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Discrete Clifford analysis: an overview

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ABSTRACT

We give an account of our current research results in the development of a higher dimensional discrete function theory in a Clifford algebra context. On the simplest of all graphs, the rectangular $\mathbb{Z}^m$ grid, the concept of a discrete monogenic function is introduced. To this end new Clifford bases, involving so–called forward and backward basis vectors and introduced by means of their underlying metric, are controlling the support of the involved operators. As our discrete Dirac operator is seen to square up to a mixed discrete Laplacian, the resulting function theory may be interpreted as a refinement of discrete harmonic analysis. After a proper definition of some topological concepts, function theoretic results amongst which Cauchy's theorem and a Cauchy integral formula are obtained. Finally a first attempt is made at creating a general model for the Clifford bases used, involving geometrically interpretable curvature vectors.

RESUMEN

Nosotros damos un relato de los resultados de investigación actual en el desarrollo de la teoría de funciones discretas de dimensión grande en un álgebra de Clifford. Sobre el mas simple de todos los gráficos, la red de rectangulos $\mathbb{Z}^m$, el concepto de función monogénica discreta es presentado. Con esta finalidad nuevas bases de Clifford, envolviendo las bases de vectores llamadas forward and backward, son introducidas mediante su métrica fundamental, estas controlan el soporte de los operadores envueltos. Como nuestro operador de Dirac discreto puede ser visto como un operador

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Laplaciano discreto mixto, la teoría de funciones resultante puede ser interpretada como refinamiento de análisis armónico discreto. Después de definir algunos conceptos topológicos, resultados de teoría de funciones entre los cuales el Teorema de Cauchy y la fórmula de Cauchy integral son obtenidos. Finalmente, una primera tentativa es hacer uso de un modelo general de bases de Clifford envolviendo vectores de curvatura geométricamente interpretables.

Key words and phrases: *discrete Clifford analysis, discrete function theory, discrete Cauchy formula.*

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1 Introduction to the Clifford analysis setting

Clifford analysis (see e.g. [3, 4, 14]) is a higher dimensional function theory centred around the notion of monogenic functions, i.e. null solutions of the rotation invariant vector valued Dirac operator $\partial_{\underline{x}}$, defined below. It is a popular viewpoint to consider this function theory both as a higher dimensional analogue of the theory of holomorphic functions in the complex plane and as a refinement of classical harmonic analysis. In order to clarify these statements, let us introduce the underlying framework.

To this end, let $\mathbb{R}^{0,m}$ be endowed with a non–degenerate quadratic form of signature $(0, m)$, let $(e_1, \ldots, e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$ and let $\mathbb{R}_{0,m}$ be the real Clifford algebra constructed over $\mathbb{R}^{0,m}$, see e.g. [22]. The non–commutative multiplication in $\mathbb{R}_{0,m}$ is governed by

$$e_j e_k + e_k e_j = -2\delta_{jk}, \qquad j, k = 1, \ldots, m \tag{1}$$

A basis for $\mathbb{R}_{0,m}$ is obtained by considering for each set $A = \{j_1, \ldots, j_h\} \subset \{1, \ldots, m\}$ the element $e_A = e_{j_1} \ldots e_{j_h}$, with $1 \leq j_1 < j_2 < \ldots < j_h \leq m$. For the empty set $\emptyset$ one puts $e_\emptyset = 1$, the identity element. Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$.

When allowing for complex constants, the same set of generators $(e_1, \ldots, e_m)$, still satisfying the anti–commutation rules (1), also produces the complex Clifford algebra $\mathbb{C}_m$, as well as all real Clifford algebras $\mathbb{R}_{p,q}$ of any signature $(p + q = m)$.

The Euclidean space $\mathbb{R}^{0,m}$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(x_1, \ldots, x_m)$ with the Clifford vector

$$\underline{x} = \sum_{j=1}^{m} e_j x_j$$

The multiplication of two vectors $\underline{x}$ and $\underline{y}$ is given by $\underline{x}\,\underline{y} = \underline{x} \bullet \underline{y} + \underline{x} \wedge \underline{y}$ with

$$\begin{aligned} \underline{x} \bullet \underline{y} &= -\sum_{j=1}^{m} x_j y_j = \frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\underline{x}) \\ \underline{x} \wedge \underline{y} &= \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\underline{x}) \end{aligned}$$

being the scalar valued dot product (equalling the Euclidean inner product up to a minus sign) and the bivector valued wedge product, respectively. Note that the square of a vector $\underline{x}$ is scalar valued and equals the norm squared up to a minus sign: $\underline{x}^2 = - < \underline{x}, \underline{x} > = -|\underline{x}|^2$.

Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution for which $\bar{e}_j = -e_j$, $j = 1, \ldots, m$. In particular for a vector $\underline{x}$ we have $\bar{\underline{x}} = -\underline{x}$.

The Fourier dual of the vector $\underline{x}$ is the vector valued first order differential operator

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

called Dirac operator. It is precisely this Dirac operator which underlies the notion of monogenicity of a function, a notion which may be considered as the higher dimensional counterpart of holomorphy in the complex plane. A function f defined and differentiable in an open region Ω of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ is called left–monogenic in Ω if $\partial_{\underline{x}}[f] = 0$. In what follows, we will use the concept of inner spherical monogenics; these are homogeneous polynomials $P_k(\underline{x})$ of degree k ($k \in \mathbb{N}$), which are moreover monogenic, i.e. for which it holds that $\partial_{\underline{x}}[P_k](\underline{x}) = 0$. Since the Dirac operator factorizes the Laplacian, $\Delta = -\partial_{\underline{x}}^2$, monogenicity may also be regarded as a refinement of harmonicity; in this sense, spherical monogenics can be seen as refinements of spherical harmonics. The fundamental group leaving the Dirac operator $\partial_{\underline{x}}$ invariant is the special orthogonal group $\mathrm{SO}(m)$, doubly covered by the $\mathrm{Spin}(m)$ group of the Clifford algebra $\mathbb{R}_{0,m}$. For this reason, the Dirac operator is called a rotation invariant operator. In the present context, we will refer to this setting as the continuous case, as opposed to the discrete setting treated in this paper.

