CUBO A Mathematical Journal
Vol.11, N$^{\underline{o}}$01, (101–121). March 2009

Wrap groups of fiber bundles and their structure

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ABSTRACT

This article is devoted to the investigation of wrap groups of connected fiber bundles. These groups are constructed with mild conditions on fibers. Their examples are given. It is shown, that these groups exist and for differentiable fibers have the infinite dimensional Lie groups structure, that is, they are continuous or differentiable manifolds and the composition $(f, g) \mapsto f^{-1}g$ is continuous or differentiable depending on a class of smoothness of groups. Moreover, it is demonstrated that in the cases of real, complex, quaternion and octonion manifolds these groups have structures of real, complex, quaternion or octonion manifolds respectively. Nevertheless, it is proved that these groups does not necessarily satisfy the Campbell-Hausdorff formula even locally.

RESUMEN

Este artículo es dedicado a la investigación de grupos Wrap de fibrados conexos. Estos grupos son construidos con condiciones blandas sobre las fibras, ejemplos son dados. Es demostrado que estos grupos existen y para fibras diferenciables tienen una estructura de grupo de Lie infinito dimensional, es decir, son variedades continuas o diferenciables y la composición $(f, g) \mapsto f^{-1}g$ es continua o diferenciable dependiendo de la clase de suavidad de los grupos. Además es demostrado que en el caso de variedades real, compleja, cuaternion y octonion esos grupos tienen una estructura de variedad real, compleja, cuaternion o octonion respectivamente. También es probado que estos grupos no necesariamente satisfacen la fórmula de Campbell-Hausdorff incluso localmente.

1 Introduction.

Wrap groups of fiber bundles considered in this paper are constructed with the help of families of mappings from a fiber bundle with a marked point into another fiber bundle with a marked point over the fields **R**, **C**, **H** and the octonion algebra **O**. Conditions on fibers supplied with parallel transport structures are rather mild here. Therefore, they generalize geometric loop groups of circles, spheres and fibers with parallel transport structures over them. A loop interpretation is lost in their generalizations, so they are called here wrap groups. This paper continues previous works of the author on this theme, where generalized loop groups of manifolds over **R**, **C** and **H** were investigated, but neither for fibers nor over octonions [15, 23, 21, 22].

Loop groups of circles were first introduced by Lefshetz in 1930-th and then their construction was reconsidered by Milnor in 1950-th. Lefshetz has used the C^0-uniformity on families of continuous mappings, which led to the necessity of combining his construction with the structure of a free group with the help of words. Later on Milnor has used the Sobolev's H^1-uniformity, that permitted to introduce group structure more naturally [27]. Iterations of these constructions produce iterated loop groups of spheres. Then their constructions were generalized for fibers over circles and spheres with parallel transport structures over **R** or **C** [4].

Wrap groups of quaternion and octonion fibers as well as for wider classes of fibers over **R** or **C** are defined and investigated here for the first time.

Holomorphic functions of quaternion and octonion variables were investigated in [19, 20, 17]. There specific definition of super-differentiability was considered, because the quaternion skew field has the graded algebra structure. This definition of super-differentiability does not impose the condition of right or left super-linearity of a super-differential, since it leads to narrow class of functions. There are some articles on quaternion manifolds, but practically they undermine a complex manifold with additional quaternion structure of its tangent space (see, for example, [28, 39] and references therein). Therefore, quaternion manifolds as they are defined below were not considered earlier by others authors (see also [17]). Applications of quaternions in mathematics and physics can be found in [6, 9, 10, 14].

In this article wrap groups of different classes of smoothness are considered. Henceforth, we consider not only orientable manifolds M and N, but also nonorientable manifolds.

In particular, geometric loop groups have important applications in modern physical theories (see [11, 24] and references therein). Groups of loops are also intensively used in gauge theory. Wrap groups defined below with the help of families of mappings from a manifold M into another manifold N with a dimension $dim(M) > 1$ can be used in the membrane theory which is the generalization of the string (superstring) theory.

Section 2 is devoted to the definitions of topological and manifold structures of wrap groups. The existence of these groups is proved and that they are infinite dimensional Lie groups not satisfying even locally the Campbell-Hausdorff formula (see Theorems 3, 6, 12, Corollaries 5, 8, 9 and Examples 10). In the cases of complex, quaternion and octonion manifolds it is proved that they have structures of complex, quaternion and octonion manifolds respectively.

All main results of this paper are obtained for the first time.

2 Wrap groups of fibers.

To avoid misunderstandings we first give our definitions and notations.

1.1. Note. Denote by $\mathcal{A}_r$ the Cayley-Dickson algebra such that $\mathcal{A}_0 = \mathbf{R}$, $\mathcal{A}_1 = \mathbf{C}$, $\mathcal{A}_2 = \mathbf{H}$ is the quaternion skew field, $\mathcal{A}_3 = \mathbf{O}$ is the octonion algebra. Henceforth we consider only $0 \le r \le 3$.

1.2. Definition. A canonical closed subset Q of the Euclidean space $X = \mathbf{R^n}$ or of the standard separable Hilbert space $X = l_2(\mathbf{R})$ over $\mathbf{R}$ is called a quadrant if it can be given by the condition $Q := \{x \in X : q_j(x) \ge 0\}$, where $(q_j : j \in \Lambda_Q)$ are linearly independent elements of the topologically adjoint space X^*. Here $\Lambda_Q \subset \mathbf{N}$ (with $card(\Lambda_Q) = k \le n$ when $X = \mathbf{R^n}$) and k is called the index of Q. If $x \in Q$ and exactly j of the q_i's satisfy $q_i(x) = 0$ then x is called a corner of index j.

If X is an additive group and also left and right module over $\mathbf{H}$ or $\mathbf{O}$ with the corresponding associativity or alternativity respectively and distributivity laws then it is called the vector space over $\mathbf{H}$ or $\mathbf{O}$ correspondingly.

In particular $l_2(\mathcal{A}_r)$ consisting of all sequences $x = \{x_n \in \mathcal{A}_r : n \in \mathbf{N}\}$ with the finite norm $\|x\| < \infty$ and scalar product $(x, y) := \sum_{n=1}^{\infty} x_n y_n^*$ with $\|x\| := (x, x)^{1/2}$ is called the Hilbert space (of separable type) over $\mathcal{A}_r$, where z^* denotes the conjugated Cayley-Dickson number, $zz^* =: |z|^2$, $z \in \mathcal{A}_r$. Since the unitary space $X = \mathcal{A}_r^n$ or the separable Hilbert space $l_2(\mathcal{A}_r)$ over $\mathcal{A}_r$ while considered over the field $\mathbf{R}$ (real shadow) is isomorphic with $X_{\mathbf{R}} := \mathbf{R^{2^r n}}$ or $l_2(\mathbf{R})$, then the above definition also describes quadrants in $\mathcal{A}_r^n$ and $l_2(\mathcal{A}_r)$. In the latter case we also consider generalized quadrants as canonical closed subsets which can be given by $Q := \{x \in X_{\mathbf{R}} : q_j(x + a_j) \ge 0, a_j \in X_{\mathbf{R}}, j \in \Lambda_Q\}$, where $\Lambda_Q \subset \mathbf{N}$ ($card(\Lambda_Q) = k \in \mathbf{N}$ when $dim_{\mathbf{R}} X_{\mathbf{R}} < \infty$).

1.2.2. Definition. A differentiable mapping $f : U \to U'$ is called a diffeomorphism if

(i) f is bijective and there exist continuous mappings f' and $(f^{-1})'$, where U and U' are interiors of quadrants Q and Q' in X.

In the $\mathcal{A}_r$ case with $1 \le r \le 3$ we consider bounded generalized quadrants Q and Q' in $\mathcal{A}_r^n$ or $l_2(\mathcal{A}_r)$ such that they are domains with piecewise C^∞-boundaries. We impose additional conditions on the diffeomorphism f in the $1 \le r \le 3$ case:

(ii) $\bar{\partial} f = 0$ on U,

(iii) f and all its strong (Frechét) differentials (as multi-linear operators) are bounded on U, where ∂f and $\bar{\partial} f$ are differential $(1, 0)$ and $(0, 1)$ forms respectively, $d = \partial + \bar{\partial}$ is an exterior derivative, for $2 \le r \le 3$ ∂ corresponds to super-differentiation by z and $\tilde{\partial} = \bar{\partial}$ corresponds to super-differentiation by $\tilde{z} := z^*$, $z \in U$ (see [19, 20]).

The Cauchy-Riemann Condition (ii) means that f on U is the $\mathcal{A}_r$-holomorphic mapping.

1.2.3. Definition and notation. An $\mathcal{A}_r$-manifold M with corners is defined in the usual way: it is a metric separable space modelled on $X = \mathcal{A}_r^n$ or $X = l_2(\mathcal{A}_r)$ respectively and is supposed to be of class C^∞, $0 \le r \le 3$. Charts on M are denoted (U_l, u_l, Q_l), that is, $u_l : U_l \to u_l(U_l) \subset Q_l$ is a C^∞-diffeomorphism for each l, U_l is open in M, $u_l \circ {u_j}^{-1}$ is biholomorphic for $1 \le r \le 3$ from the domain $u_j(U_l \cap U_j) \ne \emptyset$ onto $u_l(U_l \cap U_j)$ (that is, $u_j \circ u_l^{-1}$ and $u_l \circ u_j^{-1}$ are holomorphic and bijective) and $u_l \circ u_j^{-1}$ satisfy conditions $(i - iii)$ from §1.2.2, $\bigcup_j U_j = M$.

A point $x \in M$ is called a corner of index j if there exists a chart (U, u, Q) of M with $x \in U$ and $u(x)$ is of index $ind_M(x) = j$ in $u(U) \subset Q$. A set of all corners of index $j \ge 1$ is called a border ∂M of M, x is called an inner point of M if $ind_M(x) = 0$, so $\partial M = \bigcup_{j \ge 1} \partial^j M$, where $\partial^j M := \{x \in M : ind_M(x) = j\}$.

For a real manifold with corners on the connecting mappings $u_l \circ u_j^{-1} \in C^\infty$ of real charts only Condition 1.2.2(i) is imposed.

1.2.4. Terminology. In an $\mathcal{A}_r$-manifold N there exists an Hermitian metric, which in each analytic system of coordinates is the following $\sum_{j,k=1}^n h_{j,k} dz_j d\bar{z}_k$, where $(h_{j,k})$ is a positive definite Hermitian matrix with coefficients of the class C^∞, $h_{j,k} = h_{j,k}(z) \in \mathcal{A}_r$, z are local coordinates in N.

As real manifolds we shall consider Riemann manifolds.

In accordance with the definition above for internal points of N it is supposed that they can belong only to interiors of charts, but for boundary points ∂N it may happen that $x \in \partial N$ belongs to boundaries of several charts. It is convenient to choose an atlas such that $ind(x)$ is the same for all charts containing this x.

1.3.1. Remark. If M is a metrizable space and $K = K_M$ is a closed subset in M of codimension $codim_{\mathbf{R}}\, N \ge 2$ such that $M \setminus K = M_1$ is a manifold with corners over $\mathcal{A}_r$, then we call M a pseudo-manifold over $\mathcal{A}_r$, where K_M is a critical subset.

Two pseudo-manifolds B and C are called diffeomorphic, if $B \setminus K_B$ is diffeomorphic with $C \setminus K_C$ as for manifolds with corners (see also [4, 26]).

