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Dirac Type Gauge Theories – Motivations and Perspectives

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ABSTRACT

We summarize the geometrical description of a specific class of gauge theories, called "of Dirac type", in terms of Dirac type first order differential operators on twisted Clifford bundles. We show how these differential operators may be geometrically considered as being the images of sections of a specific principal fibering naturally associated with twisted Clifford bundles. Based on the notion of real Hermitian vector bundles, we discuss the most general real Dirac type operator on "particle-anti-particle" modules over an arbitrary (orientable) semi-Riemannian manifold of even dimension. This setting may be appropriate for a common geometrical description of both the Dirac and the Majorana equation.

RESUMEN

Nosotros resumimos la descripción geométrica de una clase específica de teoría gauge, llamada "de tipo Dirac", en términos del tipo de Dirac de operadores diferenciales de primer orden sobre fibrados de Clifford twisted. Mostramos como esos operadores pueden ser geométricamente considerados como siendo imágenes de secciones de una fibra principal específica naturalmente asociada con el fibrado de Clifford twisted. Basado en la noción de fibrado vectorial Hermitiano real, discutimos el más general operador de tipo Dirac real sobre módulos "partícula-anti-partícula" sobre una variedad semi-Riemanniana (orientable) arbitraria de dimensión par. Este contexto puede ser apropiado para una descripción geométrica común para las ecuaciones de Dirac y de Majorana.

Key words and phrases: *Dirac Type Differential Operators, Real Clifford Modules, General Relativity, Gauge Theories, Majorana equation*

Math. Subj. Class.: *53C05, 53C07, 70S05, 70S15, 83C05.*

1 Synopsis

In a nutshell, Dirac type gauge theories are based on the following "universal (Dirac) action functional":

$$\mathcal{I}_{\mathrm{D}} : \int_M [\langle\psi, \not{D}\psi\rangle_{\mathcal{E}} + \mathrm{tr}_\gamma curv(\not{D})] \; dvol_{\mathrm{M}} \,. \tag{1}$$

Here, $\not{D} \in \mathcal{D}(\mathcal{E})$ denotes the most general Dirac type first order differential operator, acting on the $\mathcal{C}^\infty(M)-$module of smooth sections $\psi \in \mathfrak{Sec}(M, \mathcal{E})$ on a Hermitian Clifford module bundle $\pi_{\mathcal{E}} : \mathcal{E} \longrightarrow M$ over a smooth orientable semi-Riemannian manifold (M, g_{M}) of even dimension $n = 2k \geq 2$. The notation $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ denotes a chosen Hermitian form on $\mathcal{E}$ and

$$curv(\not{D}) \in \Omega^2(M, \mathrm{End}(\mathcal{E})) \tag{2}$$

is the *curvature* of $\not{D}$.

A detailed general discussion of the geometrical background of the functional (1) and how it is related to the well-known general Lichnerowicz decomposition (c.f. [5] and [3])

$$\not{D}^2 = -\triangle_{\mathrm{B}} + V_{\mathrm{D}} \tag{3}$$

can be found in [22] and [23]. Note that in contrast to what has been stated in the latter Reference, however, the "Dirac potential" V_{D} actually reads:

$$V_{\mathrm{D}} \;=\; \gamma(curv(\not{D})) - \mathrm{ev}_g(\omega_{\mathrm{D}}^2) + \mathrm{ev}_g(\partial_{\mathrm{D}}\omega_{\mathrm{D}}) \tag{4}$$

where the "Dirac form" $\omega_{\mathrm{D}} \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$ is a certain one-form canonically associated with $\not{D}$ and "ev_g" means evaluation with respect to the metric g_{M} (c.f. below). In our discussion presented here we shall omit the quadratic term. The third term in (4) only contributes as a boundary term. As a consequence, the functional (1) differs from the "universal Dirac action" that is discussed in Reference [22] by the quadratic term (and the boundary term) in (4). To calculate the curvature of a general Dirac type operator is generally more involved than for a connection. In contrast, explicit formulae are available for the corresponding Dirac potential. Hence, (4) may be used to also obtain the curvature of a general $\not{D}$ via (neglecting the boundary term)

$$\gamma(curv(\not{D})) = V_{\mathrm{D}} + \mathrm{ev}_g(\omega_{\mathrm{D}}^2)\,. \tag{5}$$

We follow this line of reasoning to calculate the curvature of the most general Dirac type operator on "particle-anti-particle" modules in section four.

In the following we focus on a summary of some of the basic features of the universal Dirac action (1). A detailed discussion of its motivation is presented, whereby we put emphasize to its "universality" and its relation to various partial differential equations well-known from physics and geometry. In the particular case of twisted Clifford bundles we discuss how Dirac type first order differential operators can be geometrically considered as being images of sections of a principal fibering that is naturally associated with the geometry of Clifford modules. Finally, we discuss a specific class of real Hermitian Clifford modules. For these we present an explicit formula for the universal Dirac action.

