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On mapping properties of monogenic functions

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ABSTRACT

Main goal of this paper is to study the description of monogenic functions by their geometric mapping properties. At first monogenic functions are studied as general quasi-conformal mappings. Moreover, dilatations and distortions of these mappings are estimated in terms of the hypercomplex derivative. Then pointwise estimates from below and from above are given by using a generalized Bohr's theorem and a Borel-Carathéodory theorem for monogenic functions. Finally it will be shown that monogenic functions can be defined as mappings which map infinitesimal balls to special ellipsoids.

RESUMEN

El principal objetivo de este artículo es estudiar la descripción de funciones monogénicas a través de las propiedades geométricas de sus mapeos. Primero son estudiadas funciones monogénicas como aplicaciones casi-conformes generales. Además, dilataciones y distorciones de estas aplicaciones son estimadas en términos de la derivada hipercompleja. Entonces estimativas puntuales por abajo y por arriba son dadas usando un teorema de Bohr generalizado y un Teorema de Borel-Carathéodory para funciones monogénicas. Finalmente es demostrado que funciones monogénicas pueden ser definidas como mapeos que aplican bolas infinitesimales en elipsoides especiales.

Key words and phrases: *monogenic functions, quasi-conformal mappings, geometric mapping properties.*

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1 Introduction

Quaternionic analysis provides us with a spatial analogue of the one-dimensional complex function theory in the plane. Generalizing ideas from the complex case to the higher dimensional real Euclidean space, quaternionic analysis is applied to construct solutions of some classes of partial differential equations for vector-valued functions in three or four dimensions (see, e.g., [17, 18, 31, 32]), involving the Dirac operator or a generalized Cauchy-Riemann operator, respectively.

We are mainly interested in the study of spatial generalizations of holomorphic or anti-holomorphic functions. The crucial fact of such functions (transformations) is that they describe essentially mappings of the unit disk onto or into the unit disk transforming partial differential equations to other differential equations and moreover, they embrace Cauchy's inequalities, maximum modulus principle, Bohr's Theorem, Schwarz Lemma, Hadamard Theorem and others. Since they provide with the best description of the pointwise behavior from a given function, at first one has to ask whether or not those results can be generalized to "holomorphic" (resp. anti-holomorphic) functions in higher dimensions (sections 3 and 4). In addition, to deal with these results it is also necessary to study the mapping properties of such functions more detailed. It is already known, as the ideas of [7] and [5] show, that these functions preserve geometrical properties like length, distance and angles, while mapping domains to the ball. Therefore they are applied as well for the transformation of differential equations. Our main task is to characterize such mappings which map technically relevant domains to mathematically simple domains and to find out if such class of functions leave the differential equations and/or some geometrical properties invariant.

At this point it is of interest to know that in contrast to the situation in the plane, the set of conformal mappings is restricted only to the set of Möbius transformations. But the theory of generalized holomorphic functions (by historical reasons they are also called monogenic functions, cf. [8]) does not cover the set of Möbius tranformations in $\mathbb{R}^{n+1}$, and since the Möbius transformations are not monogenic, one can only expect that monogenic functions represent certain quasi-conformal mappings. On the other hand, the class of all quasi-conformal mappings is much bigger than the class of monogenic functions. The question arises if monogenic functions correspond to a special subclass of quasi-conformal mappings.

In the paper [28], the concept of monogenic-conformal mappings realized by functions in $\mathbb{R}^{n+1}$ and with values in the Clifford algebra $Cl_{0,n}$ was already considered. Together with the geometric interpretation of the hypercomplex derivative (see [17]), dilatations and distortions of these mappings can be estimated. Compared with the related work, the advantage of our approach lies in the possibility to study the description of monogenic functions by their geometric mapping properties. The local mapping properties of a monogenic function or of a real analytic function are mainly determined by the behaviour of the linear part of their Taylor expansions. According to this, in [23] the geometric behaviour of the linear part of a monogenic function was considered.

As a consequence, it is shown that monogenic functions can be defined as mappings which map infinitesimal balls to special ellipsoids and vice versa. Here we extend this result considering the all series expansion of a function.

The text is based on recent publications of the authors [19, 20, 21, 22, 23] and contains new material as well.

2 Preliminaries

Let $\mathbb{H} := \{\mathbf{a} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3,\ \ a_i \in \mathbb{R},\ i = 0,1,2,3\}$ be the algebra of the real quaternions, where the imaginary units $\mathbf{e}_i$ $(i = 1,2,3)$ are subject to the multiplication rules

$$\begin{aligned}&\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1,\\ &\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3 = -\mathbf{e}_2\mathbf{e}_1,\ \mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1 = -\mathbf{e}_3\mathbf{e}_2,\ \mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_3.\end{aligned}$$

Through this paper we shall denote by $\mathbf{Sc}(\mathbf{a}) := a_0$ the scalar part of $\mathbf{a}$ and by $\mathbf{Vec}(\mathbf{a}) := a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ its vector part. Analogously to the complex case, the (quaternion-)conjugate element of $\mathbf{a}$ is the quaternion

$$\overline{\mathbf{a}} := \mathbf{Sc}(\mathbf{a}) - \mathbf{Vec}(\mathbf{a}) = a_0 - a_1\mathbf{e}_1 - a_2\mathbf{e}_2 - a_3\mathbf{e}_3.$$

Also, we shall use the Euclidean norm $|\mathbf{a}|^2 = \mathbf{a}\overline{\mathbf{a}} = \left(a_0^2 + a_1^2 + a_2^2 + a_3^2\right)^{1/2}$. The real vector space $\mathbb{R}^3$ is to be embedded in the subset $\mathcal{A} := span_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{H}$ via the identification of each element $\mathbf{x} = (x_0, \underline{x}) = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the paravector (also called reduced quaternion)

$$\mathbf{x} := x_0 + \underline{x} = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathcal{A}.$$

As a consequence, no distinction will be made between $\mathbf{x}$ as a point in $\mathbb{R}^3$ or its correspondent reduced quaternion. Also, we emphasize that $\mathcal{A}$ is only a real vector space but not a subalgebra of $\mathbb{H}$. For more details on the real algebra of quaternions we refer e.g. to [8], [31], [18], [15].

Let now Ω be an open subset of $\mathbb{R}^3$ with piecewise smooth boundary. A quaternion-valued function or, briefly, an $\mathbb{H}$-valued function is a mapping $f : \Omega \longrightarrow \mathbb{H}$ such that

$$f(\mathbf{x}) = \sum_{i=0}^{3} f_i(\mathbf{x})\mathbf{e}_i,$$

where $\mathbf{e}_0 = 1$ and the coordinate-functions f_i $(i = 0,1,2,3)$ are real-valued in Ω. Properties such as continuity, differentiability or integrability are ascribed coordinate-wisely.

For continuously real-differentiable functions $f : \Omega \longrightarrow \mathbb{H}$, the operator

$$D = \partial_{x_0} + \mathbf{e}_1\partial_{x_1} + \mathbf{e}_2\partial_{x_2} \tag{1}$$

is called generalized Cauchy-Riemann operator. The conjugate generalized Cauchy-Riemann operator is defined by

$$\overline{D} = \partial_{x_0} - \mathbf{e}_1 \partial_{x_1} - \mathbf{e}_2 \partial_{x_2}. \tag{2}$$

A function $f : \Omega \longrightarrow \mathbb{H}$ is called *left* (resp. *right*) *monogenic* in Ω if

$$Df = 0 \ \text{ in } \ \Omega \ \ (\text{resp.}, fD = 0 \ \text{ in } \ \Omega).$$

Remark 1. *In general, left (resp. right) monogenic functions are not right (resp. left) monogenic. From now on, we refer only to left monogenic functions. For simplicity, we will call them monogenic. However, all results achieved to left monogenic functions can easily be adapted to right monogenic functions.*

The generalized Cauchy-Riemann operator (1) and its conjugate (2) factorize the Laplace operator in $\mathbb{R}^3$. In fact, it holds

$$\Delta_3 f = D\overline{D}f = \overline{D}Df,$$

which implies that any monogenic function is also a harmonic function. Analogously as in the complex one-dimensional case $\frac{1}{2}\bar{D}$ defines a derivative of monogenic functions. This was shown in [16], where $\frac{1}{2}\bar{D}f$ was called *hypercomplex derivative* of f.

A monogenic function $f : \Omega \longrightarrow \mathbb{H}$ with an identically vanishing hypercomplex derivative (i.e. a function from $\ker D \cap \ker \bar{D}$) is called hyperholomorphic constant (see again [16]). It is immediately clear that such function depends only on x_1 and x_2.