Take on M a Borel σ-additive measure ν such that ν on $M \setminus K$ coincides with the Riemann volume element and $\nu(K) = 0$, since the real shadow of M_1 has it.

The uniform space $H^t_p(M_1, N)$ of all continuous piecewise H^t Sobolev mappings from M_1 into N is introduced in the standard way [21, 22], which induces $H^t_p(M, N)$ the uniform space of continuous piecewise H^t Sobolev mappings on M, since $\nu(K) = 0$, where $\mathbf{R} \ni t \ge [m/2] + 1$, m denotes the dimension of M over $\mathbf{R}$, $[k]$ denotes the integer part of $k \in \mathbf{R}$, $[k] \le k$. Then put $H^\infty_p(M, N) = \bigcap_{t > m} H^t_p(M, N)$ with the corresponding uniformity.

For manifolds over $\mathcal{A}_r$ with $1 \le r \le 3$ take as $H^t_p(M, N)$ the completion of the family of all continuous piecewise $\mathcal{A}_r$-holomorphic mappings from M into N relative to the H_p^t uniformity, where $[m/2]+1 \le t \le \infty$. Henceforth we consider pseudo-manifolds with connecting mappings of charts continuous in M and $H_p^{t'}$ in $M \setminus K_M$ for $0 \le r \le 3$, where $t' \ge t$.

1.3.2. Note. Since the octonion algebra $\mathbf{O}$ is non-associative, we consider a non-associative subgroup G of the family $Mat_q(\mathbf{O})$ of all square $q \times q$ matrices with entries in $\mathbf{O}$. More generally G is a group which has a H_p^t manifold structure over $\mathcal{A}_r$ and group's operations are H_p^t mappings. The G may be non-associative for $r=3$, but G is supposed to be alternative, that is, $(aa)b = a(ab)$ and $a(a^{-1}b) = b$ for each $a, b \in G$.

As a generalization of pseudo-manifolds there is used the following (over $\mathbf{R}$ and $\mathbf{C}$ see [4, 34]). Suppose that M is a Hausdorff topological space of covering dimension $dim\ M = m$ supplied with a family $\{h : U \to M\}$ of the so called plots h which are continuous maps satisfying conditions $(D1-D4)$:

$(D1)$ each plot has as a domain a convex subset U in $\mathcal{A}_r^n$, $n \in \mathbf{N}$;

$(D2)$ if $h : U \to M$ is a plot, V is a convex subset in $\mathcal{A}_r^l$ and $g : V \to U$ is an H_p^t mapping, then $h \circ g$ is also a plot, where $t \ge [m/2]+1$;

$(D3)$ every constant map from a convex set U in $\mathcal{A}_r^n$ into M is a plot;

$(D4)$ if U is a convex set in $\mathcal{A}_r^n$ and $\{U_j : j \in J\}$ is a covering of U by convex sets in $\mathcal{A}_r^n$, each U_j is open in U, $h : U \to M$ is such that each its restriction $h|_{U_j}$ is a plot, then h is a plot. Then M is called an H_p^t-differentiable space.

A mapping $f : M \to N$ between two H_p^t-differentiable spaces is called differentiable if it continuous and for each plot $h : U \to M$ the composition $f \circ h : U \to N$ is a plot of N. A topological group G is called an H_p^t-differentiable group if its group operations are H_p^t-differentiable mappings.

Let E, N, F be $H_p^{t'}$-pseudo-manifolds or $H_p^{t'}$-differentiable spaces over $\mathcal{A}_r$, let also G be an $H_p^{t'}$ group over $\mathcal{A}_r$, $t \le t' \le \infty$. A fiber bundle $E(N, F, G, \pi, \Psi)$ with a fiber space E, a base space N, a typical fiber F and a structural group G over $\mathcal{A}_r$, a projection $\pi : E \to N$ and an atlas Ψ is defined in the standard way [4, 26, 35] with the condition, that transition functions are of $H_p^{t'}$ class such that for $r=3$ a structure group may be non-associative, but alternative.

Local trivializations $\phi_j \circ \pi \circ \Psi_k^{-1} : V_k(E) \to V_j(N)$ induce the $H_p^{t'}$-uniformity in the family W of all principal $H_p^{t'}$-fiber bundles $E(N, G, \pi, \Psi)$, where $V_k(E) = \Psi_k(U_k(E)) \subset X^2(G)$, $V_j(N) = \phi_j(U_j(N)) \subset X(N)$, where $X(G)$ and $X(N)$ are $\mathcal{A}_r$-vector spaces on which G and N are modelled, $(U_k(E), \Psi_k)$ and $(U_j(N), \phi_j)$ are charts of atlases of E and N, $\Psi_k = \Psi_k^E$, $\phi_j = \phi_j^N$.

If $G = F$ and G acts on itself by left shifts, then a fiber bundle is called the principal fiber bundle and is denoted by $E(N, G, \pi, \Psi)$. As a particular case there may be $G = \mathcal{A}_r^*$, where $\mathcal{A}_r^*$ denotes the multiplicative group $\mathcal{A}_r \setminus \{0\}$. If $G = F = \{e\}$, then E reduces to N.

2. Definitions. Let M be a connected H_p^t-pseudo-manifold over $\mathcal{A}_r$, $0 \le r \le 3$ satisfying the following conditions:

(i) it is compact;

(ii) M is a union of two closed subsets over $\mathcal{A}_r$ A_1 and A_2, which are pseudo-manifolds and which are canonical closed subsets in M with $A_1 \cap A_2 = \partial A_1 \cap \partial A_2 =: A_3$ and a codimension over $\mathbf{R}$ of A_3 in M is $codim_{\mathbf{R}} A_3 = 1$, also A_3 is a pseudo-manifold;

(iii) a finite set of marked points $s_{0,1}, ..., s_{0,k}$ is in $\partial A_1 \cap \partial A_2$, moreover, ∂A_j are arcwise connected $j = 1, 2$;

(iv) $A_1 \setminus \partial A_1$ and $A_2 \setminus \partial A_2$ are H^t_p-diffeomorphic with $M \setminus [\{s_{0,1}, ..., s_{0,k}\} \cup (A_3 \setminus Int(\partial A_1 \cap \partial A_2))]$ by mappings $F_j(z)$, where $j = 1$ or $j = 2$, $\infty \geq t \geq [m/2]+1$, $m = dim_{\mathbf{R}} M$ such that $H^t \subset C^0$ due to the Sobolev embedding theorem [25], where the interior $Int(\partial A_1 \cap \partial A_2)$ is taken in $\partial A_1 \cup \partial A_2$.

Instead of (iv) we consider also the case

(iv') M, A_1 and A_2 are such that $(A_j \setminus \partial A_j) \cup \{s_{0,1}, ..., s_{0,k}\}$ are $C^0([0,1], H^t_p(A_j, A_j))$-retractable on $X_{0,q} \cap A_j$, where $X_{0,q}$ is a closed arcwise connected subset in M, $j = 1$ or $j = 2$, $s_{0,q} \in X_{0,q}$, $X_{0,q} \subset K_M$, $q = 1, ..., k$, $codim_{\mathbf{R}}\ K_M \geq 2$.

Let $\hat{M}$ be a compact connected H^t_p-pseudo-manifold which is a canonical closed subset in $\mathcal{A}^l_r$ with a boundary $\partial \hat{M}$ and marked points $\{\hat{s}_{0,q} \in \partial \hat{M} : q = 1, ..., 2k\}$ and an H^t_p-mapping $\Xi : \hat{M} \to M$ such that

(v) Ξ is surjective and bijective from $\hat{M} \setminus \partial \hat{M}$ onto $M \setminus \Xi(\partial \hat{M})$ open in M, $\Xi(\hat{s}_{0,q}) = \Xi(\hat{s}_{0,k+q}) s_{0,q}$ for each $q = 1, ..., k$, also $\partial M \subset \Xi(\partial \hat{M})$.

A parallel transport structure on a $H^{t'}_p$-differentiable principal G-bundle $E(N, G, \pi, \Psi)$ with arcwise connected E and G for H^t_p-pseudo-manifolds M and $\hat{M}$ as above over the same $\mathcal{A}_r$ with $t' \geq t+1$ assigns to each H^t_p mapping γ from M into N and points $u_1, ..., u_k \in E_{y_0}$, where y_0 is a marked point in N, $y_0 = \gamma(s_{0,q})$, $q = 1, ..., k$, a unique H^t_p mapping $\mathbf{P}_{\hat{\gamma},u} : \hat{M} \to E$ satisfying conditions $(P1 - P5)$:

$(P1)$ take $\hat{\gamma} : \hat{M} \to N$ such that $\hat{\gamma} = \gamma \circ \Xi$, then $\mathbf{P}_{\hat{\gamma},u}(\hat{s}_{0,q}) = u_q$ for each $q = 1, ..., k$ and $\pi \circ \mathbf{P}_{\hat{\gamma},u} = \hat{\gamma}$

$(P2)$ $\mathbf{P}_{\hat{\gamma},u}$ is the H^t_p-mapping by γ and u;

$(P3)$ for each $x \in \hat{M}$ and every $\phi \in DifH^t_p(\hat{M}, \{\hat{s}_{0,1}, ..., \hat{s}_{0,2k}\})$ there is the equality $\mathbf{P}_{\hat{\gamma},u}(\phi(x)) = \mathbf{P}_{\hat{\gamma}\circ\phi,u}(x)$, where $DifH^t_p(\hat{M}, \{\hat{s}_{0,1}, ..., \hat{s}_{0,2k}\})$ denotes the group of all H^t_p homeomorphisms of $\hat{M}$ preserving marked points $\phi(\hat{s}_{0,q}) = \hat{s}_{0,q}$ for each $q = 1, ..., 2k$;

$(P4)$ $\mathbf{P}_{\hat{\gamma},u}$ is G-equivariant, which means that $\mathbf{P}_{\hat{\gamma},uz}(x) = \mathbf{P}_{\hat{\gamma},u}(x)z$ for every $x \in \hat{M}$ and each $z \in G$;

$(P5)$ if U is an open neighborhood of $\hat{s}_{0,q}$ in $\hat{M}$ and $\hat{\gamma}_0, \hat{\gamma}_1 : U \to N$ are $H^{t'}_p$-mappings such that $\hat{\gamma}_0(\hat{s}_{0,q}) = \hat{\gamma}_1(\hat{s}_{0,q}) = v_q$ and tangent spaces, which are vector manifolds over $\mathcal{A}_r$, for γ_0 and γ_1 at v_q are the same, then the tangent spaces of $\mathbf{P}_{\hat{\gamma}_0,u}$ and $\mathbf{P}_{\hat{\gamma}_1,u}$ at u_q are the same, where $q = 1, ..., k$, $u = (u_1, ..., u_k)$.

Two $H_p^{t'}$-differentiable principal G-bundles E_1 and E_2 with parallel transport structures $(E_1, \mathbf{P}_1)$ and $(E_2, \mathbf{P}_2)$ are called isomorphic, if there exists an isomorphism $h : E_1 \to E_2$ such that $\mathbf{P}_{2,\hat{\gamma},u}(x) = h(\mathbf{P}_{1,\hat{\gamma},h^{-1}(u)}(x))$ for each H_p^t-mapping $\gamma : M \to N$ and $u_q \in (E_2)_{y_0}$, where $q = 1, ..., k$, $h^{-1}(u) = (h^{-1}(u_1), ..., h^{-1}(u_k))$.

Let $(S^M E)_{t,H} : (S^{M,\{s_{0,q}:q=1,...,k\}} E; N, G, \mathbf{P})_{t,H}$ be a set of H_p^t-closures of isomorphism classes of H_p^t principal G fiber bundles with parallel transport structure.