Our work is organized as follows: The second section is addressed to present some detailed discussion of the motivation for the Dirac action and how it is related to various well-known "field equations", like Yang-Mills and Einstein's equation of gravity. In the third section we discuss how the Dirac action may be regarded as a functional of the metric and (endomorphism valued) super fields. The fourth section is addressed to the Dirac action on the geometrical background of (a specific class of) real Hermitian Clifford modules which may allow to incorporate the geometrical description of the Majorana equation in terms of the universal Dirac action. Finally, in the fives section we present some outlook.

2 Motivation: Four equations and one action

To get started, let us call in mind that the two most profound equations in classical physics are provided by the Maxwell equations of electrodynamics:

$$dF = 0\,, \tag{6}$$

$$d{*}F = j_{\rm elm} \tag{7}$$

and the Einstein equation of gravity:

$$Ric(g_{\rm M}) - \frac{1}{2}scal(g_{\rm M}) = \lambda_{\rm grav}\tau\,. \tag{8}$$

Here, the "electromagnetic Field strength" is geometrically represented by a (closed) two-form $F \in \Omega^2(M)$ on a given four-dimensional, orientable semi-Riemannian manifold $(M, g_{\rm M})$ with index of $g_{\rm M}$ equals ± 2. Accordingly, the two-form $*F$ denotes the Hodge-dual of F with respect to $g_{\rm M}$ and a chosen orientation of M. Moreover, the differential operator d is the usual exterior derivative. We stress, that in the case of the Maxwell equations (6–7) the metric structure $g_{\rm M}$ on the manifold M is supposed to be fixed.

In contrast, in the case of Einstein's theory of gravity the gravitational field is supposed to be fully described in terms of the metric structure $g_{\rm M}$ on M. However, only those metric structures are physically admissible which satisfy Einstein's field equation (8). The tensor $Ric \in \mathfrak{Sec}(M, {\rm End}TM)$ denotes the "Ricci-tensor" and $scal \in \mathcal{C}^\infty(M)$ its trace the so-called "Ricci-scalar". For $TM \twoheadrightarrow M$

being the tangent bundle of M, the bundle $\mathrm{End}TM \twoheadrightarrow M$ is the associated bundle of endomorphisms on TM (over the identity on M).

The right-hand side of the Maxwell equation (7) denotes the "electrically charged matter current". Similarly, the right-hand side of the Einstein equation (8) is the "energy-momentum current" associated with any form of energy and matter. The numerical constant λ_{grav} is called the "gravitation coupling constant". It carries a physical dimension in contrast to the electromagnetic coupling constant which is purely numerical (approximately 1/137).

In classical physics these source terms for non-trivial electromagnetic field strength and gravitational fields are supposed to be given objects, reflecting the physical situation at hand. Of course, as a special case one may consider the physical situation where (part of) space-time (M, g_{M}) is filled only with an electromagnetic field that is generated by electrically charged matter whose support is outside the considered region of space-time. Then, within this region j_{elm} vanishes identically and τ is a unique function of F such that the pair (g_{M}, F) is physically admissible provided it is a solution of the coupled Einstein-Maxwell equations.

In a so-called "semi-classical" description of matter (i.e. within a certain approximation of a full quantum description), the classical Maxwell and Einstein equations are supplemented by the (gauge covariant) Dirac equation

$$(i\not{\partial}_{\text{A}} - m)\psi = 0\,. \tag{9}$$

In particular, the electromagnetic current

$$\begin{aligned} *j_{\text{elm}} &= \sum_{\mu,\nu=0}^{3} q_{\text{elm}}\langle\psi, \gamma(e^\mu)\psi\rangle_{\mathcal{E}}\, g_{\text{M}}(e_\mu, e_\nu)e^\nu \\ &\equiv q_{\text{elm}}\langle\psi, \gamma(e^\mu)\psi\rangle_{\mathcal{E}}\, g_{\text{M}}(e_\mu, e_\nu)e^\nu \end{aligned} \tag{10}$$

becomes a (quadratic) function of $\psi \in \mathfrak{Sec}(M, \mathcal{E})$ such that the triple (g_{M}, F, ψ) is physically admissible if and only if it fulfills the now coupled Einstein-Maxwell-Dirac equation (6–9). Here, when appropriate units are used the parameters $(m, q_{\text{elm}}) \in \mathbb{R}_+ \times \mathbb{Z}$ are physically interpreted as "mass" and "electric charge" of the matter described in terms of the "matter field" ψ (e.g. an electron).