Recently, several authors have shown interest in finding an appropriate framework for the development of discrete counterparts of the basic notions and concepts of Clifford analysis, see a.o. [15, 16, 9, 10, 12]. Some, yet not all, of these contributions are explicitly oriented towards the numerical treatment of problems from potential theory and boundary value problems, rather than towards discrete function theoretic results, see also [17, 18]. In this paper, however, we will abandon the path of possible applications in order to focus on the fundamental features of a concrete model for a Clifford algebra framework in which discrete Dirac operators and the corresponding discrete function theories can be developed, see also [5, 6]. Seen the above mentioned connection between continuous Clifford analysis and complex analysis in the plane, special attention should be paid to the important property of the discrete Dirac operator factorizing a discrete Laplacian. This was also the case in the study of holomorphic functions on $\mathbb{Z}^2$, see e.g. [13, 19, 8] and, more recently [20, 21].

Discrete mathematics always involve graphs; here, we will only consider the simplest of all graphs in Euclidean space, namely the one corresponding to the rectangular $\mathbb{Z}^m$ grid.

2 Definition of a discrete Dirac operator

As announced above, we will consider the natural graph corresponding to the equidistant grid $\mathbb{Z}^m$; thus a Clifford vector $\underline{x}$ as introduced above will now only show integer co–ordinates. For the pointwise discretization of the partial derivatives $\frac{\partial}{\partial x_j}$ we then introduce the traditional one–sided forward and backward differences, respectively given by

$$\Delta_j^+[f](\underline{x}) = f(\ldots, x_j+1, \ldots) - f(\ldots, x_j, \ldots) = f(\underline{x}+e_j) - f(\underline{x}), \quad j = 1, \ldots, m$$

$$\Delta_j^-[f](\underline{x}) = f(\ldots, x_j, \ldots) - f(\ldots, x_j-1, \ldots) = f(\underline{x}) - f(\underline{x}-e_j), \quad j = 1, \ldots, m$$

We then first introduce a discrete Laplacian by its usual definition for an arbitrary connected graph.

Definition 1. *Let f be a function defined on the vertices of a connected graph and let $\underline{x}$ be such an arbitrary vertex. Then the action of the discrete Laplace operator on f at $\underline{x}$ is defined by*

$$\Delta f(\underline{x}) = \sum_{\underline{y}\sim\underline{x}} \big(f(\underline{y}) - f(\underline{x})\big) = \sum_{\underline{y}\sim\underline{x}} f(\underline{y}) - \big(\#\mathcal{N}_{\underline{x}}\big)\ f(\underline{x})$$

where the notation $\underline{y} \sim \underline{x}$ means that there is an edge in the graph under consideration which links the vertex $\underline{y}$ to $\underline{x}$, and where $\mathcal{N}_{\underline{x}}$ stands for the neighbourhood of $\underline{x}$ with respect to the graph, i.e. the set of all points $\underline{y} \sim \underline{x}$.

In the present case, with respect to the $\mathbb{Z}^m$ neighbourhood of $\underline{x}$, the above definition explicitly reads

$$\Delta^*[f](\underline{x}) = \sum_{j=1}^{m} \left[\Delta_j^+[f](\underline{x}) - \Delta_j^-[f](\underline{x})\right] = \sum_{j=1}^{m} \left[f(\underline{x}+e_j) + f(\underline{x}-e_j)\right] - 2mf(\underline{x}) \tag{2}$$

where we have denoted the corresponding discrete Laplacian by Δ^*; it is usually called the *star Laplacian* and involves the values of the considered function at the midpoints of the faces of the unit cube centred at $\underline{x}$. Clearly, with respect to the same grid, but changing the graph, other discrete Laplacians may be defined, involving e.g. the function values at the vertices of the cube (the cross Laplacian), or at the midpoints of the "edges".

For now, we restrict ourselves to the star Laplacian (2); note that it can also be written as

$$\Delta^*[f](\underline{x}) = \sum_{j=1}^{m} \Delta_j^+ \Delta_j^- [f](\underline{x}) = \sum_{j=1}^{m} \Delta_j^- \Delta_j^+ [f](\underline{x})$$

When passing to the Dirac operator, we cannot simply combine each discretized partial derivative, be it forward or backward, with the corresponding basis vector e_j, $j = 1, \ldots, m$, since such attempts do not serve our aim at developing a discrete function theory in which the notion of discrete monogenicity implies discrete harmonicity, as has been shown in [5]. Instead, an alternative approach is followed, in which the basis vectors will carry an orientation, just like the forward and backward differences do. To this end, we need to embed the Clifford algebra $\mathbb{R}_{0,m}$ into a bigger one, with an underlying vector space of the double dimension, e.g. $\mathbb{C}_{2m}$, where we consider $2m$ vectors e_j^+ and e_j^-, $j = 1, \ldots, m$, satisfying the following anti–commutator relations:

$$e_j^+ e_k^+ + e_k^+ e_j^+ = -2g_{jk}^+, \quad e_j^- e_k^- + e_k^- e_j^- = -2g_{jk}^-, \quad e_j^+ e_k^- + e_k^- e_j^+ = -2M_{jk}$$

where the symmetric tensors (g_{jk}^+), (g_{jk}^-) and the general tensor (M_{jk}) determine the corresponding metric, see also [12]. Three subsequent assumptions on this metric will now significantly reduce the degrees of freedom in the choice of the metric scalars.