3. Theorems. 1. *The uniform space $(S^M E)_{t,H}$ from §2 has the structure of a topological alternative monoid with a unit and with a cancelation property and the multiplication operation of H_p^l class with $l = t' - t$ ($l = \infty$ for $t' = \infty$). If N and G are separable, then $(S^M E)_{t,H}$ is separable. If N and G are complete, then $(S^M E)_{t,H}$ is complete.*

2. *If G is associative, then $(S^M E)_{t,H}$ is associative. If G is commutative, then $(S^M E)_{t,H}$ is commutative. If G is a Lie group, then $(S^M E)_{t,H}$ is a Lie monoid.*

3. *The $(S^M E)_{t,H}$ is non-discrete, locally connected and infinite dimensional for $dim_{\mathbf{R}}(N \times G) > 1$.*

Proof. If there is a homomorphism $\theta : G \to F$ of $H_p^{t'}$-differentiable groups, then there exists an induced principal F fiber bundle $(E \times^\theta F)(N, F, \pi^\theta, \Psi^\theta)$ with the total space $(E \times^\theta F)(E \times F)/\mathcal{Y}$, where $\mathcal{Y}$ is the equivalence relation such that $(vg, f)\mathcal{Y}(v, \theta(g)f)$ for each $v \in E$, $g \in G$, $f \in F$. Then the projection $\pi^\theta : (E \times^\theta F) \to N$ is defined by $\pi^\theta([v, f]) = \pi(v)$, where $[v, f] := \{(w, b) : (w, b)\mathcal{Y}(v, f), w \in E, b \in F\}$ denotes the equivalence class of (v, f).

Therefore, each parallel transport structure $\mathbf{P}$ on the principal G fiber bundle $E(N, G, \pi, \Psi)$ induces a parallel transport structure $\mathbf{P}^\theta$ on the induced bundle by the formula $\mathbf{P}^\theta_{\hat{\gamma},[u,f]}(x) = [\mathbf{P}_{\hat{\gamma},u}(x), f]$.

Define multiplication with the help of certain embeddings and isomorphisms of spaces of functions. Mention that for each two compact canonical closed subsets A and B in $\mathcal{A}_r^l$ Hilbert spaces $H^t(A, \mathbf{R}^m)$ and $H^t(B, \mathbf{R}^m)$ are linearly topologically isomorphic, where $l, m \in \mathbf{N}$, hence $H_p^t(A, N)$ and $H_p^t(B, N)$ are isomorphic as uniform spaces. Let $H_p^t(M, \{s_{0,1}, ..., s_{0,k}\}; W, y_0) := \{(E, f) : E = E(N, G, \pi, \Psi) \in W, f = \mathbf{P}_{\hat{\gamma},y_0} \in H_p^t : \pi \circ f(s_{0,q}) = y_0 \forall q = 1, ..., k; \pi \circ f = \hat{\gamma}, \gamma \in H_p^t(M, N)\}$ be the space of all $H_p^{t'}$ principal G fiber bundles E with their parallel transport H_p^t-mappings $f = \mathbf{P}_{\hat{\gamma},y_0}$, where W is as in §1.3.2. Put $\omega_0 = (E_0, \mathbf{P}_0)$ be its element such that $\gamma_0(M) = \{y_0\}$, where $e \in G$ denotes the unit element, $E_0 = N \times G$, $\pi_0(y, g) = y$ for each $y \in N$, $g \in G$, $\mathbf{P}_{\hat{\gamma}_0,u} = \mathbf{P}_0$.

The mapping $\Xi : \hat{M} \to M$ from §2 induces the embedding

$\Xi^* : H_p^t(M, \{s_{0,1}, ..., s_{0,k}\}; W, y_0) \hookrightarrow H_p^t(\hat{M}, \{\hat{s}_{0,1}, ..., \hat{s}_{0,2k}\}; W, y_0)$,
where $\hat{M}$ and $\hat{A}_1$ and $\hat{A}_2$ are retractable into points.

Let as usually $A \vee B := \rho(\mathcal{Z})$ be the wedge sum of pointed spaces $(A, \{a_{0,q} : q = 1, ..., k\})$ and $(B, \{b_{0,q} : q = 1, ..., k\})$, where $\mathcal{Z} := [A \times \{b_{0,q} : q = 1, ..., k\} \cup \{a_{0,q} : q = 1, ..., k\} \times B] \subset A \times B$, ρ is a continuous quotient mapping such that $\rho(x) = x$ for each $x \in \mathcal{Z} \setminus \{a_{0,q} \times b_{0,j}; q, j = 1, ..., k\}$ and $\rho(a_{0,q}) = \rho(b_{0,q})$ for each $q = 1, ..., k$, where A and B are topological spaces with marked points $a_{0,q} \in A$ and $b_{0,q} \in B$, $q = 1, ..., k$. Then the wedge product $g \vee f$ of two elements $f, g \in H^t_p(M, \{s_{0,1}, ..., s_{0,k}\}; N, y_0)$ is defined on the domain $M \vee M$ such that $(f \vee g)(x \times b_{0,q}) = f(x)$ and $(f \vee g)(a_{0,q} \times x) = g(x)$ for each $x \in M$, where to f, g there correspond $f_1, g_1 \in H^t_p(\hat{M}, \{\hat{s}_{0,1}, ..., \hat{s}_{0,2k}\}; N, y_0)$ such that $f_1 f \circ \Xi$ and $g_1 = g \circ \Xi$.

Let $(E_j, \mathbf{P}_{\hat{\gamma}_j, u^j}) \in H^t_p(M, \{s_{0,1}, ..., s_{0,k}\}; W, y_0)$, $j = 1, 2$, then take their wedge product $\mathbf{P}_{\hat{\gamma}, u^1} := \mathbf{P}_{\hat{\gamma}_1, u^1} \vee \mathbf{P}_{\hat{\gamma}_2, v}$ on $M \vee M$ with $v_q = u_q g^{-1}_{2,q} g_{1,q+k} = y_0 \times g_{1,q+k}$ for each $q = 1, ..., k$ due to the alternativity of G, $\gamma = \gamma_1 \vee \gamma_2$, where $\mathbf{P}_{\hat{\gamma}_j, u^j}(\hat{s}_{j,0,q}) y_0 \times g_{j,q} \in E_{y_0}$ for every j and q. For each $\gamma_j : M \to N$ there exists $\tilde{\gamma}_j : M \to E_j$ such that $\pi \circ \tilde{\gamma}_j = \gamma_j$. Denote by $\mathbf{m} : G \times G \to G$ the multiplication operation. The wedge product $(E_1, \mathbf{P}_{\hat{\gamma}_1, u^1}) \vee (E_2, \mathbf{P}_{\hat{\gamma}_2, u^2})$ is the principal G fiber bundle $(E_1 \times E_2) \times^{\mathbf{m}} G$ with the parallel transport structure $\mathbf{P}_{\hat{\gamma}_1, u^1} \vee \mathbf{P}_{\hat{\gamma}_2, v}$.

The uniform space $H^t_p(J, A_3; W, y_0) := \{(E, f) \in H^t_p(J, W) : \pi \circ f(A_3) = \{y_0\}\}$ has the H^t_p-manifold structure and has an embedding into $H^t_p(M, \{s_{0,1}, ..., s_{0,k}\}; W, y_0)$ due to Conditions $2(i - iii)$, where either $J = A_1$ or $J = A_2$. This induces the following embedding $\chi^* : H^t_p(M \vee M, \{s_{0,q} \times s_{0,q} : q = 1, ..., k\}; W, y_0) \hookrightarrow H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0)$.

Analogously considering $H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0) = \{f \in H^t(M, W) : f(X_{0,q}) = \{y_0\}, q = 1, ..., k\}$ and $H^t_p(J, A_3 \cup \{X_{0,q} : q = 1, ..., k\}; W, y_0)$ in the case (iv') instead of (iv) we get the embedding $\chi^* : H^t_p(M \vee M, \{X_{0,q} \times X_{0,q} : q = 1, ..., k\}; W, y_0) \hookrightarrow H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)$. Therefore, $g \circ f := \chi^*(f \vee g)$ is the composition in $H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0)$.

There exists the following equivalence relation $R_{t,H}$ in $H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)$: $f R_{t,H} h$ if and only if there exist nets $\eta_n \in Dif H^t_p(M, \{X_{0,q} : q = 1, ..., k\})$, also f_n and $h_n \in H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)$ with $\lim_n f_n = f$ and $\lim_n h_n = h$ such that $f_n(x) = h_n(\eta_n(x))$ for each $x \in M$ and $n \in \omega$, where ω is a directed set and convergence is considered in $H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)$. Henceforward in the case $2(iv)$ we get $s_{0,q}$ instead of $X_{0,q}$ in the case $2(iv')$.

Thus there exists the quotient uniform space $H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)/R_{t,H} =: (S^M E)_{t,H}$. In view of [30, 31] $Dif H^t_p(M)$ is the group of diffeomorphisms for $t \geq [m/2] + 1$. The Lebesgue measure λ in the real shadow of $\hat{M}$ by the mapping Ξ induces the measure λ^{Ξ} on M which is equivalent to ν, since Ξ is the H^t_p-mapping from the compact space onto the compact space, $\lambda(\partial \hat{M}) = 0$ and $\Xi : \hat{M} \setminus \partial \hat{M} \to M$ is bijective.

Due to Conditions $(P1 - P5)$ each element $f = \mathbf{P}_{\hat{\gamma}, u}$ up to a set Q_M of measure zero, $\nu(Q_M) = 0$, is given as $f \circ \Xi^{-1}$ on $M \setminus Q_M$, where $\pi \circ f = \hat{\gamma}$, $\hat{\gamma} = \gamma \circ \Xi$. Denote $f \circ \Xi^{-1}$ also by f. Thus, for each $(E, f) \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0)$ the image $f(M)$ is compact and connected in E.

Therefore, for each partition Z there exists $\delta > 0$ such that for each partition Z^* with $\sup_i \inf_j dist(M_i, M^*{}_j) < \delta$ and $(E, f) \in H^t(M, W; Z)$, $f(s_{0,q}) = u_q$, there exists $(E, f_1) \in H^t(M, W; Z^*)$ with $f_1(s_{0,q}) = u_q$ for each $q = 1, ..., k$ such that $f R_{t,H} f_1$, where M_i and M^*_j are canonical closed pseudo-submanifolds in M corresponding to partitions Z and Z^*, $H^t(M, W; Z)$ denotes the space of all continuous piecewise H^t-mappings from M into W subordinated to the partition Z such that Z and Z^* respect H^t_p structure of M.

Hence there exists a countable subfamily $\{Z_j : j \in \mathbf{N}\}$ in the family of all partitions Υ such that $Z_j \subset Z_{j+1}$ for each j and $\lim_j \tilde{d}iamZ_j = 0$. Then

(i) $str-ind\{H^t(M,\{s_{0,q} : q = 1,...,k\}; W, y_0; Z_j); h^{Z_i}_{Z_j}; \mathbf{N}\}/R_{t,H} = (S^M E)_{t,H}$ is separable if N and G are separable, since each space $H^t_p(M,\{s_{0,q} : q = 1,...,k\}; W, y_0; Z_j)$ is separable.