Additionally, we introduce the following notations: $B_R := B_R(0)$ will denote the ball of radius R in $\mathbb{R}^3$ centered at the origin, $S_R = \partial B_R$ its boundary and $d\sigma_R$ (resp. dV_R) the Lebesgue measure on S_R (resp. B_R). For simplicity, in the case $R = 1$ we omit R in the notations. We will also denote by $L_2(S_R; \mathbb{X}; \mathbb{R})$ (resp. $L_2(B_R; \mathbb{X}; \mathbb{R})$) the $\mathbb{R}$-linear Hilbert space of square integrable functions on S_R (resp. B_R) with values in $\mathbb{X}$ ($\mathbb{X} = \mathbb{R}$ or $\mathcal{A}$). In the case $\mathbb{X} = \mathbb{R}$ we abbreviate $L_2(S_R; \mathbb{R}; \mathbb{R})$ (resp. $L_2(B_R; \mathbb{R}; \mathbb{R})$) briefly by $L_2(S_R)$ (resp. $L_2(B_R)$). Also, the real-valued inner product in $L_2(S_R; \mathcal{A}; \mathbb{R})$ (resp. $L_2(B_R; \mathcal{A}; \mathbb{R})$) is given by

$$\langle f, g \rangle_{L_2(S_R;\mathcal{A};\mathbb{R})} = \int_{S_R} \mathbf{Sc}(\overline{f}g) d\sigma_R, \tag{3}$$

respectively,

$$\langle f, g \rangle_{L_2(B_R;\mathcal{A};\mathbb{R})} = \int_{B_R} \mathbf{Sc}(\overline{f}g) dV_R, \tag{4}$$

for any $f, g \in L_2(S_R; \mathcal{A}; \mathbb{R})$ (resp. $L_2(B_R; \mathcal{A}; \mathbb{R})$). Each homogeneous harmonic polynomial P_n of degree n can be written in spherical coordinates as

$$P_n(x) = R^n P_n(\omega), \ \omega \in S_R, \tag{5}$$

its restriction, $P_n(\omega)$, to the boundary of the ball B_R is called *spherical harmonic*[1] of degree n. From (5), it is clear that a homogeneous polynomial is determined by its restriction to S_R. Denoting by $\mathcal{H}_n(S_R)$ the space of real-valued spherical harmonics of degree n on S_R, it is well-known (see [2] and [29]) that

$$\dim \mathcal{H}_n(S_R) = 2n + 1.$$

It is also known (see [2] and [29]) if $n \neq m$, the spaces $\mathcal{H}_n(S_R)$ and $\mathcal{H}_m(S_R)$ are orthogonal in $L_2(S_R)$.

Let us denote the homogeneous monogenic polynomials of degree n by H_n. In an analogous way to the spherical harmonics, the restriction of H_n to the boundary of the ball B_R is called *spherical monogenic*[2] of degree n. Now let $M^+(\Omega; \mathcal{A}; n)$ be the space of $\mathcal{A}$-valued homogeneous monogenic polynomials of degree n in $\Omega \subset \mathbb{R}^3$. In [27], it is shown that the space $M^+(\Omega; \mathcal{A}; n)$ has dimension $2n + 3$. Later, this result was generalized for arbitrary higher dimensions by R. Delanghe in [12].

Consider, for each $n \in \mathbb{N}_0$, a basis $\{H_n^\nu : \nu 1, ..., \dim M^+(\Omega; \mathcal{A}; n)\}$ of $M^+(\Omega; \mathcal{A}; n)$. Since the coordinates of H_n^ν are harmonic, for arbitrary $n, k = 0, 1, ...$, we have

$$\|H_n^\nu\|^2_{L_2(B_R;\mathcal{A};\mathbb{R})} = \frac{R^{2n+3}}{2n+3} \|H_n^\nu\|^2_{L_2(S;\mathcal{A};\mathbb{R})}. \tag{6}$$

3 Homogeneous Monogenic Polynomials

In [9] and [10], a special $\mathbb{R}$-linear complete orthonormal system of $\mathcal{A}$-valued homogeneous monogenic polynomials defined in the unit ball of $\mathbb{R}^3$ is explicitly constructed. The main idea of this construction is based on the already referred factorization of the Laplace operator. The authors took a system of real-valued homogeneous harmonic polynomials and applied the $\overline{D}$ operator in order to obtain a system of $\mathcal{A}$-valued homogeneous monogenic polynomials.

For an easier description, we introduce spherical coordinates

$$x_0 = r\cos\theta,\ x_1 = r\sin\theta\cos\varphi,\ x_2 = r\sin\theta\sin\varphi,$$

where $0 < r < \infty$, $0 < \theta \leq \pi$, $0 < \varphi \leq 2\pi$. Each point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}$ admits a unique representation $\mathbf{x} = r\boldsymbol{\omega}$, where $r|\mathbf{x}|$ and $|\boldsymbol{\omega}| = 1$. Therefore, $\omega_i = \frac{x_i}{r}$ for $i = 0, 1, 2$.

As described, the homogeneous monogenic polynomials (solid spherical monogenics)

$$\{X_n^{0,\dagger},\ X_n^{m,\dagger},\ Y_n^{m,\dagger} : m = 1, ..., n+1\}, \tag{7}$$

[1] This restriction is also called surface spherical harmonic by some authors (see [8]).
[2] Such restriction is also called surface inner spherical monogenic (see [8]).

formed by the extensions to the ball of $\{X_n^0, X_n^m, Y_n^m : m = 1, ..., n+1\}$ are obtained by applying the operator $\frac{1}{2}\overline{D}$ to the system of homogeneous harmonic polynomials

$$U_{n+1}^{0,\dagger}, U_{n+1}^{m,\dagger}, V_{n+1}^{m,\dagger}, \quad m = 1, \ldots, n+1,$$

with the notations

$$\begin{array}{rcl} U_{n+1}^{0,\dagger} & := & r^{n+1}U_{n+1}^{0} = r^{n+1}P_{n+1}(\cos\theta) \\ U_{n+1}^{m,\dagger} & := & r^{n+1}U_{n+1}^{m} = r^{n+1}P_{n+1}^{m}(\cos\theta)\cos m\varphi \\ V_{n+1}^{m,\dagger} & := & r^{n+1}V_{n+1}^{m} = r^{n+1}P_{n+1}^{m}(\cos\theta)\sin m\varphi, \ m = 1, ..., n+1. \end{array}$$

Hereby P_{n+1} stands for the Legendre polynomial of degree $n+1$ and the functions P_{n+1}^m are the *associated Legendre functions*[3]. The set $\{U_{n+1}^0, U_{n+1}^m, V_{n+1}^m : m = 1, \ldots, n+1\}$ denotes the standard orthogonal basis of spherical harmonics of degree $n+1$ in $\mathbb{R}^3$ (considered, e.g., in [34]) with respect to the inner product

$$\langle f, g \rangle_{L_2(S)} = \int_S f g \, d\sigma.$$

Moreover, their norms are given by

$$\begin{array}{rcl} \| U_{n+1}^0 \|_{L_2(S)} & = & 2\sqrt{\dfrac{\pi}{2n+3}} \\ \| U_{n+1}^m \|_{L_2(S)} & = & \| V_{n+1}^m \|_{L_2(S)} = \sqrt{\dfrac{2\pi}{2n+3}\dfrac{(n+1+m)!}{(n+1-m)!}}. \end{array}$$

We begin by considering the following norm estimates already obtained in [9] which will be used later on.

Proposition 1. *For a given fixed $n \in \mathbb{N}_0$, the spherical monogenics X_n^0, X_n^m and Y_n^m ($m = 1, \ldots, n+1$) are orthogonal to each other with respect to the inner product* (3) *and their norms are given by*

$$\begin{array}{rcl} \| X_n^0 \|_{L_2(S;\mathcal{A};\mathbb{R})} & = & \sqrt{\pi(n+1)} \\ \| X_n^m \|_{L_2(S;\mathcal{A};\mathbb{R})} & = & \| Y_n^m \|_{L_2(S;\mathcal{A};\mathbb{R})} = \sqrt{\dfrac{\pi}{2}(n+1)\dfrac{(n+1+m)!}{(n+1-m)!}}. \end{array}$$

Remark 2. *A similar result can be obtained for the homogeneous monogenic polynomials* (7) *if one takes into account relation* (6).

[3]These functions were introduced in 1877 by Ferrers. For that reason, some authors (c.f. [34]) call them *Ferrers functions*.

In the second and third sections we will look more closely to the pointwise behavior of a given function. For that reason in what follows we present pointwise estimates of our basis polynomials (7), already obtained by the authors in [21].

Proposition 2. *Let $n \in \mathbb{N}_0$. For the homogeneous monogenic polynomials* (7) *the following estimates hold:*

$$\begin{aligned}
|X_n^{0,\dagger}(\mathbf{x})| &\leq \frac{1}{2\sqrt{\pi}}(n+1)(2r)^n \parallel X_n^0 \parallel_{L_2(S;\mathcal{A};\mathbb{R})} \\
|X_n^{m,\dagger}(\mathbf{x})| &\leq \frac{1}{2\sqrt{\pi}}(n+1)(2r)^n \parallel X_n^m \parallel_{L_2(S;\mathcal{A};\mathbb{R})} \\
|Y_n^{m,\dagger}(\mathbf{x})| &\leq \frac{1}{2\sqrt{\pi}}(n+1)(2r)^n \parallel Y_n^m \parallel_{L_2(S;\mathcal{A};\mathbb{R})},
\end{aligned}$$

with $m = 1, \ldots, n+1$.

An interesting point to note here is that the real part of these polynomials are again related with the set $\{U_n^0, U_n^m, V_n^m : m = 1, \ldots, n\}$. These relations are given in the next theorem:

Theorem 1. *Given a fixed $n \in \mathbb{N}_0$, we have the following relations:*

$$\begin{aligned}
\mathbf{Sc}(X_n^0) &= \frac{(n+1)}{2}\ U_n^0(\theta,\varphi) \\
\mathbf{Sc}(X_n^m) &= \frac{(n+m+1)}{2}\ U_n^m(\theta,\varphi) \\
\mathbf{Sc}(Y_n^m) &= \frac{(n+m+1)}{2}\ V_n^m(\theta,\varphi),
\end{aligned}$$

for $m = 1, \ldots, n$.