Assumption 1. *The forward and the backward basis vector in each particular cartesian direction add up to the traditional basis vector in that direction, i.e.* $e_j^+ + e_j^- = e_j$, $j = 1, \ldots, m$.

Assumption 2. *There are no preferential cartesian directions, or: all cartesian directions play the same role in the metric. This assumption will be referred to as the principle of dimensional democracy and may be seen as a kind of rotational invariance.*

Assumption 3. *The positive and negative orientations of any cartesian direction play an equivalent role. This assumption may be interpreted as a kind of reflection invariance.*

On the basis of the second and third assumptions, one may put $g_{11}^+ = g_{22}^+ = \ldots = g_{mm}^+ = g_{11}^- = g_{22}^- = \ldots = g_{mm}^- = \lambda$, where $g_{jj}^\pm = -(e_j^\pm)^2$, $j = 1, \ldots, m$, and $M_{11} = M_{22} = \ldots = M_{mm} = \mu$, where $2M_{jj} = -(e_j^+ e_j^- + e_j^- e_j^+)$, $j = 1, \ldots, m$. Furthermore, also $g_{jk}^\pm$ and M_{jk}, for $j \neq k$, should be independent of their subscripts, whence we put $g_{jk}^\pm = g$ and $M_{jk} = M_{kj} = M$, $j, k = 1, \ldots, m$, $j \neq k$. The first assumption, combined with the traditional Clifford multiplication rules, then leads to the additional conditions $\lambda + \mu = \frac{1}{2}$ and $g + M = 0$. Summarizing, the forward and backward basis vectors e_j^+ and e_j^-, $j = 1, \ldots, m$, will submit to the following multiplication rules:

- $e_j^+ e_k^+ + e_k^+ e_j^+ = e_j^- e_k^- + e_k^- e_j^- = -2g$, $j \neq k$
- $e_j^+ e_k^- + e_k^- e_j^+ = 2g$, $j \neq k$
- $(e_j^+)^2 = (e_j^-)^2 = -\lambda$, $j = 1, \ldots, m$
- $e_j^+ e_j^- + e_j^- e_j^+ = 2\lambda - 1$, $j = 1, \ldots, m$

We are now led to the definition of our discrete Dirac operator.

Definition 2. *The discrete Dirac operator ∂ is the first order, Clifford vector valued difference operator given by*

$$\partial = \partial^+ + \partial^-$$

where the forward and backward discrete Dirac operators ∂^+ and ∂^- are respectively given by

$$\partial^+ = \sum_{j=1}^{m} e_j^+ \Delta_j^+ \quad \textit{and} \quad \partial^- = \sum_{j=1}^{m} e_j^- \Delta_j^-$$

We obtain, using the above multiplication rules, that

$$\partial^2 = -\lambda \sum_{j=1}^{m} (\Delta_j^+ \Delta_j^+ + \Delta_j^- \Delta_j^-) + (2\lambda - 1) \sum_{j=1}^{m} \Delta_j^+ \Delta_j^- + g \sum_{j \neq k} (2\Delta_j^+ \Delta_k^- - \Delta_j^- \Delta_k^- - \Delta_j^+ \Delta_k^+)$$

If we require the support of ∂^2 to remain at least in the unit cube centred at $\underline{x}$, the isotropy of the forward and backward basis vectors needs to be imposed, i.e. we have to put $\lambda = (e_j^+)^2 = (e_j^-)^2 = 0$ as in [12], whence in our case it follows in addition that $\mu = \frac{1}{2}$, or $e_j^+ e_j^- + e_j^- e_j^+ = -1$, $j = 1, \ldots, m$. One thus finally arrives at

- $e_j^+ e_k^+ + e_k^+ e_j^+ = e_j^- e_k^- + e_k^- e_j^- = -2g$, $j \neq k$
- $e_j^+ e_k^- + e_k^- e_j^+ = 2g$, $j \neq k$
- $(e_j^+)^2 = (e_j^-)^2 = 0$, $j = 1, \ldots, m$
- $e_j^+ e_j^- + e_j^- e_j^+ = -1$, $j = 1, \ldots, m$

see also [5]. These relations completely determine the metric of the underlying $2m$–dimensional space in terms of one free scalar parameter g, the metric tensor being given by

$$m_{jk} = \begin{cases} e_j^+ \bullet e_k^+, & j,k = 1, \ldots, m \\ e_j^+ \bullet e_k^-, & j = 1, \ldots, m, \ k = m+1, \ldots, 2m \\ e_j^- \bullet e_k^+, & j = m+1, \ldots, 2m, \ k = 1, \ldots, m \\ e_j^- \bullet e_k^-, & j,k = m+1, \ldots, 2m \end{cases}$$

or explicitly:

$$M = \left(\begin{array}{cccc|cccc} 0 & -g & \cdots & -g & -\frac{1}{2} & g & \cdots & g \\ -g & 0 & \ddots & \vdots & g & -\frac{1}{2} & \ddots & \vdots \\ \vdots & \ddots & 0 & -g & \vdots & \ddots & -\frac{1}{2} & g \\ -g & \cdots & -g & 0 & g & \cdots & g & -\frac{1}{2} \\ \hline -\frac{1}{2} & g & \cdots & g & 0 & -g & \cdots & -g \\ g & -\frac{1}{2} & \ddots & \vdots & -g & 0 & \ddots & \vdots \\ \vdots & \ddots & -\frac{1}{2} & g & \vdots & \ddots & 0 & -g \\ g & \cdots & g & -\frac{1}{2} & -g & \cdots & -g & 0 \end{array}\right)$$