The space $str-ind\{H^t(M,\{s_{0,q} : q = 1,...,k\}; W, y_0; Z_j); h^{Z_i}_{Z_j}; \mathbf{N}\}$ is complete due to Theorem 12.1.4 [29], when N and G are complete. Each class of $R_{t,H}$-equivalent elements is closed in it. Then to each Cauchy net in $(S^M E)_{t,H}$ there corresponds a Cauchy net in $str-ind\{H^t(M\times[0,1],\{s_{0,q}\times e\times 0; W, y_0; Z_j\times Y_j); h^{Z_i\times Y_i}_{Z_j\times Y_j}; \mathbf{N}\}$ due to theorems about extensions of functions [25, 33, 38], where Y_j are partitions of $[0,1]$ with $\lim_j \tilde{d}iam(Y_j) = 0$, $Z_j \times Y_j$ are the corresponding partitions of $M \times [0,1]$. Hence $(S^M E)_{t,H}$ is complete, if N and G are complete.

If $f, g \in H^t(M,X)$ and $f(M) \neq g(M)$, then

(ii) $\inf_{\psi\in DifH^t_p(M,\{s_{0,q}:q=1,...,k\})} \|f\circ\psi - g\|_{H^t(M,X)} > 0$. Thus equivalence classes $< f >_{t,H}$ and $< g >_{t,H}$ are different. The pseudo-manifold $\hat{M}$ is arcwise connected. Take $\eta : [0,1] \to \hat{M}$ an H^t_p-mapping with $\eta(0) = \hat{s}_{0,q}$ and $\eta(1) = \hat{s}_{0,k+q}$, where $1 \le q \le k$. Choose in $\hat{M}$ H^t_p-coordinates one of which is a parameter along η. Therefore, for each $g_q, g_{k+q} \in G$ there exists $\mathbf{P}_{\hat{\gamma},u}$ with $\mathbf{P}_{\hat{\gamma},u}(s_{0,q}) = y_0 \times g_q$ and $\mathbf{P}_{\hat{\gamma},u}(s_{0,k+q}) = y_0 \times g_{k+q}$ for each $q = 1,...,k$. Since E and G are arcwise connected, then N is arcwise connected and $(S^M E)_{t,H}$ is locally connected for $dim_{\mathbf{R}} N > 1$. Thus, the uniform space $(S^M E)_{t,H}$ is non-discrete.

The tangent bundle $TH^t_p(M,E)$ is isomorphic with $H^t_p(M,TE)$, where TE is the $H^{t'-1}_p$ fiber bundle, $t' \ge t+1$. There is an infinite family of $f_\alpha \in H^t_p(M,TE)$ with pairwise distinct images in TE for different α such that $f_\alpha(M)$ is not contained in $\bigcup_{\beta<\alpha} f_\beta(M)$, $\alpha \in \Lambda$, where Λ is an infinite ordinal. Therefore, $T(S^M E)_{t,H}$ is an infinite dimensional fiber bundle due to (ii) and inevitably $(S^M E)_{t,H}$ is infinite dimensional.

Evidently, if $f \vee g = h \vee g$ or $g \vee f = g \vee h$ for $\{f,g,h\} \subset H^t_p(M,\{s_{0,q} : q = 1,...,k\}; W, y_0)$, then $f = h$. Thus $\chi^*(f \vee g) = \chi^*(h \vee g)$ or $\chi^*(g \vee f) = \chi^*(g \vee h)$ is equivalent to $f = h$ due to the definition of $f \vee g$ and the definition of equal functions, since χ^* is the embedding. Using the equivalence relation $R_{t,H}$ gives $< f >_{t,H} \circ < g >_{t,H} = < h >_{t,H} \circ < g >_{t,H}$ or $< g >_{t,H} \circ < f >_{t,H} = < g >_{t,H} \circ < h >_{t,H}$ is equivalent to $< h >_{t,H} = < f >_{t,H}$. Therefore, $(S^M E)_{t,H}$ has the cancelation property.

Since G is alternative, then $a_{2,q}[a^{-1}_{2,q}(a_{2,q+k}(a^{-1}_{2,q}a_{1,q+k}))]a_{2,q+k}(a^{-1}_{2,q}a_{1,q+k})$, hence $\mathbf{P}_1 \vee (\mathbf{P}_2 \vee \mathbf{P}_2) = (\mathbf{P}_1 \vee \mathbf{P}_2) \vee \mathbf{P}_2$; also $a_{2,q}[a^{-1}_{2,q}(a_{1,q+k}(a^{-1}_{1,q}a_{1,q+k}))]a_{1,q+k}(a^{-1}_{1,q}a_{1,q+k})$, consequently, $\mathbf{P}_1 \vee (\mathbf{P}_1 \vee \mathbf{P}_2) = (\mathbf{P}_1 \vee \mathbf{P}_1) \vee \mathbf{P}_2$ and inevitably for equivalence classes $(aa)b = a(ab)$ and $b(aa) = (ba)a$ for each $a, b \in (S^M E)_{t,H}$. Thus $(S^M E)_{t,H}$ is alternative.

If G is associative, then the parallel transport structure gives $(f \vee g) \vee h = f \vee (g \vee h)$ on $M \vee M \vee M$ for each $\{f,g,h\} \subset H^t_p(M,\{s_{0,q} : q = 1,...,k; W, y_0)$. Applying the embedding χ^* and the equivalence relation $R_{t,H}$ we get, that $(S^M E)_{t,H}$ is associative $< f >_\xi \circ (< g >_\xi \circ < h >_\xi) =$

$(< f >_\xi \circ < g >_\xi) \circ < h >_\xi$.

In view of Conditions $2(i - iv)$ there exists an H^t_p-diffeomoprhism of $(A_1 \setminus A_3) \vee (A_2 \setminus A_3)$ with $(A_2 \setminus A_3) \vee (A_1 \setminus A_3)$ as pseudo-manifolds (see §1.3.1). For the measure ν on M naturally the equality $\nu(A_3) = 0$ is satisfied. If M' - is the submanifold may be with corners or pseudo-manifold, accomplishing the partition $Z = Z_f$ of the manifold M, then the codimension M' in M is equal to one and $\nu(M') = 0$. For the point $s_{0,q}$ in $(M \setminus A_3) \cup \{s_{0,q}\}$ there exists an open neighborhood U having the H^t_p-retraction $F : [0, 1] \times U \to \{s_{0,q}\}$. Hence it is possible to take a sequence of diffeomorphisms $\psi_n \in DifH^t_p(M, \{s_{0,q} : q = 1, ..., k\})$ such that $\lim_{n\to\infty} diam(\psi_n(U)) = 0$.

Let w_0 be a mapping $w_0 : M \to W$ such that $w_0(M) = \{y_0 \times e\}$. Consider $w_0 \vee (E, f)$ for some $(E, f) \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0)$. If $(E, f) \in H^t_p(M, \{s_{0,q} : q = 1, .., k\}; W, y_0)$ with the natural positive $t \in \mathbf{N}$, then f is bounded relative to the uniformity of the uniform space $H^t_p(M; E)$. If U_n is a sequence of bounded open or canonical closed subsets in M such that $\lim_n diam(U_n) = 0$, then $\lim_{n\to\infty} \nu(V_n) = 0$ for the sequence of ν-measurable subsets V_n such that $V_n \subset U_n$. Therefore, for each bounded sequence $\{g_n : g_n \in H^t_p(M; E); n \in \mathbf{N}\}$ there exists the limit $\lim_{n\to\infty} g_n|_{U_n} = 0$ relative to the H^t_p uniformity, where U_n is subordinated to the partition of M into H^t submanifolds. Then if $\{g_n : g_n \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; E, y_0); n \in \mathbf{N}\}$ is a bounded sequence such that g_n converges to $g \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; N, y_0)$ on $M \setminus W_k$ for each k relative to the H^t_p-uniformity, the given open W_k in M, where $k, n \in \mathbf{N}$ and $\lim_{n\to\infty} \nu(W_n \triangle U_n) = 0$, then g_n converges to g in the uniform space $H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; E, y_0)$.

Mention that for each marked point $s_{0,q}$ in M there exists a neighborhood U of $s_{0,q}$ in M such that for each $\gamma_1 \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; N, y_0)$ there exists $\gamma_2 \in H^t_p$ such that they are $R_{t,H}$ equivalent and $\gamma_2|_U = y_0$. Therefore, if C is an arcwise connected compact subset in M of codimension $codim_{\mathbf{R}} C \geq 1$ such that $s_{0,q} \in C$, then the standard proceeding shows that for each $\gamma_1 \in H^t_p$ there exists $\gamma_2 \in H^t_p$ such that $\gamma_1 R_{t,H} \gamma_2$ and $\gamma_2|_C = y_0$. Since C is compact, then each its open covering has a finite subcovering and hence

(Y_0) there exists an open neighborhood U of C in M such that for each γ_1 there exists γ_2 such that $\gamma_1 R_{t,H} \gamma_2$ and $\gamma_2|_U = y_0$.

There exists a sequence $\eta_n \in DifH^t_p(M, \{s_{0,q} : q = 1, ..., k\})$ such that $\lim_{n\to\infty} diam(\eta_n(A_2 \setminus \partial A_2)) = 0$ and $w_n, f_n \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; E, y_0)$ with

(iii) $\lim_{n\to\infty} f_n = f$, $\lim_{n\to\infty} w_n = w_0$ and $\lim_{n\to\infty} \chi^*(f_n \vee w_n)(\eta_n^{-1}) = f$ due to $\pi \circ f(s_{0,q}) = s_{0,q}$ in the formula of differentiation of compositions of functions (over $\mathbf{H}$ and $\mathbf{O}$ see it in [19, 20, 17]).

In more details, the sequence η_n as a limit of $\eta_n(A_2)$ produces a pseudo-submanifold B in M of codimension not less than one such that B can be presented with the help of the wedge product of spheres and compact quadrants up to H^t_p-diffeomorphism with marked points $\{s_{0,q} : q = 1, ..., k\}$, but as well B may be a finite discrete set also. Then by induction the procedure can be continued lowering the dimension of B. Particularly there may be circles and curves in the case of the unit dimension. Two quadrants up to an H^t_p quotient mapping gluing boundaries produce a sphere. Thus the consideration reduces to the case of the wedge product of spheres. The case of spheres reduces to the iterated construction with circles, since the reduced product $S^1 \wedge S^n$ is H^t_p homeomorphic with S^{n+1} (see Lemma 2.27 [37] and [4]). For the particular case of the n-dimensional sphere $M_n = S^n$ take $\hat{M}_n = D^n$, where D^n is the unit ball (disk) in $\mathbf{R^n}$ or in a n dimensional over $\mathbf{R}$ subspace in $\mathcal{A}^l_r$, $D_1 = [0,1]$ for $n = 1$. But $S^n \setminus s_0$ has the retraction into the point in S^n, where $s_0 \in S^n$, $n \in \mathbf{N}$.

Therefore, $w_0 \vee (E,f)$ and (E,f) belong to the equivalence class $< (E,f) >_{t,H}$: $\{g \in H^t_p(M,\{s_{0,q} : q = 1,...,k\};W,y_0) : (E,f)R_{t,H}g\}$ due to (iii) and (Y_0). Thus, $< w_0 >_{t,H} \circ < g >_{t,H} = < g >_{t,H}$.