Proof. We just prove the relation for the spherical harmonics $\mathbf{Sc}(X_n^l)$ $(l = 0, \ldots, n)$. The proof for $\mathbf{Sc}(Y_n^m)$ $(m = 1, \ldots, n)$ is similar. Taking results from [9], the real part of the spherical monogenics X_n^l $(l = 0, ..., n)$ is given by

$$\mathbf{Sc}(X_n^l) = A^{l,n}(\theta)\cos(l\varphi)$$

with

$$A^{l,n}(\theta) = \frac{1}{2}\left(\sin^2\theta \frac{d}{dt}\left[P_{n+1}^l(t)\right]_{t=\cos\theta} + (n+1)\cos\theta P_{n+1}^l(\cos\theta)\right).$$

It is well known that the Legendre polynomials, together with the associated Legendre functions, satisfy in particular the recurrence formula

$$(1-t^2)(P_{n+1}^l(t))' = (n+l+1)P_n^l(t) - (n+1)tP_{n+1}^l(t), \tag{8}$$

for $l = 0, \ldots, n+1$. Now, making the change of variable $\cos\theta = t$ in $A^{l,n}(\theta)$ and using the previous recurrence formula, it follows immediately that

$$\begin{aligned} A^{l,n}(\arccos t) &= \frac{1}{2}\left[(1-t^2)(P^l_{n+1}(t))' + (n+1)tP^l_{n+1}(t)\right] \\ &= \frac{(n+l+1)}{2}P^l_n(t). \end{aligned}$$

Making again a change of variable $t = \cos\theta$ our statement is proved. □

As a consequence, it turns out the result:

Theorem 2. *For a fixed $n \in \mathbb{N}_0$, the spherical harmonics*

$$\left\{\mathbf{Sc}(X^0_n), \mathbf{Sc}(X^m_n), \mathbf{Sc}(Y^m_n) : \ m = 1, \ldots, n\right\}$$

are orthogonal in $L_2(S)$.

With such relations together with the norms of the spherical harmonics, we are ready to establish, as well, the L_2-norms of $\mathbf{Sc}(X^0_n)$, $\mathbf{Sc}(X^m_n)$ and $\mathbf{Sc}(Y^m_n)$ $(m = 1, ..., n)$.

Proposition 3. *For a fixed $n \in \mathbb{N}_0$, the norms of the spherical harmonics $\mathbf{Sc}(X^0_n)$, $\mathbf{Sc}(X^m_n)$ and $\mathbf{Sc}(Y^m_n)$ are given by*

$$\|\mathbf{Sc}(X^0_n)\|_{L_2(S)} = (n+1)\sqrt{\frac{\pi}{2n+1}}$$

and

$$\|\mathbf{Sc}(X^m_n)\|_{L_2(S)} = \|\mathbf{Sc}(Y^m_n)\|_{L_2(S)}(n+1+m)\sqrt{\frac{\pi}{2}\frac{1}{(2n+1)}\frac{(n+m)!}{(n-m)!}},$$

for $m = 1, \ldots, n$.

Remark 3. *Using a different measure, we have established Theorem 2 and the previous proposition already in [21].*

For future use we need also the next results:

Proposition 4. *Given a fixed $n \in \mathbb{N}_0$, the spherical harmonics*

$$\left\{\mathbf{Sc}(X^{n+1}_n \mathbf{e}_i), \mathbf{Sc}(Y^{n+1}_n \mathbf{e}_i) : i = 1, 2\right\}$$

are orthogonal to each other with respect to the inner product (3) and their norms are given by

$$\|\mathbf{Sc}(X^{n+1}_n \mathbf{e}_i)\|_{L_2(S)} = \|\mathbf{Sc}(Y^{n+1}_n \mathbf{e}_i)\|_{L_2(S)} = \frac{1}{2}\sqrt{\pi(n+1)(2n+2)!}.$$

The case $i = 1$ was already studied by the authors in [20]. For $i = 2$ the proof is similar.

Remark 4. *The orthogonality is ensured if one takes into account the following representations* ([9], *Proposition* 3.4.3)

$$\begin{aligned} X_n^{n+1} &= -C^{n+1,n}\cos(n\varphi)\mathbf{e}_1 + C^{n+1,n}\sin(n\varphi)\mathbf{e}_2 \\ Y_n^{n+1} &= -C^{n+1,n}\sin(n\varphi)\mathbf{e}_1 - C^{n+1,n}\cos(n\varphi)\mathbf{e}_2. \end{aligned} \tag{9}$$

Since some of our further results are not only restricted to the unit ball, from now on we represent by $X_n^{0,\dagger,*_R}, X_n^{m,\dagger,*_R}, Y_n^{m,\dagger,*_R}$ $(m = 1, ..., n+1)$ the normalized basis functions $X_n^{0,\dagger}, X_n^{m,\dagger}, Y_n^{m,\dagger}$ in $L_2(B_R; \mathcal{A}; \mathbb{R})$. Based on these functions, in [9] and [10] the following orthonormal basis is constructed, therein restricted to the unit ball.

Theorem 3. *For each n, the set of $2n + 3$ homogeneous monogenic polynomials*

$$\left\{X_n^{0,\dagger,*_R}, X_n^{m,\dagger,*_R}, Y_n^{m,\dagger,*_R},\ m1, ..., n+1\right\} \tag{10}$$

forms an orthonormal basis in $M^+(B_R; \mathcal{A}; n)$ with respect to the inner product (4).

Remark 5. *The estimates stated in Proposition 2 are still valid for this new system of polynomials* (10). *In particular, taking into account relation* (6) *it follows:*

$$|X_n^{0,\dagger,*_R}(\mathbf{x})| = \sqrt{\frac{2n+3}{R^{2n+3}}}\frac{|X_n^{0,\dagger}(\mathbf{x})|}{\|X_n^{0,\dagger}\|_{L_2(S;\mathcal{A};\mathbb{R})}}$$

and moreover, from Proposition 2

$$|X_n^{0,\dagger}(\mathbf{x})| = \left|R^n X_n^{0,\dagger}\left(\frac{\mathbf{x}}{R}\right)\right| \leq R^n \frac{1}{2\sqrt{\pi}}(n+1)2^n \left|\frac{\mathbf{x}}{R}\right|^n \parallel X_n^0 \parallel_{L_2(S;\mathcal{A};\mathbb{R})}$$

for $0 < |\mathbf{x}| = r < R$.

Theorem 3 makes it possible to define the Fourier expansion of a square integrable $\mathcal{A}$-valued monogenic function in $L_2(B_R)$. Moreover, each monogenic function can be decomposed in an orthogonal sum of a monogenic "main part" (g) of the function and a hyperholomorphic constant (h). More precisely, it holds:

Lemma 1. *A monogenic L_2-function $f : \Omega \subset \mathbb{R}^3 \longrightarrow \mathcal{A}$ can be decomposed into*

$$f = f(\mathbf{0}) + g + h, \tag{11}$$

where the functions g and h have Fourier series

$$\begin{aligned} g(\mathbf{x}) &= \sum_{n=1}^{\infty}\left(X_n^{0,\dagger,*_R}(\mathbf{x})\alpha_n^0 + \sum_{m=1}^{n}\left[X_n^{m,\dagger,*_R}(\mathbf{x})\alpha_n^m + Y_n^{m,\dagger,*_R}(\mathbf{x})\beta_n^m\right]\right) \\ h(\mathbf{x}) &= \sum_{n=1}^{\infty}\left[X_n^{n+1,\dagger,*_R}(\mathbf{x})\alpha_n^{n+1} + Y_n^{n+1,\dagger,*_R}(\mathbf{x})\beta_n^{n+1}\right]. \end{aligned}$$

The associated Fourier coefficients $\alpha_n^0, \alpha_n^m, \beta_n^m$ $(m = 1, ..., n+1)$ *are real-valued.*

4 Bohr's Theorem for monogenic functions

During the last years the standard Bohr's phenomena attracted a lot of attention. In 1914, H. Bohr discovered that there exists a radius $r \in (0,1)$ such that if a power series of a holomorphic function converges in the unit disk and its sum has a modulus less than 1, then for $|z| < r$ the sum of the absolute values of its terms is again less than 1. The significance of the theorem is that such radius does not depend on the function.[4] To be more precise, the classical Bohr's Theorem says that:

Theorem 4. [6] *Let f be a bounded analytic function in the open unit disk, with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ convergent in the unit disk and with modulus less than 1. Then $\sum_{n=0}^{\infty} |a_n| r^n < 1$ for $0 \leq r < \frac{1}{3}$.*

This result, known as Bohr's inequality, is true for $0 < r < \frac{1}{3}$ and the constant $\frac{1}{3}$ cannot be improved, that is, the inequality fails for any $r \geq \frac{1}{3}$. Originally, this theorem was proved for $0 \leq r < \frac{1}{6}$, but soon improved to the sharp result by M. Riesz, I. Schur, and N. Wiener independently. In Bohr's paper [6] his own proof was published as well as a proof by Wiener based on function theory methods. Later, S. Sidon gave a different proof [35], which was subsequently rediscovered by M. Tomić [36].

Recently, multi-dimensional analogues and other generalizations of Bohr's theorem are treated by several mathematicians such as Aizenberg [1], Beneteau, Dahlner and Khavinson [3], Boas and Khavinson [4], Dineen and Timoney [14], Paulsen, Popescu and Singh [30], and many others. In several of these papers, the proof of Bohr's inequality or of Bohr-type inequalities, respectively, in the theory of holomorphic functions of one or n variables is based on the orthogonality of the powers of the complex variable(s). To use similar ideas in the quaternionic case, it seems to be natural to work with the Fourier expansion of monogenic functions. It is not so simple as in the complex case to switch between the Taylor expansion and the Fourier series of a function. The reason for this is that the Taylor expansion with respect to the Fueter variables does not give us orthogonal summands.