Its determinant reads

$$\det M = (-1)^m \frac{(1+4g)^{m-1}(1-4(m-1)g)}{4^m}$$

whence it should hold that $g \neq -\frac{1}{4}$ and $g \neq \frac{1}{4(m-1)}$, since these specific values would induce a collapse of dimension; for a further discussion of this phenomenon we refer to Section 7. Under the above conditions, ∂^2 takes the form

$$\begin{aligned} \partial^2 &= -\sum_{j=1}^{m} \Delta_j^+ \Delta_j^- + g \sum_{j \neq k} (\Delta_j^+ \Delta_k^- + \Delta_k^+ \Delta_j^- - \Delta_j^- \Delta_k^- - \Delta_j^+ \Delta_k^+) \\ &= (4(m-1)g - 1)\Delta^* - 2g \sum_{j<k} \widetilde{\Delta}_{jk} \end{aligned} \qquad (3)$$

where Δ^* is the star Laplacian (2), and

$$\widetilde{\Delta}_{jk} = f(\underline{x} + e_j + e_k) + f(\underline{x} + e_j - e_k) + f(\underline{x} - e_j + e_k) + f(\underline{x} - e_j - e_k) - 4f(\underline{x}), \quad j < k$$

each $\widetilde{\Delta}_{jk}$ being interpretable as a cross Laplacian on the corresponding (e_j, e_k) plane, see also [12]. Note however that the grid points involved in these additional terms do not respect the neighbourhood $\mathcal{N}_{\underline{x}}$ of the vertex $\underline{x}$ in the originally chosen $\mathbb{Z}^m$ graph; we will consider in the next section the particular case where this term disappears. Anyhow, observe that, if (3) is to be interpreted as a similar result to the continuous factorization $\partial_{\underline{x}}^2 = -\Delta$, then we should in fact restrict the metric scalar g to the range $[0, \frac{1}{4(m-1)}[$.

3 Special case: the star Laplacian factorized

In the special case of the above approach where $g = 0$, the defining relations for the forward and backward basis vectors reduce to

- $e_j^+ + e_j^- = e_j$, $j = 1, \ldots, m$
- $\{e_j^+, e_k^+\} = \{e_j^-, e_k^-\} = \{e_j^+, e_k^-\} = 0$, $j, k = 1, \ldots, m$, $j \neq k$

- $(e_j^+)^2 = (e_j^-)^2 = 0$, $j = 1, \ldots, m$
- $\{e_j^+, e_j^-\} = -1$, $j = 1, \ldots, m$

(with the usual notation $\{.,.\}$ for the anti–commutator). This particular choice for the metric scalar causes the second term in (3) to drop, whence we are left with a factorization of the star Laplacian, i.e. $\partial^2 = -\Delta^*$, the support of the involved operators now staying in the $\mathbb{Z}^m$ neighbourhood of $\underline{x}$. As has been remarked in [5, 11], there is a well–known model for these particular forward and backward vectors, namely the so–called Witt basis of the Clifford algebra $\mathbb{C}_{2m}$. In order to understand this model properly, provide $\mathbb{C}_{2m}$ with the structure of a Hermitean space by introducing a so–called complex structure J on the underlying orthogonal space $\mathbb{R}^{0,2m}$, i.e. $J \in \mathrm{SO}(2m)$ with $J^2 = -\mathbf{1}$. For details on the construction, we refer to [1, 2]; for our purpose the following observations are sufficient. Start from the given orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{R}^{0,m}$ and complement it with additional vectors $(e_{m+1}, \ldots, e_{2m})$ yielding an orthonormal basis of $\mathbb{R}^{0,2m}$, i.e. $e_j e_k + e_k e_j = -2\delta_{jk}$, $j, k = 1, \ldots, 2m$. Without loss of generality, the complex structure J may always be chosen such that it maps the m–dimensional subspaces spanned by $(e_1, \ldots, e_m)$ and by $(e_{m+1}, \ldots, e_{2m})$ onto each other. A commonly used choice is $J[e_j] = -e_{m+j}$ and $J[e_{m+j}] = e_j$, $j = 1, \ldots, m$, but other choices are possible as well. The Witt basis $(\mathfrak{f}_j, \mathfrak{f}_j^c)_{j=1}^m$ for the complex Clifford algebra $\mathbb{C}_{2m}$ is then obtained through the action of the projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ on the basis elements e_j:

$$\begin{aligned}
\mathfrak{f}_j &= \frac{1}{2}(e_j + i\,J[e_j]) = \frac{1}{2}(e_j - i\,e_{m+j}), \qquad j = 1, \ldots, m \\
\mathfrak{f}_j^c &= \frac{1}{2}(e_j - i\,J[e_j]) = \frac{1}{2}(e_j + i\,e_{m+j}), \qquad j = 1, \ldots, m
\end{aligned}$$

It holds that $\mathfrak{f}_j + \mathfrak{f}_j^c = e_j$, $j = 1, \ldots, m$ and moreover the Witt basis elements satisfy the Grassmann identities $\mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = \mathfrak{f}_j^c\mathfrak{f}_k^c + \mathfrak{f}_k^c\mathfrak{f}_j^c = 0$, $j, k = 1, \ldots, m$, which also implies their isotropy ($(\mathfrak{f}_j)^2 = (\mathfrak{f}_j^c)^2 = 0$, $j = 1, \ldots, m$), and the duality identities $\mathfrak{f}_j\mathfrak{f}_k^c + \mathfrak{f}_k^c\mathfrak{f}_j = -\delta_{jk}$, $j, k = 1, \ldots, m$. These properties exactly coincide with the above conditions on the vectors e_j^+ and e_j^-, so that we may put $e_j^+ = \mathfrak{f}_j$ and $e_j^- = \mathfrak{f}_j^c$, $j = 1, \ldots, m$ and we are left with the Witt discrete Dirac operator $\partial = \partial^+ + \partial^-$, with $\partial^+ = \sum_{j=1}^m \mathfrak{f}_j\,\Delta_j^+$ and $\partial^- = \sum_{j=1}^m \mathfrak{f}_j^c\,\Delta_j^+$. This setting was already mentioned in [21], however without any function theoretic aims.