The pseudo-manifold $M \vee M \setminus \{s_{0,q} \times s_{0,j} : q,j = 1,...,k\}$ has the H^t_p-diffeomorphism ψ (see definition in §1.3.1) such that $\psi(x,y) = (y,x)$ for each $(x,y) \in (M \times M \setminus \{s_{0,q} \times s_{0,j} : q,j = 1,...,k\})$. Suppose now, that G is commutative. Then $(f \vee g) \circ \psi|_{(M\times M\setminus\{s_{0,q}\times s_{0,j}:q,j=1,...,k\})} = g \vee f|_{(M\times M\setminus\{s_{0,q}\times s_{0,j}:q,j=1,...,k\})}$. On the other hand, $< f \vee w_0 >_{t,H} = < f >_{t,H} = < f >_{t,H} \circ < w_0 >_{t,H} = < w_0 >_{t,H} \circ < f >_{t,H}$, hence, $< f \vee g >_{t,H} = < f >_{t,H} \circ < g >_{t,H} = < f \vee w_0 >_{t,H} \circ < w_0 \vee g >_{t,H} = < (f \vee w_0) \vee (w_0 \vee g) >_{t,H} = < (w_0 \vee g) \vee (f \vee w_0) >_{t,H}$ due to the existence of the unit element $< w_0 >_{t,H}$ and due to the properties of ψ. Indeed, take a sequence ψ_n as above. Therefore, the parallel transport structure gives $(g \vee f)(\psi(x,y)) = (g \circ f)(y,x)$ for each $x,y \in M$, consequently, $(f \circ g)R_{t,H}(g \circ f)$ for each $f,g \in H^t_p(M,\{s_{0,q} : q = 1,...,k\};W,y_0)$. The using of the embedding χ^* gives that $(S^M E)_{t,H}$ is commutative, when G is commutative.

The mapping $(f,g) \mapsto f \vee g$ from $H^t_p(M,\{s_{0,q} : q = 1,...,k\};W,y_0)^2$ into $H^t_p(M \vee M \setminus \{s_{0,q} \times s_{0,j} : q,j = 1,...,k\};W,y_0)$ is of class H^t_p. Since the mapping χ^* is of class H^t_p, then $(f,g) \mapsto \chi^*(f \vee g)$ is the H^t_p-mapping. The quotient mapping from $H^t_p(M,\{s_{0,q} : q = 1,...,k\};W,y_0)$ into $(S^M E)_{t,H}$ is continuous and induces the quotient uniformity, $T^b(S^M E)_{t,H}$ has embedding into $(S^M T^b E)_{t,H}$ for each $1 \le b \le t' - t$, when $t' > t$ is finite, for every $1 \le b < \infty$ if $t' = \infty$, since E is the $H^{t'}_p$ fiber bundle, $T^b E$ is the fiber bundle with the base space N. Hence the multiplication $(< f >_{t,H}, < g >_{t,H}) \mapsto < f >_{t,H} \circ < g >_{t,H} = < f \vee g >_{t,H}$ is continuous in $(S^M E)_{t,H}$ and is of class H^l_p with $l = t' - t$ for finite t' and $l = \infty$ for $t' = \infty$.

4. Definition. The $(S^M E)_{t,H}$ from Theorem 3.1 we call the wrap monoid.

5. Corollary. *Let $\phi : M_1 \to M_2$ be a surjective H^t_p-mapping of H^t_p-pseudo-manifolds over the same $\mathcal{A}_r$ such that $\phi(s_{1,0,q}) = s_{2,0,a(q)}$ for each $q = 1,...,k_1$, where $\{s_{j,0,q} : q = 1,...,k_j\}$ are marked points in M_j, $j = 1,2$, $1 \le a \le k_2$, $l_1 \le k_2$, $l_1 :$ card $\phi(\{s_{1,0,q} : q = 1,...,k_1\})$. Then there exists an induced homomorphism of monoids $\phi^* : (S^{M_2}E)_{t,H} \to (S^{M_1}E)_{t,H}$. If $l_1 = k_2$, then ϕ^* is the embedding.*

Proof. Take $\Xi_1 : \hat{M}_1 \to M_1$ with marked points $\{\hat{s}_{1,0,q} : q = 1,...,2k_1\}$ as in §2, then take $\hat{M}_2$ the same $\hat{M}_1$ with additional $2(k_2 - l_1)$ marked points $\{\hat{s}_{2,0,q} : q = 1,...,2k_3\}$ such that $\hat{s}_{1,0,q} = \hat{s}_{2,0,q}$ for each $q = 1,..,k_1$, $k_3 = k_1 + k_2 - l_1$, then $\phi \circ \Xi_1 := \Xi_2 : \hat{M}_2 \to M_2$ is the desired mapping inducing the parallel transport structure from that of M_1. Therefore, each $\hat{\gamma}_2 : \hat{M}_2 \to N$ induces $\hat{\gamma}_1 : \hat{M}_1 \to N$ and to $\mathbf{P}_{\hat{\gamma}_2,u^2}$ there corresponds $\mathbf{P}_{\hat{\gamma}_1,u^1}$ with additional conditions in extra marked points, where $u^1 \subset u^2$. The equivalence class $< (E_2,\mathbf{P}_{\hat{\gamma}_2,u^2}) >_{t,H} \in (S^{M_2}E)_{t,H}$

gives the corresponding elements $<(E_1, \mathbf{P}_{\hat{\gamma}_1,u^1})>_{t,H} \in (S^{M_1}E)_{t,H}$, since $DifH^t_p(\hat{M}_1, \{\hat{s}_{0,q} : q = 1,...,2k_2\}) \subset DifH^t_p(\hat{M}_1, \{\hat{s}_{0,q} : q = 1,...,2k_3\})$. Then $\phi^*(<(E_2, \mathbf{P}_{\hat{\gamma}_2,u^2}) \vee (E_1, \mathbf{P}_{\hat{\eta}_2,v^2})>_{t,H}) = \phi^*(<(E_2, \mathbf{P}_{\hat{\gamma}_2,u^2})>_{t,H})\phi^*(<(E_1, \mathbf{P}_{\hat{\eta}_2,v^2})>_{t,H})$, since $f_2 \circ \phi(x)$ for each $x \in \Xi_1(\hat{M}_1 \setminus \partial\hat{M}_1)$ coincides with $f_1(x)$, where f_j corresponds to $\mathbf{P}_{\gamma_j, y_0 \times e}$ (see also the beginning of §3).

If $l_1 = k_2$, then $\hat{M}_1 = \hat{M}_2$ and the group of diffeomorphisms $DifH^t_p(\hat{M}_1, \{\hat{s}_{0,q} : q = 1,...,2k_1\})$ is the same for two cases, hence ϕ^* is bijective and inevitably ϕ^* is the embedding.

6. Theorems. 1. *There exists an alternative topological group $(W^ME)_{t,H}$ containing the monoid $(S^ME)_{t,H}$ and the group operation of H^l_p class with $l = t' - t$ ($l = \infty$ for $t' = \infty$). If N and G are separable, then $(W^ME)_{t,H}$ is separable. If N and G are complete, then $(W^ME)_{t,H}$ is complete.*

2. *If G is associative, then $(W^ME)_{t,H}$ is associative. If G is commutative, then $(W^ME)_{t,H}$ is commutative. If G is a Lie group, then $(W^ME)_{t,H}$ is a Lie group.*

3. *The $(W^ME)_{t,H}$ is non-discrete, locally connected and infinite dimensional for $dim_{\mathbf{R}}(N \times G) > 1$. Moreover, if there exist two different sets of marked points $s_{0,q,j}$ in A_3, $q = 1,...,k$, $j = 1, 2$, then two groups $(W^ME)_{t,H,j}$, defined for $\{s_{0,q,j} : q = 1,...,k\}$ as marked points, are isomorphic.*

4. *The $(W^ME)_{t,H}$ has a structure of an H^t_p-differentiable manifold over $\mathcal{A}_r$.*

Proof. If $\gamma \in H^t_p(M, \{s_{0,q} : q = 1,...,k\}; N, y_0)$, then for $u \in E_{y_0}$ there exists a unique $h_q \in G$ such that $\mathbf{P}_{\hat{\gamma},u}(\hat{s}_{0,q+k}) = u_q h_q$, where $h_q = g_q^{-1} g_{q+k}$, $y_0 \times g_q = \mathbf{P}_{\hat{\gamma},u}(\hat{s}_{0,q})$, $g_q \in G$. Due to the equivariance of the parallel transport structure h depends on γ only and we denote it by $h^{(E,\mathbf{P})}(\gamma) = h(\gamma) = h$, $h = (h_1, ..., h_k)$. The element $h(\gamma)$ is called the holonomy of $\mathbf{P}$ along γ and $h^{(E,\mathbf{P})}(\gamma)$ depends only on the isomorphism class of $(E, \mathbf{P})$ due to the use of $DifH^t_p(\hat{M}; \{\hat{s}_{0,q} : q = 1,...,2k\})$ and boundary conditions on $\hat{\gamma}$ at $\hat{s}_{0,q}$ for $q = 1,...,2k$.

Therefore, $h^{(E_1,\mathbf{P}_1)(E_2,\mathbf{P}_2)}(\gamma) = h^{(E_1,\mathbf{P}_1)}(\gamma)h^{(E_2,\mathbf{P}_2)}(\gamma) \in G^k$, where G^k denotes the direct product of k copies of the group G. Hence for each such γ there exists the homomorphism $h(\gamma) : (S^ME)_{t,H} \to G^k$, which induces the homomorphism $h : (S^ME)_{t,H} \to C^0(H^t_p(M, \{s_{0,q} : q = 1,...,k\}; N, y_0), G^k)$, where $C^0(A, G^k)$ is the space of continuous maps from a topological space A into G^k and the group structure $(hb)(\gamma) = h(\gamma)b(\gamma)$ (see also [4] for S^n).

Thus, it is sufficient to construct $(W^MN)_{t,H}$ from $(S^MN)_{t,H}$. For the commutative monoid $(S^MN)_{t,H}$ with the unit and the cancelation property there exists a commutative group $(W^MN)_{t,H}$. Algebraically it is the quotient group F/B, where F is the free commutative group generated by $(S^MN)_{t,H}$, while B is the minimal closed subgroup in F generated by all elements of the form $[f + g] - [f] - [g]$, f and $g \in (S^MN)_{t,H}$, $[f]$ denotes the element in F corresponding to f (see also about such abstract Grothendieck construction in [13, 36]).

By the construction each point in $(S^MN)_{t,H}$ is the closed subset, hence $(S^MN)_{t,H}$ is the topological T_1-space. In view of Theorem 2.3.11 [7] the product of T_1-spaces is the T_1-space. On the other hand, for the topological group G from the separation axiom T_1 it follows, that G is the

Tychonoff space [7, 32]. The natural mapping $\eta : (S^M N)_{t,H} \to (W^M N)_{t,H}$ is injective. We supply F with the topology inherited from the topology of the Tychonoff product $(S^M N)^{\mathbf{Z}}_{t,H}$, where each element z in F has the form $z = \sum_f n_{f,z}[f]$, $n_{f,z} \in \mathbf{Z}$ for each $f \in (S^M N)_{t,H}$, $\sum_f |n_{f,z}| < \infty$. By the construction F and F/B are T_1-spaces, consequently, F/B is the Tychonoff space. In particular, $[nf] - n[f] \in \mathsf{B}$, hence $(W^M N)_{t,H}$ is the complete topological group, if N and G are complete, while η is the topological embedding, since $\eta(f + g) = \eta(f) + \eta(g)$ for each $f, g \in (S^M N)_{t,H}$, $\eta(e) = e$, since $(z + B) \in \eta(S^M N)_{t,H}$, when $n_{f,z} \geq 0$ for each f, and inevitably in the general case $z = z^+ - z^-$, where $(z^+ + B)$ and $(z^- + B) \in \eta(S^M N)_{t,H}$.