It should be also remarked that in some papers (see, e.g., [30]) the idea is to work with the Fourier coefficients of the boundary values of a holomorphic function. We prefer here to consider (analogously to the original formulation of Bohr's theorem) only monogenic functions in the ball. It follows directly from the supposed boundedness of the monogenic functions that they are also square integrable in the ball and therefore we can work with Fourier series there. The existence of integrable boundary values needs additional assumptions.

[4]For the physicists, the notion "Bohr radius" is associated to Niels Bohr, the founder of the quantum theory and winner of the Nobel Prize in Physics in 1922.

In the remainder of this section, we collect generalizations and different modifications of this theorem (see [19, 21]) and we show that the result can be extended to the all class of monogenic functions with $|f(\mathbf{x})| < 1$ in B. In [19], the first version of a quaternionic Bohr type theorem was obtained, therein restricted to the case of functions with $f(\mathbf{0}) = \mathbf{0}$.

Theorem 5. [19] *Let f be a square integrable $\mathcal{A}$-valued monogenic function with $f(\mathbf{0}) = \mathbf{0}$ and $|f(\mathbf{x})| < 1$ in B and let*

$$\sum_{n=1}^{\infty} \left[X_n^{0,\dagger,*}\alpha_n^0 + \sum_{m=1}^{n+1} \left(X_n^{m,\dagger,*}\alpha_n^m + Y_n^{m,\dagger,*}\beta_n^m \right) \right]$$

be its Fourier expansion. Then

$$\sum_{n=1}^{\infty} \left| X_n^{0,\dagger,*}\alpha_n^0 + \sum_{m=1}^{n+1} \left(X_n^{m,\dagger,*}\alpha_n^m + Y_n^{m,\dagger,*}\beta_n^m \right) \right| < 1$$

holds in the ball $\{\mathbf{x} : |\mathbf{x}| < 0.047\}$.

This result is adapted very well to the complex situation. The absolute value is taken from all summands of the same degree n. In the complex case this is also a first important step. All the considered functions with $f(0) = 0$ are orthogonal to the constants. This is used later on to estimate all Fourier coefficients of a general holomorphic function with $|f(z)| \leq 1$ by the first Fourier coefficient (see, e.g., [30]).

However, it is important to remark that in the quaternionic context, the set of "constants" is much bigger. Hereby constants are also monogenic functions which have an identically vanishing hypercomplex derivative. Then it is immediately clear that the constant function and all monogenic functions which depend only on x_1 and x_2 are the so called *hyperholomorphic constants*. Moreover, if we, as in this paper, consider only $\mathcal{A}$-valued functions then a non-trivial hyperholomorphic constant must have values in $\mathrm{span}_{\mathbb{R}}\{\mathbf{e_1}, \mathbf{e_2}\}$. With these observations it seems to be natural to study at first the class of functions which are orthogonal to the non-trivial hyperholomorphic constants in $L_2(B; \mathcal{A}; \mathbb{R})$ with $|f(\mathbf{x})| < 1$ in B (see [21]). This approach is also supported by the fact that in Lemma 1 it is shown that each monogenic function can be decomposed in an orthogonal sum of a monogenic "main part" of the function and a hyperholomorphic constant. Remind that an orthonormal basis of the subspace of hyperholomorphic constants is given by the set $\{X_n^{n+1,\dagger}, Y_n^{n+1,\dagger}\}_{n=0}^{\infty}$.

The Fourier representation in the hypothesis of the next Theorem describes the general form of these main parts. The non-trivial hyperholomorphic constants in the decomposition do not influence the real part of the function at the origin because their image lies in $\mathrm{span}_{\mathbb{R}}\{\mathbf{e_1}, \mathbf{e_2}\}$.

Theorem 6. [21] Let f be an $\mathcal{A}$-valued monogenic function such that $f(\mathbf{x}) - f(\mathbf{0})$ is orthogonal to the hyperholomorphic constants with respect to the inner product (4) with $|f(\mathbf{x})| < 1$ in B and let

$$\sum_{n=0}^{\infty}\left[X_n^{0,\dagger,*}\alpha_n^0+\sum_{m=1}^{n}\left(X_n^{m,\dagger,*}\alpha_n^m+Y_n^{m,\dagger,*}\beta_n^m\right)\right]$$

be its Fourier expansion. Then

$$\sum_{n=0}^{\infty}\left[|X_n^{0,\dagger,*}||\alpha_n^0|+\sum_{m=1}^{n}\left(|X_n^{m,\dagger,*}||\alpha_n^m|+|Y_n^{m,\dagger,*}||\beta_n^m|\right)\right]<1$$

holds in the ball of radius r, with $0\leq r<0.004$.

Proof. We give only the main ideas of the proof. For more details see [21]. At first it is important to note that in the previous series the sum which contains the variable m runs now only from 1 to n. This fact expresses the supposed orthogonality to the hyperholomorphic constants $X_n^{n+1,\dagger}$ and $Y_n^{n+1,\dagger}$. Since the basis polynomials are homogeneous, the value of f at the origin is

$$\begin{aligned} f(\mathbf{0}) &= \sum_{n=0}^{\infty}\left(X_n^{0,\dagger,*}(\mathbf{0})\alpha_n^0+\sum_{m=1}^{n}\left[X_n^{m,\dagger,*}(\mathbf{0})\alpha_n^m+Y_n^{m,\dagger,*}(\mathbf{0})\beta_n^m\right]\right)\\ &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0\in\mathbb{R}. \end{aligned}$$

Without loss of generality we assume that $f(\mathbf{0})$ is positive (otherwise we work with $-f$). Since the associated Fourier coefficients are real-valued, the real part of f is given by

$$\mathbf{Sc}(f)=\sum_{n=0}^{\infty}\left\{\mathbf{Sc}(X_n^{0,\dagger,*})\alpha_n^0+\sum_{m=1}^{n}\left[\mathbf{Sc}(X_n^{m,\dagger,*})\alpha_n^m+\mathbf{Sc}(Y_n^{m,\dagger,*})\beta_n^m\right]\right\}.$$

Basically, the main idea of the proof is to find relations between the general Fourier coefficients and the coefficient of the zeroth term, i.e., α_0^0. Multiplying both sides of the equation

$$\mathbf{Sc}(1-f)=1-\mathbf{Sc}(f) \tag{12}$$

by the solid spherical harmonics $\{\mathbf{Sc}(X_k^{0,\dagger,*}),\mathbf{Sc}(X_k^{p,\dagger,*}),\mathbf{Sc}(Y_k^{p,\dagger,*}):p=1,...,k\}$, integrating over the ball and applying the modulus we get the following relations:

$$\begin{aligned} |\alpha_k^0| &\leq \max_B|X_k^{0,\dagger,*}|\frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(X_k^{0,\dagger,*})\|_{L_2(B)}^2}\left(2\sqrt{\frac{\pi}{3}}-\alpha_0^0\right)\\ |\alpha_k^p| &\leq \max_B|X_k^{p,\dagger,*}|\frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(X_k^{p,\dagger,*})\|_{L_2(B)}^2}\left(2\sqrt{\frac{\pi}{3}}-\alpha_0^0\right)\\ |\beta_k^p| &\leq \max_B|Y_k^{p,\dagger,*}|\frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(Y_k^{p,\dagger,*})\|_{L_2(B)}^2}\left(2\sqrt{\frac{\pi}{3}}-\alpha_0^0\right),\ p=1,...,k. \end{aligned}$$

With some calculations, applying Propositions 1-3 we arrive at

$$\begin{aligned}
&\frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 \;+\; \sum_{n=1}^{\infty}\left[|X_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n}\left(|X_n^{m,\dagger,*}||\alpha_n^m| + |Y_n^{m,\dagger,*}||\beta_n^m|\right)\right] \\
\leq\;& \frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 \;+\; \frac{1}{\sqrt{3\pi}}\left(2\sqrt{\frac{\pi}{3}} - \alpha_0^0\right)\sum_{n=1}^{\infty}(4r)^n(n+1)^4(2n+3).
\end{aligned}$$

The principal significance is that

$$\frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 \;+\; \sum_{n=1}^{\infty}\left[|X_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n}\left(|X_n^{m,\dagger,*}||\alpha_n^m| + |Y_n^{m,\dagger,*}||\beta_n^m|\right)\right] < 1$$

if $\frac{2}{3}\sum\limits_{n=1}^{\infty}(4r)^n(n+1)^4(2n+3) < 1$. We see that the last series converges for $r < \frac{1}{4}$, and therefore, the inequality is satisfied for $0 \leq r < 0.004$. □

As we have seen, the set $\{X_n^{n+1,\dagger}, Y_n^{n+1,\dagger}\}$ belonging to h play a special role. In order to extend the previous result, next we present some important properties of this function and/or of its coordinates. For simplicity we restrict ourselves to the unit ball.

Lemma 2. *The hyperholomorphic constant h can be written as*

$$h = h_1\mathbf{e}_1 + h_2\mathbf{e}_2,$$

where its coordinates have Fourier series

$$\begin{aligned}
h_1(\underline{x}) &= \sum_{n=1}^{\infty}\left([X_n^{n+1,\dagger,*}(\underline{x})]_1\alpha_n^{n+1} + [Y_n^{n+1,\dagger,*}(\underline{x})]_1\beta_n^{n+1}\right) \\
h_2(\underline{x}) &= \sum_{n=1}^{\infty}\left([X_n^{n+1,\dagger,*}(\underline{x})]_2\alpha_n^{n+1} + [Y_n^{n+1,\dagger,*}(\underline{x})]_2\beta_n^{n+1}\right).
\end{aligned}$$

Moreover, the following properties hold:

Proposition 5. *The harmonic functions h_1 and h_2 are orthogonal with respect to the inner product* (3).