4 Discrete monogenic functions

In order to define discrete monogenicity, one first needs some discrete topology. So, consider a bounded set $B \subset \mathbb{Z}^m$ and its characteristic function

$$\psi_B(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} \in B \\ 0 & \text{if } \underline{x} \notin B \end{cases}$$

as well as the discrete operator

$$\check{\partial} = \sum_{j=1}^m e_j^+ \Delta_j^- + \sum_{j=1}^m e_j^- \Delta_j^+$$

The vector valued function

$$\psi_B \check{\partial} = \sum_{j=1}^{m} e_j^+ \Delta_j^- [\psi_B] + \sum_{j=1}^{m} e_j^- \Delta_j^+ [\psi_B]$$

is called the oriented boundary of B. Observe that $\mathrm{supp}(\psi_B \check{\partial})$ contains points which do not belong to B. In fact, it consists of all vertices the $\mathbb{Z}^m$ neighbourhood of which contains points of both B and $\mathrm{co}(B) \equiv \mathbb{Z}^m \setminus B$. In addition to this definition of the boundary, one may then also define the interior of B (respectively the exterior of B) to be the set of all points of B (respectively of $\mathrm{co}(B)$) which do not belong to $\mathrm{supp}(\psi_B \check{\partial})$. Each bounded set $B \subset \mathbb{Z}^m$ thus gives rise to a partition of $\mathbb{Z}^m$ into its interior, its exterior and the support of its oriented boundary.

The above concepts now allow to give a definition of a discrete monogenic function.

Definition 3. *Let B be a bounded set in $\mathbb{Z}^m$ and let the Clifford algebra valued function f be defined on $B \cup \mathrm{supp}(\psi_B \check{\partial})$. The f is called discrete (left) monogenic in B if and only if it holds that $\partial[f](\underline{x}) = 0$ for all $\underline{x} \in B$.*

Defined in this way, discrete monogenicity constitutes a proper generalization to higher dimension of discrete holomorphy in the Isaacs or the Ferrand sense, see [13, 19]. Moreover, it may be seen as a refinement of discrete harmonicity, since the right hand side of (3) can be interpreted as a generalized discrete Laplacian, also called mixed Laplacian, see [12], which even coincides with the star Laplacian when $g = 0$.

5 Some function theoretic results

We consider Clifford algebra valued functions defined on $\mathbb{Z}^m$.

Then, first of all, a discrete version of Leibniz's rule is obtained by direct calculation. Observe that, as compared to its continuous counterpart, it contains an extra term, which fortunately will turn out to become small when considering finer grids.

Lemma 1 (Leibniz's rule)**.** *Let f and g be Clifford algebra valued functions defined on $\mathbb{Z}^m$. Then*

(i) $\Delta_j^{\pm}[fg] = (\Delta_j^{\pm} f)\, g + f\,(\Delta_j^{\pm} g) \pm (\Delta_j^{\pm} f)\,(\Delta_j^{\pm} g)$;

(ii) if f is scalar–valued, then
$$[fg]\,\check{\partial} = g\,(f\check{\partial}) + f\,(g\check{\partial}) + \sum_{j=1}^{m} \left((\Delta_j^+ f)(\Delta_j^+ g) e_j^- - (\Delta_j^- f)(\Delta_j^- g) e_j^+ \right).$$

Next, the integral of a discrete function f is quite naturally defined as

$$\int f = \sum_{\underline{x} \in \mathbb{Z}^m} f(\underline{x})$$

where, in order to ensure integrability, integrands are required to have compact supports. The following results were then directly obtained, see [6].

Lemma 2 (partial integration). *Let f and g be Clifford algebra valued functions defined on $\mathbb{Z}^m$, where at least one of both has compact support, then*

$$\int f \Delta_j^{\pm}[g] = - \int \Delta_j^{\mp}[f]\, g$$

Lemma 3 (Stokes' theorem). *Let f and g be Clifford algebra valued functions defined on $\mathbb{Z}^m$, where at least one of both has compact support, then*

$$\int f\,(\partial g) = - \int (f \check{\partial})\, g \quad \textit{and} \quad \int f\,(\check{\partial} g) = - \int (f \partial)\, g$$

Observe that the domains of integration on both sides of the formulae in the above lemmata need not to be the same.

On account of Stokes' theorem, one now easily arrives at a first fundamental result.