Using plots and $H_p^{t'}$ transition mappings of charts of N and $E(N, G, \pi, \Psi)$ and equivalence classes relative to $DifH_p^t(M, \{s_{0,q} : q = 1, ..., k\})$ we get, that $(W^M E)_{t,H}$ has the structure of the H_p^t-differentiable manifold, since $t' \geq t$.

The rest of the proof and the statements of Theorems 6(1-4) follows from this and Theorems 3(1-3) and [21, 22].

7. Definition. The $(W^M E)_{t,H} = (W^{M,\{s_{0,q}:q=1,...,k\}} E; N, G, \mathbf{P})_{t,H}$ from Theorem 6.1 we call the wrap group.

8. Corollary. *There exists the group homomorphism* $h : (W^M E)_{t,H} \to C^0(H_p^t(M, \{s_{0,q} : q = 1, ..., k\}; N, y_0), G^k)$.

Proof follows from §6 and putting $h^{f^{-1}}(\gamma)(h^f(\gamma))^{-1}$.

9. Corollary. *If M_1 and M_2 and ϕ satisfy conditions of Corollary 5, then there exists a homomorphism* $\phi^* : (W^{M_2} E)_{t,H} \to (W^{M_1} E)_{t,H}$. *If $l_1 = k_2$, then ϕ^* is the embedding.*

10. Remarks and examples. Consider examples of M which satisfy sufficient conditions for the existence of wrap groups $(W^M E)_{t,H}$. Take M, for example, D_R^n, $S_R^n \setminus V$ with $s_0 \in \partial V$, $D_R^n \setminus Int(D_b^n)$ with $s_0 \in \partial D_b^n$ and $0 < b < R < \infty$, where S_R^n denotes the sphere of the dimension $n > 1$ over $\mathbf{R}$ and radius R, V is H_p^t-diffeomorphic with the interior $Int(D_R^n)$ of the n-dimensional ball $D_R^n := \{x \in \mathbf{R^n} : \sum_{k=1}^n x_k^2 \leq R\}$ or in n dimensional over $\mathbf{R}$ subspace in $\mathcal{A}_r^l$ and is the proper subset in $S_R^n := \{x \in \mathbf{R^{n+1}} : \sum_{k=1}^{n+1} x_k^2 = R\}$. Instead of sphere it is possible to take an H_p^t pseudo-manifold Q^n homeomorphic with a sphere or a disk, particularly, Milnor's sphere. Indeed, divide M by the equator $\{x_1 = 0\}$ into two parts A_1 and A_2 and take $A_3 = \{x \in M : x_1 = 0\} \cup P$, where $s_0 \in \partial A_1 \cap \partial A_2$, while $P = \emptyset$, $P = \partial V$, $P = \partial D_b^n$ correspondingly. Then take also V and D_b^n such that their equators would be generated by the equator $\{x_1 = 0\}$ in S_R^n or D_R^n respectively or more generally Q^n.

Take then $M = Q^n \setminus \bigcup_{k=1}^l V_k$, where V_k are H_p^t-diffeomorphic to interiors of bounded quadrants in $\mathbf{R^n}$ or in n dimensional subspace in $\mathcal{A}_r^a$, where $l > 1$, $l \in \mathbf{N}$, $\partial V_k \cap \partial V_j = \{s_0\}$ and $V_k \cap V_j = \emptyset$ for each $k \neq j$, $diam(V_k) \leq b < R/3$. In more details it is possible make a specification such that if l is even, then $[l/2] - 1$ among V_k are displayed above the equator and the same amount below it, two of V_k have equators, generated by equators $\{x_1 = 0\}$ in Q^n. If l odd, then $[(l-1)/2]$ among V_k are displayed above and the same amount below it, one of V_k has equator generated by that of $\{x_1 = 0\}$ in Q^n, $s_0 \in \bigcap_k \partial V_k \cap \{x \in M : x_1 = 0\}$.

Divide M by the equator $\{x_1 = 0\}$ into two parts A_1 and A_2 and let $A_3 = \{x \in M : x_1 = 0\} \cup P$, where $P = \bigcup_{k=1}^{l} \partial V_k$. Then either $A_1 \setminus A_3$ and $A_2 \setminus A_3$ are H_p^t diffeomorphic as pseudo-manifolds or manifolds with corners and H_p^t diffeomorphic with $M \setminus [\{s_0\} \cup (A_3 \setminus Int(\partial A_1 \cap \partial A_2))] =: D$ or $2(iv')$ is satisfied, since the latter topological space D is obtained from Q^n by cutting a non-void connected closed subset, $n > 1$, consequently, D is retractable into a point.

In a case of a usual manifold M the point $s_0 \in \partial M$ (for $\partial M \neq \emptyset$) may be a critical point, but in the case of a manifold with corners this s_0 is the corner point from ∂M, since for $x \in \partial M$ there is not less than one chart (U, u, Q) such that $u(x) \in \partial Q$, $M \setminus \partial M = \bigcup_k u_k^{-1}(Int(Q_k))$, $\partial M \subset \bigcup_k u_k^{-1}(\partial Q_k)$. Further, if M satisfies Conditions $2(i-v)$ or $(i-iii, iv', v)$, then $M \times D_R^m = P$ also satisfies them for $m \geq 1$, since D_R^m is retractable into the point, taking as two parts $A_j(K) = A_j(M) \times D_R^m$ of P, where $j = 1, 2$, $A_j(M)$ are pseudo-submanifolds of M. Then $A_1(P) \cap A_2(P) = (A_1(M) \cap A_2(M)) \times D_R^m$ and it is possible to take $A_3(P) = A_3(M) \times D_R^m$, $s_0(P) \in s_0(M) \times \{x \in D_R^m : x_1 = 0\}$. In particular, for $M = S^1$ and $m = 1$ this gives the filled torus.

This construction can be naturally generalized for non-orientable manifolds, for example, the Möbius band L, also for $M := L \setminus (\bigcup_{j=1}^{\beta} V_j)$ with the diameter b_j of V_j less than the width of L, where each V_j is H_p^t diffeomorphic with an interior of a bounded quadrant in $\mathbf{R^2}$, $s_{0,q} \in \partial L \cap (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_j)$, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$, since ∂L is diffeomorphic with S^1, also $S^1 \setminus \{s_{0,q}\}$ is retractable into a point, consequently, A_1 and A_2 are retractable into a point. For L take $\hat{M} = I^2$, then take a connected curve $\hat{\eta}$ consisting of the left side $\{0\} \times [0,1]$ joined by a straight line segment joining points $\{0,1\}$ and $\{1,0\}$ and then joined by the right side $\{1\} \times [0,1]$. This gives the proper cutting of $\hat{M}$ which induces the proper cutting of L and of M with $A_3 \supset \eta \cup \partial L$ up to an H_p^t diffeomorphism, where $\eta := \Xi(\hat{\eta})$, hence the Möbius band L and M satisfy Conditions $2(i-iii, iv', v)$.

Take a quotient mapping $\phi : I^2 \to S^1$ such that $\phi(\{s_{0,1}, s_{0,2}\}) = s_0 \in S^1$, $s_{0,1} = (0,0)$, $s_{0,2} = (0,1) \in I^2$, where $I = [0,1]$, hence there exists the embedding $\phi^* : (W^{S^1, s_0} E)_{t,H} \hookrightarrow (W^{I^2, \{s_{0,1}, s_{0,2}\}} E)_{t,H}$.

The Klein bottle K has $\hat{M} = I^2$ with twisting equivalence relation on ∂I^2 so it satisfies sufficient conditions. Moreover, K is the quotient $\phi : Z \to K$ of the cylinder Z with twisted equivalence relation of its ends S^1 using reflection relative to a horizontal diameter. Thus $A_3 \supset \phi(S^1)$. Therefore, there exists the embedding $\phi^* : (W^{K, \{s_0\}} E)_{t,H} \to (W^{Z, \{s_{0,1}, s_{0,2}\}} E)_{t,H}$, where $s_{0,1}, s_{0,2} \in \partial Z$, $\phi(\{s_{0,1}, s_{0,2}\}) = s_0$.

Take a pseudo-manifold Q^n H_p^t-diffeomorphic with S^n for $n \geq 2$, cut from it β non-intersecting open domains $V_1, ..., V_\beta$ H_p^t-diffeomorphic with interiors of bounded quadrants in $\mathbf{R^n}$, $s_{0,q} \in \bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_j$, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$. Then glue for $V_1, ..., V_l$, $1 \leq l \leq \beta$, by boundaries of slits H_p^t-diffeomorphic with S^{m-1} the reduced product $L \vee S^{n-2}$, since $\partial L = S^1$, $S^1 \wedge S^{n-2}$ is H_p^t-diffeomorphic with S^{n-1} [37]. We get the non-orientable H_p^t-pseudo-manifold M, satisfying sufficient conditions.

Since the projective space $\mathbf{R}P^n$ is obtained from the sphere by identifying diametrically opposite points. Then take M H^t_p-diffeomorphic with $\mathbf{R}P^n$ for $n > 1$ also M with cut $V_1, ..., V_\beta$ H^t_p-diffeomorphic with open subsets in $\mathbf{R}P^n$, $s_{0,q} \in (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_j) \cap \{x \in M : x_1 = 0\}$, $V_j \cap V_l = \emptyset$ for each $j \ne l$, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$. Then Conditions $2(i - v)$ or $(i - iii, iv', v)$ are also satisfied for $\mathbf{R}P^n$ and M.

In view of Proposition 2.14 [37] about H-groups $[X, x_0; K, k_0]$ there is not any expectation or need on rigorous conditions on a class of acceptable M for constructions of wrap groups $(W^M E)_{t,H}$.

If M_1 is an analytic real manifold, then taking its graded product with generators $\{i_0, ..., i_{2^r-1}\}$ of the Cayley-Dickson algebra gives the $\mathcal{A}_r$ manifold (see [19, 17, 18]). Particularly this gives $l2^r$ dimensional torus in $\mathcal{A}_r^l$ for the l dimensional real torus $\mathbf{T}_2 = (S^1)^l$ as M_1.

Consider $\mathbf{T}_2$. It can be slit along a closed curve (loop) C H^∞_p-diffeomorphic with S^1 and marked points $s_{0,q} \in C \subset \mathbf{T}_2$ such that C rotates on the surface of $\mathbf{T}_2 = S^1_R \times S^1_b$ on angle π around S^1_b while C rotates on 2π around S^1_R, such that C rotates on 4π around S^1_R that return to the initial point on C, where $0 < b < R < \infty$, $q = 1, ..., k$, $k \in \mathbf{N}$. Therefore, the slit along C of $\mathbf{T}_2$ is the non-orientable band which inevitably is the Möbius band with twice larger number of marked points $\{s^L_{0,j} : j = 1, ..., 2k\} \subset \partial L$.

