. Joyce, *Hypercomplex algebraic geometry*, Quart. J. Math. Oxford **49** (1998), 129–162.

[7] V.V. Kravchenko and M.V. Shapiro, *Integral representations for spatial models of mathematical physics*, Harlow: Longman, 1996.

[8] W. Królikowski, *On Fueter–Hurwitz regular mappings*, Diss. Math. **353** (1996), 1–91.

[9] W. Królikowski and R.M. Porter, *Regular and biregular functions in the sense of Fueter—some problems*, Ann. Polon. Math. **59** (1994), 53–64.

[10] A. Lichnerowicz, *Applications harmoniques et variétés kähleriennes.* (French) 1968/1969 Symposia Mathematica, Vol. III (INDAM, Rome, 1968/69) pp. 341–402 Academic Press, London. Symp. Math. **III**, Bologna, 341–402, 1970.

[11] M. Naser, *Hyperholomorphe Funktionen*, Sib. Mat. Zh. **12**, 1327–1340 (Russian). English transl. in Sib. Math. J. **12**, (1971) 959–968.

[12] K. Nōno, *α-hyperholomorphic function theory*, Bull. Fukuoka Univ. Ed. III **35** (1985), 11–17.

[13] A. Perotti, *Holomorphic functions and regular quaternionic functions on the hyperkähler space* $\mathbb{H}$, Proceedings of the 5th ISAAC Congress, Catania 2005, World Scientific Publishing Co. (in press) (arXiv:0711.4440v1).

[14] A. Perotti, *Quaternionic regular functions and the $\overline{\partial}$-Neumann problem in* $\mathbb{C}^2$, Complex Variables and Elliptic Equations **52** No. 5 (2007), 439–453.

[15] A. Perotti, *Every biregular function is biholomorphic*, Advances in Applied Clifford Algebras, in press.

[16] J. Ryan, Clifford analysis, in *Lectures on Clifford (geometric) algebras and applications.* Edited by Ablamowicz and Sobczyk. Birkhäuser Boston, Inc., Boston, MA, 53–89, 2004.

[17] M.V. Shapiro and N.L. Vasilevski, *Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. I. ψ-hyperholomorphic function theory*, Complex Variables Theory Appl. **27** no.1 (1995), 17–46.

[18] M.V. Shapiro and N.L. Vasilevski, *Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. II: Algebras of singular integral operators and Riemann type boundary value problems*, Complex Variables Theory Appl. **27** no.1 (1995), 67–96.

[19] A. Sudbery, *Quaternionic analysis*, Mat. Proc. Camb. Phil. Soc. **85** (1979), 199–225.