Theorem 1 (Cauchy's theorem). *Let f be a Clifford algebra valued function defined on $\mathbb{Z}^m$, which is discrete left monogenic in the bounded set B, then*

$$\int (\psi_B\, \check{\partial})\, f = 0$$

Corollary 1. *If B is a bounded set in $\mathbb{Z}^m$, then*

$$\int \psi_B\, \check{\partial} = 0$$

Clearly, for the further development of this function theory, a Cauchy integral formula is essential. So, assume that E is the fundamental solution of operator $\check{\partial}$, i.e.

$$E(\underline{x})\, \check{\partial} = \delta(\underline{x}) = \left\{ \begin{array}{ll} 0, & \underline{x} \neq \underline{0} \\ 1 & \underline{x} = \underline{0} \end{array} \right\} = \prod_{j=1}^{m} \delta_{0\, x_j} \tag{4}$$

and

$$E(\underline{x} - \underline{y})\, \check{\partial} = \delta(\underline{x} - \underline{y}) = \left\{ \begin{array}{ll} 0, & \underline{x} \neq \underline{y} \\ 1 & \underline{x} = \underline{y} \end{array} \right\} = \prod_{j=1}^{m} \delta_{x_j\, y_j} \tag{5}$$

For further use, we then define

$$GT(\underline{x}, \underline{y}) = \sum_{j=1}^{m} \left(\Delta_j^{+}[\psi_B(\underline{x})] \Delta_j^{+}[E(\underline{x} - \underline{y})]\, e_j^{-} - \Delta_j^{-}[\psi_B(\underline{x})] \Delta_j^{-}[E(\underline{x} - \underline{y})]\, e_j^{+} \right) \tag{6}$$

The following results were then obtained in [6].

Theorem 2 (Cauchy–Pompeiu formula)**.** *Let B be a bounded set in $\mathbb{Z}^m$ and let f be a Clifford algebra valued function defined on $B \cup supp(\psi_B\, \check{\partial})$, then for all points $\underline{y} \in B$ it holds that*

$$-f(\underline{y}) = \int \psi_B(\underline{x})\, E(\underline{x}-\underline{y})\partial f(\underline{x}) + \int E(\underline{x}-\underline{y})\,(\psi_B\check{\partial})\, f(\underline{x}) + \int GT(\underline{x},\underline{y})\, f(\underline{x})$$

while for all points $\underline{y} \in co(B)$:

$$0 = \int \psi_B(\underline{x})\, E(\underline{x}-\underline{y})\partial f(\underline{x}) + \int E(\underline{x}-\underline{y})\,(\psi_B\check{\partial})\, f(\underline{x}) + \int GT(\underline{x},\underline{y})\, f(\underline{x})$$

where $GT(\underline{x},\underline{y})$ is given by (6).

The first and the second term at the right hand side in the above formulae are 'traditional' terms, representing a volume integral over the bounded set B and a surface integral over the oriented boundary of B, respectively. On the contrary, the third term is an additional one, arising due to the grid (and more precisely: it originates from the additional term already arising in Leibniz's rule). We call this term the 'grid tension' term, which explains the notation $GT(\underline{x},\underline{y})$, introduced above.

Theorem 3 (Cauchy's integral formula)**.** *Let B be a bounded set in $\mathbb{Z}^m$ and let the function f be discrete monogenic on B, then for all points $\underline{y} \in B$ it holds that*

$$-f(\underline{y}) = \int E(\underline{x}-\underline{y})\,(\psi_B\check{\partial})\, f(\underline{x}) + \int GT(\underline{x},\underline{y})\, f(\underline{x})$$

while for all points $\underline{y} \in co(B)$:

$$0 = \int E(\underline{x}-\underline{y})\,(\psi_B\check{\partial})\, f(\underline{x}) + \int GT(\underline{x},\underline{y})\, f(\underline{x})$$

where $GT(\underline{x},\underline{y})$ is given by (6).

Obviously, in the above results, an essential role is played by the so–called fundamental solution $E(\underline{x})$, defined by (4)–(5). In order to obtain $E(\underline{x})$ explicitly, we will pass to frequency space by means of the discrete–time Fourier transform, defined for a discrete Clifford algebra valued function $f(\underline{x})$ with compact support as follows:

$$\mathcal{F}[f(\underline{x})](\underline{\xi}) = \int f(\underline{x})\,\exp(-i\langle\underline{\xi},\underline{x}\rangle) = \sum_{\underline{x}\in\mathbb{Z}^m} \exp(-i\langle\underline{\xi},\underline{x}\rangle)\, f(\underline{x}), \qquad \underline{\xi} \in \mathbb{Z}^m \tag{7}$$

and yielding a periodic function of $\underline{\xi}$ with period $(2\pi)^m$. Elementary properties of this discrete–time Fourier transform are listed in the following lemma.

Lemma 4. *Let $f(\underline{x})$ be a Clifford algebra valued function defined on $\mathbb{Z}^m$ with compact support and let its discrete–time Fourier transform be given by (7), then it holds that*

- $\mathcal{F}[f(\underline{x} \pm e_j)](\underline{\xi}) = \exp(\pm i\xi_j)\,\mathcal{F}[f(\underline{x})](\underline{\xi})$;

- $\mathcal{F}[\Delta_j^{\pm} f(\underline{x})](\underline{\xi}) = \mp(1-\exp(\pm i\xi_j))\,\mathcal{F}[f(\underline{x})](\underline{\xi})$;
- $\mathcal{F}[f(\underline{x})\,\check{\partial}](\underline{\xi}) = \mathcal{F}[f(\underline{x})](\underline{\xi})\,G(\underline{\xi})$, *where*

$$G(\underline{\xi}) = \sum_{j=1}^{m}\left[(1-\exp(-i\xi_j))\,e_j^{+} + (\exp(i\xi_j)-1)\,e_j^{-}\right] \tag{8}$$

- $\mathcal{F}[\delta(\underline{x})](\underline{\xi}) = 1$.

On account of these calculus rules, it was then obtained in [6] that

$$\hat{E}(\underline{\xi}) \equiv \mathcal{F}[E(\underline{x})](\underline{\xi}) = \frac{G(\underline{\xi})}{(G(\underline{\xi}))^2}, \qquad \text{wherever } G(\underline{\xi}) \neq 0 \tag{9}$$

with $G(\underline{\xi})$ being given by (8).