Therefore, for $M = \mathbf{T}_2$ as $\hat{M}$ take a quadrant in $\mathbf{R^2}$ with $2k$ pairwise opposite marked points $\hat{s}_{0,q}$ and $\hat{s}_{0,q+k}$ on the boundary of $\hat{M}$, $q = 1, ..., k$, $k \in \mathbf{N}$. Suitable gluing of boundary points in $\partial\hat{M}$ gives the mapping $\Xi : \hat{M} \to \mathbf{T}_2$, $\Xi(\hat{s}_{0,q}) = \Xi(\hat{s}_{0,q+k}) = s_{0,q}$, $q = 1, ..., k$. Proper cutting of $\hat{M}$ into $\hat{A}_j$, $j = 1, 2$, or of L induces that of $\mathbf{T}_2$. Thus we get a pseudo-submanifold $A_3(\mathbf{T}_2) =: A_3 \supset C$, while A_1 and A_2 are retractable into a marked point $s_{0,q} \in C$ for each q, hence $\mathbf{T}_2$ satisfies Conditions $2(i - iii, iv', v)$. In view of Corollary 9 there exists the embedding $\phi^* : (W^{\mathbf{T}_2,\{s_{0,q}:q=1,...,k\}}E)_{t,H} \to (W^{L,\{s^L_{0,q}:q=1,...,2k\}}E)_{t,H}$, where $\phi : L \to \mathbf{T}_2$ is the quotient mapping with $\phi(\{s^L_{0,q}, s^L_{0,q+k}\})\{s_{0,q}\}$, $q = 1, ..., k$.

For the n-dimensional torus $\mathbf{T}_n$ in $\mathcal{A}_r^a$ with $n > 2$ take a $n-1$-dimensional surface B such that each its projection into $\mathbf{T}_2$ is H^t_p-diffeomorphic with C for a loop C as above. Therefore, the slit along B up to a H^t_p-diffeomorphism gives $M_0 := L \times I^{n-2}$ for even n or $M_0 := S^1 \times I^{n-1}$ for odd n, where $I = [0, 1]$. Since I^m is retractable into a point, where $m \ge 1$. Thus we lightly get for $\mathbf{T}_n$ a pseudo-submanifold $A_3 \supset B$ and two A_1 and A_2 retractable into points and satisfying sufficient Conditions $2(i - iii, iv', v)$, where $\hat{M} = I^n$ up to a H^t_p-diffeomorphism, $s_{0,q} \in B \subset A_3 := A_3(\mathbf{T}_n)$, $\{s^{M_0}_{0,q}, s^{M_0}_{0,q+k}\} \subset \partial M_0$, $q = 1, ..., k$, $k \in \mathbf{N}$. Proper cutting of $\hat{M}$ into $\hat{A}_j$, $j = 1, 2$, induces that of $\mathbf{T}_n$. Thus there exists an H^t_p quotient mapping $\phi : M_0 \to \mathbf{T}_n$ with $\phi(\{s^{M_0}_{0,q}, s^{M_0}_{0,q+k}\}) = \{s_{0,q}\}$ and the embedding $\phi^* : (W^{\mathbf{T}_n,\{s_{0,q}:q=1,...,k\}}E)_{t,H} \hookrightarrow (W^{M_0,\{s^{M_0}_{0,q}:q=1,...,2k\}}E)_{t,H}$ due to Corollary 9.

More generally cut from $\mathbf{T}_n$ open subsets V_j which are H^t_p diffeomorphic with interiors of bounded quadrants in $\mathbf{R^n}$ embedded into $\mathcal{A}_r^l$, $j = 1, ..., \beta$, such that $s_{0,q} \in B \cap (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_j)$, $V_j \cap V_i = \emptyset$ for each $j \ne i$, $V_j \cap B = \emptyset$ for each j, where B is defined up to an H^t_p diffeomorhism, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$, that gives the manifold M_2. Then from M_0 cut analogously corresponding $V_{j,b}$, such that $s_{0,q} \in B \cap (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_{j,1})$, $s_{0,q+k} \in B \cap (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_{j,2})$,

$V_{j,b_1} \cap V_{i,b_2} = \emptyset$ for each $j \ne i$ or $b_1 \ne b_2$, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$, $j = 1, ..., \beta$, $b = 1, 2$, that produces the manifold M_1. We choose $V_{j,b}$ such that for the restriction $\phi : M_1 \to M_2$ of the mapping ϕ there is the equality $\phi(V_{j,1} \cup V_{j,2}) = V_j$ for each j, $\phi(\{s_{0,q}^{M_1}, s_{0,q+k}^{M_1}\}) = \{s_{0,q}\}$. This gives the embedding $\phi^* : (W^{M_2, \{s_{0,q}: q=1,...,k\}}E)_{t,H} \hookrightarrow (W^{M_1, \{s_{0,q}^{M_1}: q=1,...,2k\}}E)_{t,H}$.

Another example is M_3 obtained from the previous M_2 with $2k$ marked points and 2β cut out domains V_j, when $s_{0,q}$ is identified with $s_{0,q+k}$ and each ∂V_j is glued with $\partial V_{j+\beta}$ for each $j \in \lambda_q \subset \{d : a_1 + ... + a_{q-1} + 1 \le d \le a_1 + ... + a_q\}$, $q = 1, ..., k$, $k \in \mathbf{N}$, by an equvalence relation υ. Such M_3 is obtained from the torus $\mathbf{T}_{n,m}$ with m holes instead of one hole in the standard torus $\mathbf{T}_{n,1} = \mathbf{T}_n$ cutting from it V_j with $j \in \{1, ..., 2\beta\} \setminus (\bigcup_{q=1,...,k} \lambda_q)$, where $m = m_1 + ... + m_k$, $m_q := card(\lambda_q)$. For $\mathbf{T}_n$ and M_2 the surface B is H_p^t diffeomorphic with $(\partial L) \times I^{n-2}$ for even n or $S^1 \times I^{n-1}$ for odd n. Take $A_3 \supset B \cup (\bigcup_{j \in \lambda_q} \upsilon(\partial V_j))$, it is arcwise connected and contains all marked points. Therefore, M_3 satisfies conditions of §2 and there exists the embedding $\upsilon^* : (W^{M_3, \{s_{0,q}^{M_3}: q=1,...,k\}}E)_{t,H} \hookrightarrow (W^{M_2, \{s_{0,q}^{M_2}: q=1,...,2k\}}E)_{t,H}$. This also induces the embedding $(W^{\mathbf{T}_{n,m}, \{s_{0,q}^{\mathbf{T}_{n,m}}: q=1,...,k\}}E)_{t,H} \hookrightarrow (W^{\mathbf{T}_n, \{s_{0,q}^{\mathbf{T}_n}: q=1,...,2k-1\}}E)_{t,H}$ such that each element $g \in (W^{\mathbf{T}_{n,m}, \{s_{0,q}^{\mathbf{T}_{n,m}}: q=1,...,k\}}E)_{t,H}$ can be presented as a product $g = (..(g_1 g_2)...g_m)$ of m elements $g_j \in (W^{\mathbf{T}_n, \{s_{0,q}^{\mathbf{T}_n}: q=1,...,2k-1\}}E)_{t,H}$, $g_j =< f_j >_{t,H}$, $supp(\pi \circ f_j) \subset B_j$, $B_1 \cup ... \cup B_m = \mathbf{T}_n$, $B_i \cap B_j = \partial B_i \cap \partial B_j$ for each $i \ne j$, each B_j is a canonical closed subset in $\mathbf{T}_n$, $s_{0,1} \in B_1$, $s_{0,2q}, s_{2q+1} \in B_d$ for $m_1 + ... + m_0 + 1 \le d \le m_1 + ... + m_q$, $q = 1, ..., k-1$, where $m_0 := 0$.

Evidently, in the general case for different manifolds M and N wrap groups may be non isomorphic. For example, as M_1 take a sphere S^n of the dimension $n > 1$, as M_2 take $M_1 \setminus K$, where K is up to an H_p^t-diffeomorphism the union of non intersecting interiors B_j of quadrants of diameters $d_1, ..., d_s$ much less, than 1, $K = B_1 \cup ... \cup B_l$, $l \in \mathbf{N}$. Let N be a δ-enlargement for M_2 in $\mathbf{R^{n+1}}$ relative to the metric of the latter Euclidean space, where $0 < \delta < \min(d_1, ..., d_l)/2$. Then the groups $(W^{M_1}N)_{t,H}$ and $(W^{M_2}N)_{t,H}$ are not isomorphic. This lightly follows from the consideration of the element $b :=< f >_{t,H} \in (W^{M_2}N)_{t,H}$, where $f : M_2 \to N$ is the identity embedding induced by the structure of the δ-enlargement.

Recall, that for orientable closed manifolds A and B of the same dimension m the degree of the continuous mapping $f : A \to B$ is defined as an integer number $deg(f) \in \mathbf{Z}$ such that $f_*[A] = deg(f)[B]$, where $[A] \in H_m(A)$ or $[B] \in H_m(B)$ denotes a generator, defined by the orientation of A or B respectively [5]. Consider mappings $f_j : S^n \to N$ such that $V_j \supset \partial B_j \cap N$, where V_j is a domain in $\mathbf{R^{n+1}}$ bounded by the hyper-surface $f_j(B_j)$, f_j is w_0 on each B_i with $i \ne j$, while the degree of the mapping f_j from S^n onto $f_j(S^n)$ is equal to one. If there would be an isomorphism $\theta : (W^{M_2}N)_{t,H} \to (W^{M_1}N)_{t,H}$, then $\theta(b)$ would have a non trivial decomposition into the sum of non canceling non zero additives, which is induced by mappings $f_j : S^n \to N$. Nevertheless, an element b in $(W^{M_2}N)_{t,H}$ has not such decomposition.

If two groups G_1 and G_2 are not isomorphic, then certainly $(W^M E; N, G_1, \mathbf{P})_{t,H}$ and $(W^M E; N, G_2, \mathbf{P})_{t,H}$ are not isomorphic.

The construction of wrap groups can be spread on locally compact non compact M satisfying conditions $2(ii - iv)$ or (ii, iii, iv') changing (v) such that $\hat{M}$ is locally compact non-compact H_p^t-domain in $\mathcal{A}_r^l$, its boundary $\partial\hat{M}$ may happen to be void. For this it is sufficient to restrict the family of functions to that of with compact supports $f : M \to W$ relative to $w_0 : M \to W$, that is $supp_{w_0}(f) := cl_M\{x \in M : f(x) \neq y_0 \times e\}$ is compact, $cl_M A$ denotes the closure of a subset A in M. Then classes of equivalent elements are given with the help of closures of orbits of the group of all H_p^t diffeomorphisms g with compact supports preserving marked points $DifH_{p,c}^t(M, \{s_{0,q} : q = 1, ..., k\})$ that is $supp_{id}(g) := cl_M\{x \in M : g(x) \neq x\}$ are compact, where $id(x) = x$ for each $x \in M$. Then wrap groups $(W^M E)_{t,H}$ for manifolds M such as hyperboloid of one sheet, one sheet of two-sheeted hyperboloid, elliptic hyperboloid, hyperbolic paraboloid and so on in larger dimensional manifolds over $\mathcal{A}_r$. For non compact locally compact manifolds it is possible also consider an infinite countable discrete set of marked points or of isolated singularities. These examples can be naturally generalized for certain knotted manifolds arising from the given above.

Milnor and Lefshetz have used for $M = S^1$ and $G = \{e\}$ the diffeomorphism group preserving an orientation and a marked point of S^1. So their loop group $L(S^1, N)$ may be non-commutative. The iterated loop group $L(S^1, L(S^{n-1}, N))$ is isomorphic with $L(S^n, N)$, where the latter group is supplied with the uniformity from the iterated loop group, so n times iterated loop group of S^1 gives loop group of S^n [4]. For $dim_{\mathbf{R}} M > 1$ orientation preservation loss its significance. Here above it was used the diffeomorphism group without any demands on orientation preservation of M such that two copies of M in the wedge product already are not distinguished in equivalence classes and for commutative G it gives a commutative wrap group.