Proof. Because relation (6) we just need to ensure the orthogonality in $L_2(S)$. For technical reasons, we rewrite the function h as follows

$$h(\underline{x}) = \sum_{n=1}^{\infty}\sqrt{2n+3}\,r^n\left(X_n^{n+1,*}(\theta,\varphi)\alpha_n^{n+1} + Y_n^{n+1,*}(\theta,\varphi)\beta_n^{n+1}\right).$$

By definition of the real-valued inner product in $L_2(S)$, using representation (9) and the previous expression we have

$$< h_1, h_2 >_{L_2(S)} = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \frac{\sqrt{2n+3}\, r^n}{\|X_n^{n+1}\|_{L_2(S;\mathcal{A};\mathbb{R})}} \frac{\sqrt{2n'+3}\, r^{n'}}{\|X_{n'}^{n'+1}\|_{L_2(S;\mathcal{A};\mathbb{R})}} \int_S A_n(\theta,\varphi) B_{n'}(\theta,\varphi) d\sigma$$

where

$$\begin{aligned} A_n(\theta,\varphi) &= -C^{n+1,n}(\theta)\left(\cos(n\varphi)\alpha_n^{n+1} + \sin(n\varphi)\beta_n^{n+1}\right) = A_n(\theta)A_n(\varphi) \\ B_{n'}(\theta,\varphi) &= C^{n'+1,n'}(\theta)\left(\sin(n'\varphi)\alpha_{n'}^{n'+1} - \cos(n'\varphi)\beta_{n'}^{n'+1}\right) = B_{n'}(\theta)B_{n'}(\varphi). \end{aligned}$$

If the degrees of the summands in the series of h_1 and h_2 are different ($n \neq n'$) then the orthogonality is ensured. Therefore it is only necessary to prove that the previous integral vanishes for $n = n'$. And since

$$\int_S A_n(\theta,\varphi) B_n(\theta,\varphi) d\sigma = \int_0^{\pi} A_n(\theta) B_n(\theta) \sin\theta d\theta \int_0^{2\pi} A_n(\varphi) B_n(\varphi) d\varphi,$$

it is enough to prove that the second integral on the right-hand side is zero. Moreover,

$$\begin{aligned} \int_0^{2\pi} A_n(\varphi) B_n(\varphi) d\varphi &= \left[(\alpha_n^{n+1})^2 - (\beta_n^{n+1})^2\right] \int_0^{2\pi} \cos(n\varphi)\sin(n\varphi) d\varphi \\ &+ \alpha_n^{n+1}\beta_n^{n+1}\left(\int_0^{2\pi} \sin^2(n\varphi) d\varphi - \int_0^{2\pi} \cos^2(n\varphi) d\varphi\right) \\ &= 0. \end{aligned}$$

□

Proposition 6. *The harmonic functions h_1 and h_2 verify the following relation:*

$$\sup_B |h_1(\underline{x})| = \sup_B |h_2(\underline{x})|.$$

The proof follows immediately from representation (9).

We are thus led to the following generalization of Theorem 6.

Theorem 7. *Let f be an $\mathcal{A}$-valued monogenic function with $|f(\mathbf{x})| < 1$ in B and let*

$$\sum_{n=0}^{\infty} \left[X_n^{0,\dagger,*}\alpha_n^0 + \sum_{m=1}^{n+1} \left(X_n^{m,\dagger,*}\alpha_n^m + Y_n^{m,\dagger,*}\beta_n^m \right) \right]$$

be its Fourier expansion. Then

$$\sum_{n=0}^{\infty} \left[|X_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n+1} \left(|X_n^{m,\dagger,*}||\alpha_n^m| + |Y_n^{m,\dagger,*}||\beta_n^m| \right) \right] < 1$$

holds in the ball of radius r, with $0 \leq r < 0.004$.

Proof. Considering f written as in (11) (Lemma 1)

$$f(\mathbf{x}) = f(\mathbf{0}) + g(\mathbf{x}) + h(\underline{x}),$$

with $f(\mathbf{0}) = g(\mathbf{0}) + h(\underline{0})$. The study of the function g was already considered in Theorem 6. We showed that

$$\begin{aligned} |g(\mathbf{x})| &\leq \sum_{n=1}^{\infty} \left[|X_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n} \left(|X_n^{m,\dagger,*}||\alpha_n^m| + |Y_n^{m,\dagger,*}||\beta_n^m| \right) \right] \\ &\leq \frac{1}{\sqrt{3\pi}} \left(2\sqrt{\frac{\pi}{3}} - |\alpha_0^0| \right) \sum_{n=1}^{\infty} (4r)^n (n+1)^4 (2n+3). \end{aligned}$$

We consider now the function h written as Fourier series

$$h(\underline{x}) = \sum_{n=1}^{\infty} \left(X_n^{n+1,\dagger,*}(\underline{x}) \alpha_n^{n+1} + Y_n^{n+1,\dagger,*}(\underline{x}) \beta_n^{n+1} \right).$$

The proof for the function h follows the same idea as the previous one. According to its Fourier expansion, in this case we should find relations with α_0^1 and/or β_0^1. However, it is important to note that the function h has no real part, and therefore, it is not possible to apply straight the previous idea. Because h lies in $\mathrm{span}_{\mathbb{R}}\{\mathbf{e_1}, \mathbf{e_2}\}$, the interesting point is that multiplying h at right by $\mathbf{e}_1$ (resp., by $\mathbf{e}_2$) the real part is different of zero, standing then for $-h_1$ (resp., $-h_2$). Moreover, since there are two different Fourier coefficients in the zeroth term, it is natural to ask: which coefficient should be compared, α_0^1 or β_0^1? Multiplying the function h at right by $\mathbf{e}_1$ we obtain

$$\begin{aligned} \tilde{h}(\underline{0}) + \tilde{h}(\underline{x}) &:= h\mathbf{e}_1(\underline{0}) + h\mathbf{e}_1(\underline{x}) \\ &= (X_0^{1,\dagger,*}\mathbf{e}_1)\alpha_0^1 + (Y_0^{1,\dagger,*}\mathbf{e}_1)\beta_0^1 \\ &+ \sum_{n=1}^{\infty} \left[(X_n^{n+1,\dagger,*}\mathbf{e}_1)(\underline{x})\alpha_n^{n+1} + (Y_n^{n+1,\dagger,*}\mathbf{e}_1)(\underline{x})\beta_n^{n+1} \right] \end{aligned}$$

where

$$(X_0^{1,\dagger,*}\mathbf{e}_1)\alpha_0^1 + (Y_0^{1,\dagger,*}\mathbf{e}_1)\beta_0^1 = \frac{1}{2\sqrt{\pi}}\alpha_0^1 + \frac{1}{2\sqrt{\pi}}\beta_0^1\mathbf{e}_3.$$

For this case, taking into account the previous assumption, it is obviously that we should consider for zeroth term the coefficient α_0^1. In a similar way, considering a new function $\tilde{\tilde{h}} := h\mathbf{e}_2$, β_0^1 should be surely considered.

Having disposed of this preliminary step let us return to the proof. It follows easily multiplying h at right by $\mathbf{e}_1$

$$\begin{aligned} \mathbf{Sc}(\tilde{h}) &= \mathbf{Sc}(h\mathbf{e}_1) \\ &= \sum_{n=1}^{\infty} \left[\mathbf{Sc}(X_n^{n+1,\dagger,*}\mathbf{e}_1)\alpha_n^{n+1} + \mathbf{Sc}(Y_n^{n+1,\dagger,*}\mathbf{e}_1)\beta_n^{n+1} \right]. \end{aligned}$$

Taking into account Proposition 6, for $0 < \delta_2 \leq 1$ we multiply both sides of the equation

$$\mathbf{Sc}\left(\frac{\delta_2}{2} - \tilde{h}\right) = \frac{\delta_2}{2} - \mathbf{Sc}(\tilde{h})$$

by the orthonormal solid spherical harmonics $\mathbf{Sc}(X_k^{k+1,\dagger,*}\mathbf{e}_1)$ (resp. $\mathbf{Sc}(Y_k^{k+1,\dagger,*}\mathbf{e}_1)$), integrating over the ball and taking the modulus it follows

$$\begin{aligned} |\alpha_k^{k+1}| &\leq \max_B |X_k^{k+1,\dagger,*}| \frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(X_k^{k+1,\dagger,*})\|^2_{L_2(B)}} \left(\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1|\right) \\ |\beta_k^{k+1}| &\leq \max_B |X_k^{k+1,\dagger,*}| \frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(X_k^{k+1,\dagger,*})\|^2_{L_2(B)}} \left(\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1|\right) \end{aligned}$$

Now with some calculations, using Propositions 1-3 and applying the Maximum modulus principle, the previous inequalities can be rewritten as follows

$$\begin{aligned} |\alpha_k^{k+1}| &\leq \frac{2}{\sqrt{3}} 2^k \sqrt{2k+3}(k+1) \left(\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1|\right) \\ |\beta_k^{k+1}| &\leq \frac{2}{\sqrt{3}} 2^k \sqrt{2k+3}(k+1) \left(\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1|\right). \end{aligned}$$

In a similar way, from the study of the function $\tilde{\tilde{h}}$ we obtain

$$\begin{aligned} |\alpha_k^{k+1}| &\leq \frac{2}{\sqrt{3}} 2^k \sqrt{2k+3}(k+1) \left(\sqrt{\frac{\pi}{3}}\delta_2 - |\beta_0^1|\right) \\ |\beta_k^{k+1}| &\leq \frac{2}{\sqrt{3}} 2^k \sqrt{2k+3}(k+1) \left(\sqrt{\frac{\pi}{3}}\delta_2 - |\beta_0^1|\right). \end{aligned}$$