In Section 7, $G(\underline{\xi})$ and $\hat{E}(\underline{\xi})$ are obtained even more explicitly, when passing to a concrete model for the Clifford forward and backward bases.

6 Discrete monogenic polynomials

Here our aim is to establish a notion of discrete spherical monogenic, i.e. the discrete counterpart of a monogenic homogeneous polynomial. To this end, one should observe that, for polynomials, it is not necessary to distinguish between the continuous and the discrete world. Indeed, if a polynomial is defined in the continuous variable $\underline{x} \in \mathbb{R}^m$, then it is trivially defined on $\mathbb{Z}^m$. Conversely, for each polynomial $P(\underline{x})$, there exists a number N such that, if $P(\underline{x})$ is defined on a subset $A \subset \mathbb{Z}^m$, with $|A| = N$, then $P(\underline{x})$ is well–defined in the whole of $\mathbb{R}^m$. So we are able to use at the same time derivatives and differences of polynomials.

For further use, we list a few auxiliary results in this respect, see also [7].

Lemma 5. *The operators $\Delta_j^{\pm} - \partial_{x_j}$, $j = 1, \ldots, m$, turn a homogeneous polynomial of degree k into a polynomial of degree $(k-2)$.*

Corollary 2. *The operator $\partial - \partial_{\underline{x}}$ turns a homogeneous polynomial of degree k into a polynomial of degree $(k-2)$.*

Corollary 3. *A homogeneous polynomial of degree k is left monogenic if and only if the discrete Dirac operator ∂ turns it into a polynomial of degree $(k-2)$.*

Proposition 1. *Let $L_k(\underline{x})$ be a polynomial of degree k, and let $P_k(\underline{x})$ be its homogeneous part of degree k, i.e. let*

$$L_k(\underline{x}) = P_k(\underline{x}) + R_{k-1}(\underline{x})$$

the meaning of $R_{k-1}(\underline{x})$ *being obvious. If* $L_k(\underline{x})$ *is discrete monogenic, i.e.* $\partial[L_k](\underline{x}) = 0$*, then* $P_k(\underline{x})$ *is an inner spherical monogenic, i.e.* $\partial_{\underline{x}}[P_k](\underline{x}) = 0$.

Corollary 4. *A homogeneous discrete monogenic polynomial is automatically an inner spherical monogenic.*

Although, fortunately, the converse is not true, the above corollary nevertheless indicates that it makes no sense to define an inner spherical discrete monogenic to be a discrete monogenic homogeneous polynomial. The question thus raises if an inner spherical monogenic can be completed, possibly uniquely, to a discrete monogenic polynomial of the same degree. The answer is given in the proposition below.

Proposition 2. *Let* $P_k(\underline{x})$ *be an inner spherical monogenic of degree* k*. Then there exists a unique polynomial* R_{k-2} *of degree* $k-2$*, such that*

$$Q_k(\underline{x}) = P_k(\underline{x}) - \underline{x}\, R_{k-2}(\underline{x})$$

is a discrete monogenic polynomial of degree k.

This induces the following fundamental result.

Theorem 4. *A discrete monogenic polynomial* $L_k(\underline{x})$ *of degree* k *may be uniquely decomposed as*

$$L_k(\underline{x}) = Q_k(\underline{x}) + L_{k-1}(\underline{x})$$

where $Q_k(\underline{x})$ *is a discrete monogenic polynomial of degree* k *showing the specific form*

$$Q_k(\underline{x}) = P_k(\underline{x}) - \underline{x}\, R_{k-2}(\underline{x})$$

$P_k(\underline{x})$ *being an inner spherical monogenic, and where* $L_{k-1}(\underline{x})$ *is a discrete monogenic polynomial of degree* $(k-1)$.

The above observations now give rise to the following definition.

Definition 4. *A discrete monogenic polynomial* $Q_k(\underline{x})$ *of degree* k*, showing the specific form*

$$Q_k(\underline{x}) = P_k(\underline{x}) - \underline{x}\, R_{k-2}(\underline{x})$$

$P_k(\underline{x})$ *being an inner spherical monogenic, is called an inner spherical discrete monogenic of degree* k.

By subsequent application of the above theorem, we may now conclude the following.

Corollary 5. *For each discrete monogenic polynomial* $L_k(\underline{x})$ *of degree* k*, there exists a unique set of inner spherical discrete monogenics* $(Q_j(\underline{x}))_{j=0}^{k}$*, such that*

$$L_k(\underline{x}) = Q_k(\underline{x}) + Q_{k-1}(\underline{x}) + \ldots + Q_1(\underline{x}) + Q_0(\underline{x})$$

7 A model for the forward and backward basis vectors

In Section 2 we have introduced our discrete Dirac operator with respect to the $\mathbb{Z}^m$ graph, a crucial role in its definition being played by the so–called forward and backward Clifford basis vectors e_j^+ and e_j^-, $j = 1, \ldots, m$, for which we have already provided a concrete model in the special case treated in Section 3. In this section, a feasible model is given in the general case where the metric scalar g does not equal zero, see also [5].