Mention for comparison homotopy groups. The group $\pi_q(X)$ for a topological space X with a marked point x_0 in view of Proposition 17.1 (b) [2] is commutative for $q > 1$. For $q = 1$ the fundamental group $\pi_1(X)$ may be non-commutative, but it is always commutative in the particular case, when $X = G$ is an arcwise connected topological group (see §49(G) in [32]).

11. Proposition. *Let $L(S^1, N)$ be an H_p^1 loop group in the classical sense. Then the iterated loop group $L(S^1, L(S^1, N))$ is commutative.*

Proof. Consider two elements $a, b \in L(S^1, L(S^1, N))$ and two mappings $f \in a$, $g \in b$, $(f(x))(y) = f(x, y) \in N$, where $x, y \in I = [0, 1] \subset \mathbf{R}$, $e^{2\pi x} \in S^1$. An inverse element d^{-1} of $d \in L(S^1, N)$ is defined as the equivalence class $d^{-1} =< h^- >$, where $h \in d$, $h^-(x) : h(1 - x)$. Then

(1) $f(x, 1-y) = (f(x))(1-y) \in a^{-1}$ and $g(x, 1-y) = (g(x))(1-y) \in b^{-1}$ for $L(S^1, L(S^1, N))$ and symmetrically

(2) $(f(y))(1 - x) = f(1 - x, y) \in a^{-1}$ and $(g(y))(1 - x) = g(1 - x, y) \in b^{-1}$. On the other hand, $f \vee g$ corresponds to ab, and $g \vee f$ corresponds to ba, where the reduced product $S^1 \wedge S^1$ is H_p^t-diffeomorphic with S^2 in the sense of pseudo-manifolds up to critical subsets of codimension not less than two.

Consider $(S^1 \vee S^1) \wedge (S^1 \vee S^1)$ and $(f \vee w_0) \vee (w_0 \vee g)$ and $(g \vee w_0) \vee (w_0 \vee f)$ and the iterated equivalence relation $R_{1,H}$. This situation corresponds to $\hat{M} = I^2$ divided into four quadrats by segments $\{1/2\} \times [0,1]$ and $[0,1] \times \{1/2\}$ with the corresponding domains for f, g and w_0 in the considered wedge products, where $< f \vee w_0 >=< w_0 \vee f >=< f >$ is the same class of equvalent elements.

Since $G = \{e\}$, $(ab)^{-1} = b^{-1}a^{-1}$, then $g(1-x,y) \vee f(1-x,y)$ is in the same class of equivalent elements as $g(x,1-y) \vee f(x,1-y)$. But due to inclusions $(1,2)$ $< g(1-x,y) \vee f(1-x,y) >=< f(x,y) \vee g(x,y) >^{-1}$ and $< f(x,y) \vee g(x,y) >=< g(x,1-y) \vee f(x,1-y) >^{-1}$ and $< h(x,y) >^{-1} =< h(x,1-y) >=< h(1-x,y) >$ for $h \in ab$, consequently, $< h(x,y) >=< h(1-x,1-y) >$ and $< (f \vee g)(x,1-y) >< f(x,1-y) \vee g(x,1-y) > \in (ab)^{-1}$, since $(x,y) \mapsto (1-x,1-y)$ interchange two spheres in the wedge product $S^2 \vee S^2$. Hence $a^{-1}b^{-1} = b^{-1}a^{-1}$ and inevitably $ab = ba$.

12. Theorem. *Let M and N be connected both either C^∞ Riemann or $\mathcal{A}_r$ holomorphic manifolds with corners, where M is compact and $dimM \geq 1$ and $dimN > 1$. Then $(W^M N)_{t,H}$ has no any nontrivial continuous local one parameter subgroup g^b for $b \in (-\epsilon, \epsilon)$ with $\epsilon > 0$.*

Proof. Suppose the contrary, that $\{g^b : b \in (-\epsilon,\epsilon)\}$ with $\epsilon > 0$ is a local nontrivial one parameter subgroup, that is, $g^b \neq e$ for $b \neq 0$. Then to g^δ for a marked $0 < \delta < \epsilon$ there corresponds $f = f_\delta \in H^\infty_p$ such that $< f >_{t,H} = g^\delta$, where $f \in H^t_p$. If $f(U) = \{y_0 \times e\}$ for a sufficiently small connected open neighborhood U of $s_{0,q}$ in M, then there exists a sequence $f \circ \psi_n$ in the equivalence class $< f >_{t,H}$ with a family of diffeomorphisms $\psi_n \in DifH^t_p(M; \{s_{0,q} : q = 1,...,k\})$ such that $\lim_{n\to\infty} diam\psi_n(U) = 0$ and $\bigcap_{n=1}^\infty \psi_n(U) = \{s_{0,q}\}$. If $h(x) \neq y_0$, then in view of the continuity of h there exists an open neighborhood P of x in M such that $y_0 \notin h(P)$. Consider the covariant differentiation ∇ on the manifold M (see [12]). The set S_h of points, where $\nabla^k h$ is discontinuous is a submanifold of codimension not less than one, hence of measure zero relative to the Riemann volume element in M. For others points x in M, $x \in M \setminus S_h$, all $\nabla^k h$ are continuous.

Take then open $V = V(f)$ in M such that $V \supset U$ and $\nabla^k_\nu f|_{\partial V} \neq 0$ for some $k \in \mathbf{N}$, where $\nabla_\nu f(x) := \lim_{z\to x, z\in M\setminus V} \nabla_\nu f(z)$, ν is a normal (perpendicular) to ∂V in M at a point x in the boundary ∂V of V in M. Practically take a minimal $k = k(x)$ with such property. Since M is compact and $\partial V := cl(V) \cap cl(M \setminus V)$ is closed in M, then ∂V is compact. The function $x \mapsto k(x) \in \mathbf{N}$ is continuous, since f and $\nabla^l f$ for each l are continuous. But $\mathbf{N}$ is discrete, hence each $\partial_q V := \{x \in \partial V : k(x) = q\}$ is open in V. Therefore, ∂V is a finite union of $\partial_q V$, $1 \leq q \leq q_m$, where $q_m : \max_{x\in\partial V} k(x) < \infty$ for $f = f_\delta$, since ∂V is compact. Thus, there exists a subset $\lambda \subset \{1,...,q_m\}$ such that $\partial V = \bigcup_{q\in\lambda} \partial_q V$ and $\partial_q V \neq \emptyset$ for each $q \in \lambda$. If $\nabla^l f(x) = 0$ for $l = 1,...,k(x)-1$ and $\nabla^{k(x)} f(x) \neq 0$, then $\nabla^{k(x)} f(\psi(y)) = \nabla^{k(x)}(\psi(y)).(\nabla\psi(y))^{\otimes k(x)} \neq 0$ for $y \in M$ such that $\psi(y) = x$, since $\nabla\psi(y) \neq 0$, where $\psi \in DifH^\infty_p(M; \{s_{0,q} : q = 1,...,k\})$.

We can take $\epsilon > 0$ such that $\{g^b : b \in (-\epsilon,\epsilon)\} \subset U$, where $U = -U$ is a connected symmetric open neighborhood of e in $(W^M N)_{t,H}$. Since $g^{b_1} + g^{b_2} = g^{b_1+b_2}$ for each $b_1, b_2, b_1 + b_2 \in (-\epsilon,\epsilon)$, then $\lim_{t\to 0} g^b = e$ for the local one parameter subgroup and in particular $\lim_{m\to\infty} g^{1/m} = e$, where $m \in \mathbf{N}$. Take $\delta = \delta_m = 1/m$ and $f = f_m \in H^\infty_p$ such that $< f_m >_{t,H} = g^{1/m}$. On the other hand, $jg^{1/m} = g^{j/m}$ for each $j < m\epsilon$, $j \in \mathbf{N}$, hence $f_{j/m}(M) = f_{1/m}(M)$ for each $j < m\epsilon$, since $f \vee h(M \vee M) = f(M) \vee h(M)$ and using embedding η of $(S^M N)_{t,H}$ into $(W^M N)_{t,H}$.

The function $|\nabla_\nu^{k(x)} f_\delta(x)|$ for $x \in \partial V$ is continuous by δ due to the Sobolev embedding theorem [25], $0 < \delta < \epsilon$, consequently, $\inf_{x \in \partial V} |\nabla_\nu^{k(x)} f_\delta(x)| > 0$, since ∂V is compact. We can choose a family f_δ such that $z^{(l)}(\delta, x) := \nabla^l f_\delta(x)$ is continuous for each $0 \le l \le k_0$ by $(\delta, x) \in (-\epsilon, \epsilon) \times M$, since $\{g^b : b \in (-\epsilon, \epsilon)\}$ is the continuous by b one parameter subgroup, where $k_0 := q_m(\delta_0)$. Therefore, for this family there exists a neighborhood $[-\epsilon + c, \epsilon - c]$ such that $\delta_0 \in [-\epsilon + c, \epsilon - c] \subset (-\epsilon, \epsilon)$ with $0 < c < \epsilon/3$ such that $q_m(\delta) \le k_0$ for each $\delta \in [-\epsilon + c, \epsilon - c]$ with a suitable choice of $V(f_\delta)$, since $\mathbf{N}$ is discrete. On the other hand, $\sup_{x \in \partial V(f_\delta), 0 < \delta \le \epsilon - c} |\nabla_\nu^{k(x)} f_\delta(x)| \le \sup_{x \in M, 0 < \delta \le \epsilon - c} |\nabla_\nu^{k(x)} f_\delta(x)| =: B < \infty$, since M and $[-\epsilon + c, \epsilon - c]$ are compact.

Therefore, for this family there exists a neighborhood $[-\epsilon+c, \epsilon-c]$ such that $\delta_0 \in [-\epsilon+c, \epsilon-c] \subset (-\epsilon, \epsilon)$ with $0 < c < \epsilon/3$ such that $q_m(\delta) \le k_0$ for each $\delta \in [-\epsilon + c, \epsilon - c]$ with a suitable choice of $V(f_\delta)$, since $\mathbf{N}$ is discrete.

Then $\underline{\lim}_{\delta \to 0, \delta > 0} |\nabla_\nu^{k(x)} f_\delta(x)| =: b > 0$ for $x \in \partial V$ with a suitable choice of $V = V(f_\delta)$, since M is connected, $dim M \ge 1$ and $\inf_{m \in \mathbf{N}} diam f_{j/m}(M) > 0$ for a marked $\delta_0 = j/m_0 < \epsilon$ with $j, m > m_0 \in \mathbf{N}$ mutually prime, $(j, m) = 1$, $(j, m_0) = 1$. To $< f_{l/m} >_{t,H}$ there corresponds $< f_{1/m} >_{t,H} \vee ... \vee < f_{1/m} >_{t,H} =: < f_{1/m} >_{t,H}^{\vee l}$ which is the l-fold wedge product. Thus there exists $C = const > 0$ for M such that $|\nabla_\nu^{k(x)} f_{l/m}(x)| \ge Cl \inf_{y \in \partial V(f_{1/m})} |\nabla_\nu^{k(y)} f_{1/m}(y)| \ge Clb$, where $C > 0$ is fixed for a chosen atlas $At(M)$ with given transition mappings $\phi_i \circ \phi_j^{-1}$ of charts.

Consider $\delta_0 \le l/m < \epsilon - c$ and m and l tending to the infinity. Then this gives $B \ge Clb$ for each $l \in \mathbf{N}$, that is the contradictory inequality, hence $(W^M N)_{t,H}$ does not contain any non trivial local one parameter subgroup.

Received: January 2008. Revised: August 2008.

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