Using the previous inequalities we end with

$$\begin{aligned} |h(\underline{x})| &\leq \sum_{n=1}^{\infty} \left(|X_n^{n+1,\dagger,*}||\alpha_n^{n+1}| + |Y_n^{n+1,\dagger,*}||\beta_n^{n+1}|\right) \\ &\leq \frac{1}{\sqrt{3\pi}} \left(2\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1| - |\beta_0^1|\right) \sum_{n=1}^{\infty} (4r)^n (n+1)^2 (2n+3). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} |f(\mathbf{x})| &\leq \frac{1}{2}\sqrt{\frac{3}{\pi}} |\alpha_0^0 - \alpha_0^1 \mathbf{e}_1 - \beta_0^1 \mathbf{e}_2| \\ &+ \frac{1}{\sqrt{3\pi}} \left(2\sqrt{\frac{\pi}{3}}(1+\delta_2) - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|\right) \sum_{n=1}^{\infty} (4r)^n (n+1)^4 (2n+3), \end{aligned}$$

and since $|f(\mathbf{x})| < 1$ it is clear that

$$\frac{1}{\sqrt{3\pi}} \left(\frac{2\sqrt{\frac{\pi}{3}}(1+\delta_2) - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|}{2\sqrt{\frac{\pi}{3}} - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|}\right) \sum_{n=1}^{\infty} (4r)^n (n+1)^4 (2n+3) < 1.$$

Of crucial importance is the fact that the coefficient

$$\frac{2\sqrt{\frac{\pi}{3}} - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|}{2\sqrt{\frac{\pi}{3}}(1+\delta_2) - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|}$$

is bounded from above by 1. A simple calculation shows that the last series converges for $r < \frac{1}{4}$, and therefore, the inequality is satisfied for $0 \leq r < 0.004$. $\square$

This shows that such a radius exists in the three-dimensional Euclidean ball. It has to be studied how the estimate for the Bohr radius can be improved.

5 Real part Theorems for monogenic functions

In the remainder of this section, we refer to the results as "real part theorems" in honor to the first assertion of such a kind, the classical (improved) Hadamard's real part theorem (1892). His work on functions of a complex variable was one of the first to examine the general theory of analytic functions. Since then, the acceptance of his work is worldwide. Looking back to all of these years one can say that time has shown that his topic has a wide range of applications. Some important indicators for such a development is that based on it, many recent results with strong applications are still coming out. Moreover, they provide with the best description of the pointwise behavior of analytic functions from a given space. A lot of results and extended list of references concerning these and other fundamental inequalities, as well as their applications, can be found in the book by Kresin and Maz'ya (see [33]).

In the complex case, Hadamard's real part theorem contains only the modulus of the function in the left-hand side of an inequality and bounds the growth of a function by the growth of its real part. More precisely, for $r < R$ the inequality

$$|f(z) - f(0)| \leq \frac{Cr}{R-r} \sup_{|\xi| \leq R} |\mathbf{Re}\,(f(\xi) - f(0))\,|, \tag{13}$$

holds for analytic functions on a closed disk of radius R centered at the origin. Such an inequality appeared first in 1892 (see [24]) with $C = 4$. Later, Borel and Carathéodory found the sharp constant $C = 2$. As a matter of this fact, a more general estimate for $|f(z)|$ with $f(0) \neq 0$ was noticed by Carathéodory (see Landau [25, 26])

$$|f(z)| \leq \frac{2r}{R-r} \sup_{|\xi| \leq R} |\mathbf{Re} f(\xi)| + \frac{R+r}{R-r}|f(0)|. \tag{14}$$

Having in mind the type of inequalities we want to prove, we will restrict ourselves to the three-dimensional case. The frequent use of quaternionic analysis in the study of three-dimensional problems motivate us to consider functions defined in $\mathbb{R}^3$ and with values in the reduced quaternions. It is also known that already in the four-dimensional case (quaternion-valued functions) there are a lot of non-trivial monogenic functions with vanishing scalar part. For such functions we cannot get the result as desired.

In [20, 22] it is shown that it is possible to generalize Borel-Carathéodory and Hadamard's real part theorems to monogenic functions, therein restricted to the unit ball in the Euclidean space $\mathbb{R}^3$. In this section we generalize these results for an arbitrary ball of radius R, analogously to the complex case.

Remark 6. *Several proofs of Bohr's inequality are based on estimating all Fourier coefficients by the first one. We will observe that the proof of both theorems follows from this idea. Here we consider relations between the Fourier coefficients of the function and the Fourier coefficients of its real part.*

The referred relations come with the next lemma:

Lemma 3. *Let f be a square integrable $\mathcal{A}$-valued monogenic function and $n \in \mathbb{N}_0$. Then the Fourier coefficients of f are given by*

$$\begin{aligned}
\alpha_n^0 &= \frac{\|X_n^{0,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{0,\dagger})\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(f)\mathbf{Sc}(X_n^{0,\dagger})dV_R \\
\alpha_n^m &= \frac{\|X_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{m,\dagger})\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(f)\mathbf{Sc}(X_n^{m,\dagger})dV_R \\
\beta_n^m &= \frac{\|Y_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{m,\dagger})\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(f)\mathbf{Sc}(Y_n^{m,\dagger})dV_R, \quad m = 1, ..., n \\
\alpha_n^{n+1} &= \frac{\|X_n^{n+1,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{n+1,\dagger}\mathbf{e}_1)\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(h\mathbf{e}_1)\mathbf{Sc}(X_n^{n+1,\dagger}\mathbf{e}_1)dV_R \\
\beta_n^{n+1} &= \frac{\|Y_n^{n+1,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{n+1,\dagger}\mathbf{e}_1)\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(h\mathbf{e}_1)\mathbf{Sc}(Y_n^{n+1,\dagger}\mathbf{e}_1)dV_R.
\end{aligned}$$

Originally the Fourier coefficients are defined by the inner product of the function f and elements of the space $M^+(\mathbb{R}^3;\mathcal{A},n)$. Now as we can see these coefficients, up to a factor, are also associated with the scalar part of f.

Proof. We give only some ideas of the proof. For more details see [20]. According to Lemma 1, f

can be written as Fourier series respecting the decomposition (11)

$$\begin{aligned} f(\mathbf{x}) = f(\mathbf{0}) \quad &+ \quad \underbrace{\sum_{n=1}^{\infty} \left(X_n^{0,\dagger,*_R}(\mathbf{x})\alpha_n^0 + \sum_{m=1}^{n} \left[X_n^{m,\dagger,*_R}(\mathbf{x})\alpha_n^m + Y_n^{m,\dagger,*_R}(\mathbf{x})\beta_n^m \right] \right)}_{=g} \\ &+ \quad \underbrace{\sum_{n=1}^{\infty} \left[X_n^{n+1,\dagger,*_R}(\underline{x})\alpha_n^{n+1} + Y_n^{n+1,\dagger,*_R}(\underline{x})\beta_n^{n+1} \right]}_{=h}. \end{aligned}$$

We will present the proof only for the coefficients α_n^0 of g. The remaining coefficients α_n^m and β_n^m $(m = 1, ..., n)$ are obtained in a similar way. As described, we aim to compare each Fourier coefficient α_n^0 with $\mathbf{Sc}(f)$. We have seen several times before that

$$\mathbf{Sc}(f) = \sum_{n=0}^{\infty} \left(\mathbf{Sc}(X_n^{0,\dagger,*_R})\alpha_n^0 + \sum_{m=1}^{n} \left[\mathbf{Sc}(X_n^{m,\dagger,*_R})\alpha_n^m + \mathbf{Sc}(Y_n^{m,\dagger,*_R})\beta_n^m \right] \right).$$

Multiplying both sides of the previous expression by the solid spherical harmonics $\{\mathbf{Sc}(X_k^{0,\dagger,*_R}), \mathbf{Sc}(X_k^{p,\dagger,*_R}), \mathbf{Sc}$ $p = 1, ..., k\}$ $(k \geq 1)$ and integrating over the ball we get the desired relations.

For the study of the coefficients α_n^{n+1} and β_n^{n+1} we multiply the equation

$$\mathbf{Sc}(h\mathbf{e}_1) = \sum_{n=0}^{\infty} \left[\mathbf{Sc}(X_n^{n+1,\dagger,*_R}\mathbf{e}_1)\alpha_n^{n+1} + \mathbf{Sc}(Y_n^{n+1,\dagger,*_R}\mathbf{e}_1)\beta_n^{n+1} \right]$$

by the orthogonal solid spherical harmonics $\mathbf{Sc}(X_k^{k+1,\dagger,*_R}\mathbf{e}_1)$ (resp.
$\mathbf{Sc}(Y_k^{k+1,\dagger,*_R}\mathbf{e}_1)$) $(k \geq 1)$ and integrating over the ball carries our results. $\square$

Lemma 4. *Let f be a square integrable $\mathcal{A}$-valued monogenic function. For each $n \in \mathbb{N}_0$, the Fourier coefficients satisfy the inequalities*

$$\begin{aligned} |\alpha_n^0| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|X_n^{0,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{0,\dagger})\|_{L_2(B_R)}} \sup_{|\xi|\leq R} |\mathbf{Sc}\left(f(\xi)\right)| \\ |\alpha_n^m| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|X_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{m,\dagger})\|_{L_2(B_R)}} \sup_{|\xi|\leq R} |\mathbf{Sc}\left(f(\xi)\right)| \\ |\beta_n^m| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|Y_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{m,\dagger})\|_{L_2(B_R)}} \sup_{|\xi|\leq R} |\mathbf{Sc}\left(f(\xi)\right)|, \quad m = 1, ..., n \\ |\alpha_n^{n+1}| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|X_n^{n+1,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{n+1,\dagger}\mathbf{e}_1)\|_{L_2(B_R)}} \sup_{|\xi|\leq R} |\mathbf{Sc}\left(h\mathbf{e}_1(\xi)\right)| \\ |\beta_n^{n+1}| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|Y_n^{n+1,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{n+1,\dagger}\mathbf{e}_1)\|_{L_2(B_R)}} \sup_{|\xi|\leq R} |\mathbf{Sc}\left(h\mathbf{e}_1(\xi)\right)|. \end{aligned}$$