To this end so–called curvature vectors B_j, $j = 1, \ldots, m$ are introduced, by means of which one puts

$$e_j^+ = \frac{1}{2}(e_j + B_j) \quad \text{and} \quad e_j^- = \frac{1}{2}(e_j - B_j), \qquad j = 1, \ldots, m$$

meanwhile ensuring that $e_j^+ + e_j^- = e_j$, $j = 1, \ldots, m$. As these forward and backward Clifford vectors should satisfy the relations derived in Section 2, it should hold that

$$\begin{cases} B_j^2 = +1 \\ \{e_j, B_j\} = 2(e_j \bullet B_j) = 0 \end{cases} \qquad j = 1, \ldots, m \tag{10}$$

and that

$$\begin{cases} \{e_k, B_j\} &= & 2(e_k \bullet B_j) &= & 0 \\ \{B_k, B_j\} &= & 2(B_k \bullet B_j) &= & -8g \end{cases} \qquad j, k = 1, \ldots, m,\ j \neq k \tag{11}$$

Note that the second condition in (10) and the first one in (11) together express the orthogonality of the space spanned by the curvature vectors and the original m-dimensional space with basis $(e_1, \ldots, e_m)$. As a consequence, the curvature vectors may be written explicitly as

$$B_j = \sum_{\ell=1}^{m} b_j^{(\ell)}(ie_{m+\ell}) = \sum_{\ell=1}^{m} b_j^{(\ell)} \epsilon_\ell, \qquad j = 1, \ldots, m$$

where $\epsilon_\ell^2 = (ie_{m+\ell})^2 = +1$, $\ell = 1, \ldots, m$ and $\sum_{\ell=1}^m (b_j^{(\ell)})^2 = 1$, $j = 1, \ldots, m$. Note that here the Clifford dot product of any two curvature vectors equals their Euclidean inner product, these inner products all being equal to the same scalar $-4g$. $(B_1, \ldots, B_m)$ may thus be interpreted as a set of vectors on the unit sphere S^{m-1} of $\mathbb{R}^m$, containing two by two the same fixed angle α, with $\cos(\alpha) = -4g$. To this end the metric scalar g needs to be restricted to the interval $]-\frac{1}{4}, \frac{1}{4}]$, creating then a kind of 'umbrella' of vectors, which will open and close according to varying g. In particular, if $g = 0$ then $\alpha = \frac{\pi}{2}$, in agreement with the Witt case of Section 3.

The above relations (10)–(11) are summarized in the metric tensor $\widetilde{M}$:

$$\widetilde{M} = \left(\begin{array}{ccccc|ccccc} -1 & 0 & \cdots & \cdots & 0 & & & & & \\ 0 & -1 & 0 & \ddots & \vdots & & & & & \\ \vdots & 0 & -1 & 0 & \vdots & & & \mathbf{0} & & \\ \vdots & \ddots & 0 & -1 & 0 & & & & & \\ 0 & \cdots & \cdots & 0 & -1 & & & & & \\ \hline & & & & & +1 & -4g & \cdots & \cdots & -4g \\ & & & & & -4g & +1 & -4g & \ddots & \vdots \\ & & \mathbf{0} & & & \vdots & -4g & +1 & -4g & \vdots \\ & & & & & \vdots & \ddots & -4g & +1 & -4g \\ & & & & & -4g & \cdots & \cdots & -4g & +1 \end{array}\right)$$

its entries being equal to the Clifford dot products of the vectors $(e_1, \ldots, e_m, B_1, \ldots, B_m)$, in this specific order. Its determinant equalling $(-1)^m(1+4g)^{m-1}(1-4(m-1)g)$, we are again confronted with the non–admissible values $-\frac{1}{4}$ and $\frac{1}{4(m-1)}$ for the metric scalar g, already obtained in Section 2. Indeed, in those cases we no longer dispose of a basis for a $2m$–dimensional space: instead, for $g = \frac{1}{4(m-1)}$ we have that rank $(\widetilde{M}) = 2m-1$, while for $g = -\frac{1}{4}$ we have rank $(\widetilde{M}) = m+1$. We will further comment on this from a geometrical point of view. To this end, first take $g = -\frac{1}{4}$. Here $B_k \bullet B_j = \langle B_k, B_j \rangle = +1$, $j, k = 1, \ldots, m$, whence their contained angle α is zero. So, the 'umbrella' completely closes, all curvature vectors coincide and the dimension of the space spanned by them becomes 1, in accordance with the rank of the metric tensor. In the case where $g = \frac{1}{4(m-1)}$, the rank of $\widetilde{M}$ shows that the space spanned by the curvature vectors should be $(m-1)$–dimensional, i.e. they should be on the intersection of the unit sphere S^{m-1} with a hyperplane in m–dimensional space. In [5], the contained angle of the vectors in this situation has been explicitly determined for dimensions $m = 3$ and $m = 4$, showing that it indeed corresponds to the given value of g.

Remark 1. *It is worth noting that, in this concrete model for the forward and backward Clifford bases, one has*

$$G(\underline{\xi}) = \sum_{j=1}^{m} \left[(1 - \cos \xi_j)\, B_j + i\, \sin \xi_j\, e_j\right]$$

and

$$(G(\underline{\xi}))^2 = 4 \sum_{j=1}^{m} \sin^2 \frac{\xi_j}{2} - 32g \sum_{j<k} \sin^2 \frac{\xi_j}{2} \sin^2 \frac{\xi_k}{2}$$

whence the fundamental solution $\hat{E}(\underline{\xi})$ in frequency space, (9), explicitly reads

$$\hat{E}(\underline{\xi}) = \frac{1}{4} \frac{\sum_{j=1}^{m} [(1-\cos\xi_j)\, B_j + i \sin\xi_j\, e_j]}{\sum_{j=1}^{m} \sin^2 \frac{\xi_j}{2} - 8g \sum_{j<k} \sin^2 \frac{\xi_j}{2} \sin^2 \frac{\xi_k}{2}}$$

This explicit expression also allows to investigate when the denominator of $\hat{E}(\underline{\xi})$ will be zero (i.e., when $G(\underline{\xi}) = 0$), see [6] for the treatment of low dimensional cases.

It is our intention to extend this first model in a forthcoming paper, taking into account generalized curvature tensors controlling the support of all involved operators.

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