The previous inequalities are basic results to prove the following theorem:

Theorem 8 (Real-Part Theorem)**.** *Let f be a square integrable $\mathcal{A}$-valued monogenic function in B_R. Then, for $0 \le r < \frac{R}{2}$ we have the inequality*

$$\begin{aligned} |f|_r \;\; &\le \;\; |f(\mathbf{0})| + \sqrt{\frac{2}{3}} \frac{8r}{(R-2r)^3} \left(A_1(r,R) \sup_{|\xi| \le R} |\mathbf{Sc}\,(f(\xi))\,| \right. \\ &+ \;\; \left. A_2(r,R) \sup_{|\xi| \le R} |\mathbf{Sc}\,(h\mathbf{e}_1(\xi))\,| \right), \end{aligned}$$

where

$$|f|_r = \max_{|\mathbf{x}|=r} |f(\mathbf{x})|$$

and

$$\begin{aligned} A_1(r,R) &= \frac{8r^2(2R-r)}{R-2r} + 6R^2 \\ A_2(r,R) &= 4r^2 - 6rR + 3R^2. \end{aligned}$$

Proof. Considering f written as in (11) and taking into account the maximum modulus principle we have

$$|f|_r \le |f(\mathbf{0})| + |g|_R + |h|_R.$$

Let us start with the study of the function g. Using the previous lemma it follows that

$$\begin{aligned} |g|_R \;\le\; & 2\sqrt{\frac{\pi}{3}} \sqrt{R^3} \sup_{|\xi| \le R} |\mathbf{Sc}\,(f(\xi))\,| \sum_{n=1}^{\infty} \left[|X_n^{0,\dagger,*_R}| \frac{\|X_n^0\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^0)\|_{L_2(B_R)}} \right. \\ + \;\; & \left. \sum_{m=1}^{n} \left(|X_n^{m,\dagger,*_R}| \frac{\|X_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{m,\dagger})\|_{L_2(B_R)}} + |Y_n^{m,\dagger,*_R}| \frac{\|Y_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{m,\dagger})\|_{L_2(B_R)}} \right) \right]. \end{aligned}$$

Applying Proposition 2 and taking into account Remark 5 it follows

$$|g|_R \le 2\sqrt{\frac{2}{3}} \sup_{|\xi| \le R} |\mathbf{Sc}\,(f(\xi))\,| \sum_{n=1}^{\infty} \left(\frac{2r}{R}\right)^n (n+1)^2(n+2).$$

In the same way, we can study the function h.

$$|h|_R \;=\; |\tilde{h}|_R \;\le\; 2\sqrt{\frac{2}{3}} \sup_{|\xi| \le R} |\mathbf{Sc}\,(h\mathbf{e_1}(\xi))\,| \sum_{n=1}^{\infty} \left(\frac{2r}{R}\right)^n (n+1)(n+2).$$

Finally, we obtain

$$\begin{aligned} |f|_r \;\le\; |f(\mathbf{0})| \;\; &+ \;\; 2\sqrt{\frac{2}{3}} \sup_{|\xi| \le R} |\mathbf{Sc}\,(f(\xi))\,| \sum_{n=1}^{\infty} \left(\frac{2r}{R}\right)^n (n+1)^2(n+2) \\ &+ \;\; 2\sqrt{\frac{2}{3}} \sup_{|\xi| \le R} |\mathbf{Sc}\,(h\mathbf{e_1}(\xi))\,| \sum_{n=1}^{\infty} \left(\frac{2r}{R}\right)^n (n+1)(n+2). \end{aligned}$$

Now, note that the last series are convergent for $0 \le r < \frac{R}{2}$. □

The previous theorem states that a monogenic L_2-function $f : \Omega \subset \mathbb{R}^3 \longrightarrow \mathcal{A}$ is bounded by a combination of its real part and one of its other components. This result is a more general estimate but, in fact, it is not a complete analogy to the complex case. Therein, an analytic function is only bounded by its real part. However, restricting ourselves to the class of functions which are orthogonal to the subspace of the non-trivial hyperholomorphic constants in $L_2(B_R; \mathcal{A}; \mathbb{R})$, we get a stronger result:

Corollary 1. *Let $\tilde{f}$ be a square integrable $\mathcal{A}$-valued monogenic function in B_R orthogonal to the non-trivial hyperholomorphic constants with respect to the inner product* (3)*. Then, for $0 \le r < \frac{R}{2}$ we have the following inequality:*

$$|\tilde{f}|_r \le |\tilde{f}(\mathbf{0})| + \frac{8rA(r,R)}{(R-2r)^4} \sup_{|\xi| \le R} |\mathbf{Sc}(\tilde{f}(\xi))|$$

where

$$A(r,R) \quad = \quad 8r^2(2R-r) + 6R^2(R-2r).$$

Remark 7. *Replacing $\tilde{f}(\mathbf{x})$ by $f(\mathbf{x}) - f(0)$ in the resulting relation from the previous corollary, we arrive at*

$$|f(\mathbf{x}) - f(\mathbf{0})|_r \le \frac{8rA(r,R)}{(R-2r)^4} \sup_{|\xi| \le R} |\mathbf{Sc}\left(f(\xi) - f(\mathbf{0})\right)|$$

with $A(r,R)$ as in the previous theorem, which is a refinement of Hadamard's real part theorem.

We observe that in the constants of our estimates, the factor $1/(R-2r)$ occurs and not $R-r$ as it could be expected from the complex case. As we will see in the next section, to explain this is because monogenic functions do not map balls to balls in the small but balls to ellipsoids.

6 First applications

As in the case of holomorphic functions in the complex plane we have to ask if also the growth of the derivative (here of the hypercomplex derivative) can be bounded by the growth of the function. If this is possible then, consequently, the behaviour of the derivative can be estimated by the scalar part of the monogenic function. The following theorem gives a first result.

Theorem 9. *Let f be a square integrable $\mathcal{A}$-valued monogenic function in B_R. Then, for $0 \le r < \frac{R}{2}$ we have the following inequality:*

$$\left|(\frac{1}{2}\overline{D})f(\mathbf{x})\right|_r \quad \le \quad \frac{8}{\sqrt{3}} \frac{R^2(2R^2+7rR+2r^2)}{(R-2r)^5} \sup_{|\xi| \le R} |\mathbf{Sc}\left(f(\xi)\right)|.$$

Proof. We consider f written as in (11) (Lemma 1). Since the referred series is convergent in L_2, it converges uniformly to f in each compact subset of B_R. Also the series of all partial derivatives converges uniformly to the corresponding partial derivatives of f in compact subsets of B_R. Applying the hypercomplex derivative $\frac{1}{2}\overline{D}$ term by term to the series, it follows formally

$$
\begin{aligned}
(\frac{1}{2}\overline{D})f = \sum_{n=1}^{\infty}\Bigg[(\frac{1}{2}\overline{D})X_n^{0,\dagger,*_R}\alpha_n^0 + \sum_{m=1}^{n}\left((\frac{1}{2}\overline{D})X_n^{m,\dagger,*_R}\alpha_n^m + (\frac{1}{2}\overline{D})Y_n^{m,\dagger,*_R}\beta_n^m\right)\\
+ \quad (\frac{1}{2}\overline{D})X_n^{n+1,\dagger,*_R}\alpha_n^{n+1} + (\frac{1}{2}\overline{D})Y_n^{n+1,\dagger,*_R}\beta_n^{n+1}\Bigg]
\end{aligned}
$$

The proof follows from the idea applied in Theorem 8 with the estimates of the Fourier coefficients from Lemma 4 and taking into account the following equalities for the homogeneous monogenic polynomials (7) and their derivatives:

$$
\begin{aligned}
(\frac{1}{2}\overline{D})X_n^{l,\dagger} &= (n+l+1)X_{n-1}^{l,\dagger}\\
(\frac{1}{2}\overline{D})Y_n^{m,\dagger} &= (n+m+1)Y_{n-1}^{m,\dagger},
\end{aligned}
$$

for $l = 0, ..., n$ and $m = 1, ..., n$. □

7 $\mathcal{M}$-conformal mappings

The concept of monogenic-conformal mappings described by paravector-valued real differentiable functions in $\Omega \subset \mathbb{R}^{n+1}$ and with values in the Clifford algebra $Cl_{0,n}$ (in the Cauchy-Riemann sense) was introduced by Malonek in [28]. Let $z^* \in S$ be a fixed point and $\{S_m\}$ a regular sequence of subdomains which is shrinking to z^* if m tends to infinity and whereby z^* belongs to all S_m. In [28] it is shown that a function F realizes locally in the neighborhood of a fixed point $z = z^*$ a left $\mathcal{M}$-conformal mapping if and only if F is left monogenic and its left derivative is different from zero.

This result is described by the limit of a "quotient" of a 2-form (surface area) and a 3-form (volume). However, the geometric properties of such a result are not directly visible. Here we show that the description of monogenic functions can be now formulated easily by accessible geometric mapping properties.

For simplicity in what follows we focus our attention on the case of the Dirac operator. Let $\mathbb{R}^{0,3}$ be the real vector space $\mathbb{R}^3$ endowed with a quadratic form of signature $(0,3)$ and let $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$ be an associated orthonormal basis for $\mathbb{R}^{0,3}$. Then $\mathbb{R}^{0,3}$ generates the Clifford algebra $\mathbb{R}_{0,3}$ which is a real linear associative algebra of dimension 2^3 and with identity 1. The multiplication in $\mathbb{R}_{0,3}$ is given according to the multiplication rules

$$
\begin{aligned}
&\varepsilon_i^2 = -1, \quad i = 0,1,2\\
&\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0, \quad i \neq j, \quad 0 \leq i,j \leq 2.
\end{aligned}
$$

We introduce the Dirac operator

$$\partial = \varepsilon_0 \partial_{x_0} + \varepsilon_1 \partial_{x_1} + \varepsilon_2 \partial_{x_2}. \tag{15}$$

We are mainly interested in the case of vector-valued functions $F : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}_{0,3}$ defined as

$$F(x) = F_0(x)\varepsilon_0 + F_1(x)\varepsilon_1 + F_2(x)\varepsilon_2, \tag{16}$$

where its coordinates F_i $(i = 0, 1, 2)$ are real-valued functions defined in Ω. Continuously real-differentiable functions $F : \Omega \longrightarrow \mathbb{R}_{0,3}$ which satisfy

$$\partial F = 0 \Longleftrightarrow \begin{cases} \partial_{x_0} F_0 + \partial_{x_1} F_1 + \partial_{x_2} F_2 = 0 \\ \partial_{x_0} F_1 - \partial_{x_1} F_0 = 0 \\ \partial_{x_0} F_2 - \partial_{x_2} F_0 = 0 \\ \partial_{x_1} F_2 - \partial_{x_2} F_1 = 0 \end{cases}, \tag{17}$$

are said to be (left) monogenic in Ω. Moreover, as $\partial^2 = -\Delta$, where Δ is the Laplace operator in $\mathbb{R}^3$, (left) monogenic functions in Ω are also harmonic in Ω.

Let $F : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^{0,3}$ be an arbitrary real-differentiable function. Clearly then,

$$F(\mathbf{x}) := F(\mathbf{0}) + f(\mathbf{x}) + R(\mathbf{x})$$

being f its linear part and R stands for the rest (degree $n \geq 2$). Under the above hypotheses, we claim that f is a general linear function given as in (16), where the coordinates f_i $(i = 0, 1, 2)$ are given by

$$\begin{aligned} f_0(x) &= a_0 x_0 + a_1 x_1 + a_2 x_2 \\ f_1(x) &= b_0 x_0 + b_1 x_1 + b_2 x_2 \\ f_2(x) &= c_0 x_0 + c_1 x_1 + c_2 x_2. \end{aligned} \tag{18}$$

Let us denote by $\mathcal{E}$ the ellipsoid generated by the quadratic form

$$\mathcal{E} := \{(x_0, x_1, x_2) : \frac{x_0^2}{\alpha^2} + \frac{x_1^2}{\beta^2} + \frac{x_2^2}{\gamma^2} = 1\}, \tag{19}$$

where α, β and γ are the lengths of the semi-axes. The next theorem characterizes the local mapping properties of the linear part f of an arbitrary function F.

Theorem 10. *Let f be a linear function. Then, the function f is monogenic if and only if it maps a ball to an ellipsoid centered at the origin with the property that the reciprocal of the length of one semi-axis is equal to the sum of the reciprocals of the lengths of the other two semi-axes.*

Proof. For simplicity we just prove the sufficient condition. The necessary condition can be found in [23]. Suppose that there exists a linear analytic function f which maps the unit ball B to an arbitrarily oriented ellipsoid ε with the referred property. We rotate this ellipsoid transforming it to an ellipsoid ε^* such that the directions of its semi-axes $y^{(0)}$, $y^{(1)}$, $y^{(2)}$ coincide with the directions of the standard coordinate system (y_0, y_1, y_2). Such an ellipsoid is given by

$$\varepsilon^* : \{(y_0, y_1, y_2) : \frac{y_0^2}{\alpha^2} + \frac{y_1^2}{\beta^2} + \frac{y_2^2}{\gamma^2} \leq 1\}.$$

Let $\tilde{f}$ be the function whose image represents ε^*. We denote by $\tilde{D}$ the associated matrix to $\tilde{f}$

$$\tilde{D} = \begin{pmatrix} \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 \\ \tilde{b}_0 & \tilde{b}_1 & \tilde{b}_2 \\ \tilde{c}_0 & \tilde{c}_1 & \tilde{c}_2 \end{pmatrix}.$$

We remind that ε^* preserves the orientation, and therefore, it holds the property $\tilde{D}y^{(i)} = \lambda y^{(i)}$ $(i = 0, 1, 2)$. It is easily seen that $\tilde{D}$ is a diagonal matrix and moreover, its elements satisfy the equation $\tilde{a}_0 + \tilde{b}_1 + \tilde{c}_2 = 0$. In this case the associated function is given by

$$\tilde{f}(x) = -(\tilde{b}_1 + \tilde{c}_2)x_0\mathbf{e}_0 + \tilde{b}_1 x_1 \mathbf{e}_1 + \tilde{c}_2 x_2 \mathbf{e}_2.$$

It is easy to check that this function is monogenic (with respect to the Dirac operator). Now, as it was described before, we apply a rotation R to ε^* in order to obtain ε. Roughly speaking, for the rotation

$$R = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix}$$

such that $R^T R = I = RR^T$, we have then that $R\tilde{D}R^T := A$. Note that the original function f is now represented by the symmetric matrix A, in fact $AX = f(x)$ for $X = (x_0\ x_1\ x_2)^T$, being its coordinates given by

$$\begin{aligned} f_0(x) &= (\tilde{a}_0 r_{1,1}^2 + \tilde{b}_1 r_{1,2}^2 + \tilde{c}_2 r_{1,3}^2)x_0 + (\tilde{a}_0 r_{1,1} r_{2,1} + \tilde{b}_1 r_{1,2} r_{2,2} + \tilde{c}_2 r_{1,3} r_{2,3})x_1 \\ &+ (\tilde{a}_0 r_{1,1} r_{3,1} + \tilde{b}_1 r_{1,2} r_{3,2} + \tilde{c}_2 r_{1,3} r_{3,3})x_2 \\ f_1(x) &= (\tilde{a}_0 r_{1,1} r_{2,1} + \tilde{b}_1 r_{1,2} r_{2,2} + \tilde{c}_2 r_{1,3} r_{2,3})x_0 + (\tilde{a}_0 r_{2,1}^2 + \tilde{b}_1 r_{2,2}^2 + \tilde{c}_2 r_{2,3}^2)x_1 \\ &+ (\tilde{a}_0 r_{2,1} r_{3,1} + \tilde{b}_1 r_{2,2} r_{3,2} + \tilde{c}_2 r_{2,3} r_{3,3})x_2 \\ f_2(x) &= (\tilde{a}_0 r_{1,1} r_{3,1} + \tilde{b}_1 r_{1,2} r_{3,2} + \tilde{c}_2 r_{1,3} r_{3,3})x_0 \\ &+ (\tilde{a}_0 r_{2,1} r_{3,1} + \tilde{b}_1 r_{2,2} r_{3,2} + \tilde{c}_2 r_{2,3} r_{3,3})x_1 + (\tilde{a}_0 r_{3,1}^2 + \tilde{b}_1 r_{3,2}^2 + \tilde{c}_2 r_{3,3}^2)x_2. \end{aligned}$$

It is easy to check, as desired, that the function f is monogenic. □

Remark 8. *By a simple linear transformation of variables the previous theorem can be extended to a (small) ball with radius R.*

Remark 9. *The composition of a linear function with a translation allows to extend the result to an ellipsoid centered at an arbitrary point $\tilde{x}$. This composition preserves the monogenicity.*

Next one has to show that Theorem 10 can be generalized to arbitrary real-analytic functions which have the described local mapping properties. A monogenic function with non-vanishing linear part will map in the small balls to the special class of ellipsoids. Non-vanishing linear part means that all directional first derivatives of the function are different from zero. Equivalently this can be characterized by the Jacobian determinant. For details and further relations to the hypercomplex derivative see the paper [11].

Theorem 11. *Let F be a real-analytic function. Then, the function F is monogenic if and only if it maps locally a ball to an ellipsoid with the property that the reciprocal of the length of one semi-axis is equal to the sum of the reciprocals of the lengths of the other two semi-axes.*

Proof. If the function is monogenic then there is almost nothing to prove. The local mapping properties at a point x are determined by the linear part of the Taylor expansion at x. Theorem 10 leads to the stated result.

If a real-analytic function has the supposed local mapping properties then we have to expand the function at x in a real Taylor series. Applying ideas from [15], Chapter II, paragraph 5.2.2 we can show after a longer calculation which we will skip here that the linear part of the Taylor expansion satisfies at each point of the domain the Dirac equation and so the function must be monogenic. □

Monogenic functions as null solutions of the Dirac operator can be mapped to monogenic (or anti-monogenic) functions in the sense of satisfied Cauchy-Riemann equations (for details see, e.g., [13]). This transformation is an isometry and so it becomes clear that the here discussed mapping properties of monogenic functions remain true under this transformation.

The result of Theorem 11 allows to describe the monogenic functions as a special class of quasi-conformal mappings. If we visualize quasi-conformal mappings in $\mathbb{R}^3$ by points, given by the lengths of the semi-axes of the associated ellipsoids, then the monogenic functions (with non-vanishing Jacobian determinant) can be seen as a two-dimensional manifold in $\mathbb{R}^3$.

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