In (10), $e_0, \ldots, e_3 \in \mathfrak{Sec}(M, TM)$ is a local orthonormal frame with respect to g_{M} and $e^0, \ldots, e^3 \in \mathfrak{Sec}(M, T^*M)$ its (local) dual frame.

Note that henceforth we will make use of Einstein's summation convention whenever local expressions come up like in (10).

Geometrically, the matter field ψ is usually considered as a section of a twisted spinor bundle

$$\pi_{\mathcal{E}} : \mathcal{E} = S \otimes W \longrightarrow M \tag{11}$$

over a semi-Riemannian spin-manifold $(M, g_{\text{M}}, \Lambda_{\text{Spin}})$ with Λ_{Spin} being a chosen spin structure on M. The Hermitian vector bundle $W = P \times_\rho V \twoheadrightarrow M$ is an associated vector bundle of a given principal G-bundle $G \hookrightarrow P \twoheadrightarrow M$ that represents the so-called "internal gauge degrees of freedom" of matter. Here, $\rho : G \to GL(V)$ is a unitary representation of G on a Hermitian vector space V which servers a the typical fiber of the twisting bundle $W \twoheadrightarrow M$.

In the case of electromagnetism the (semi-simple real) Lie group G equals the unitary group $U(1)$ with Lie-algebra Lie$G = i\mathbb{R}$. Accordingly, the gauge covariant Dirac operator

$$i\partial\!\!\!/_{\mathrm{A}} \quad \equiv \quad i\gamma \circ (\partial_{\mathrm{A}}) i\gamma \circ (\partial^{\mathrm{S}} \otimes \mathrm{Id}_{\mathrm{W}} + \mathrm{Id}_{\mathrm{S}} \otimes \partial^{\mathrm{W}}) \tag{12}$$

is given by

- The covariant derivative with respect to the spin connection on the spinor bundle $S \twoheadrightarrow M$:
$$\partial^{\mathrm{S}}_{\mu} \overset{\mathrm{loc.}}{=} \partial_{\mu} + \frac{1}{4}[\gamma^a, \gamma^b]\, \omega^{\mathrm{LC}}_{\mu ab} \tag{13}$$
with $\omega^{\mathrm{LC}} \in \Omega^1(M, so(p,q))$ being the Levi-Civita form that is determined by g_{M};
- The gauge covariant derivative on the Hermitian vector bundle $W \twoheadrightarrow M$:
$$\partial^{\mathrm{W}}_{\mu} \overset{\mathrm{loc.}}{=} \partial_{\mu} + \rho'(A_{\mu}) \tag{14}$$
with $\rho'(A) \equiv \rho' \circ A \in \Omega^1(M, \mathrm{End}(W))$ being a (local) $U(1)-$gauge potential represented on W. The Lie-algebra representation ρ' : Lie$G \to \mathrm{End}(V)$ is the derived representation with respect to the underlying group representation ρ.

Hence, locally the gauge covariant Dirac operator reads:

$$i\partial\!\!\!/_{\mathrm{A}} \overset{\mathrm{loc.}}{=} i\gamma^{\mu} \left(\partial_{\mu} + \frac{1}{4}\omega^{\mathrm{LC}}_{\mu ab}\, [\gamma^a, \gamma^b] \otimes \mathrm{Id}_{\mathrm{W}} + \mathrm{Id}_{\mathrm{S}} \otimes \rho'(A_{\mu}) \right). \tag{15}$$

Here and in the expression (10) the symbol "γ" denotes a Clifford mapping, i.e.

$$\begin{array}{rcl} \gamma : T^*M & \longrightarrow & \mathrm{End}(\mathcal{E}) \\ \omega & \mapsto & \gamma(\omega) \end{array} \tag{16}$$

satisfying $\gamma(\omega)^2 g_{\mathrm{M}}(\omega, \omega)\, \mathrm{Id}_{\mathcal{E}}$. In the following we will suppress the identity mappings $\mathrm{Id}_{\mathcal{E}}, \mathrm{Id}_{\mathrm{S}}, \mathrm{Id}_{\mathrm{W}}$ on $\mathcal{E}, S, W$ whenever this will not cause any confusion. Also, we will not make a distinction in our notation with respect to the metric on the tangent and the co-tangent bundle $T^*M \twoheadrightarrow M$. Finally, $\gamma^a \equiv \gamma(e^a)$ are the usual "gamma matrices" and $[\cdot, \cdot]$ is the ordinary commutator.

Note that the Lie algebra $so(p,q)$ is isomorphic to the Lie algebra $spin(p,q)$ of the spin group and $\frac{1}{2}[\gamma^a, \gamma^b]$ $(0 \leq a \neq b \leq 3)$ are the corresponding generators of the "spinor representation" of $so(p,q)$. Indeed, the Clifford action γ on a twisted spinor bundle (11) is simply given by the regular left action of the Clifford bundle

$$Cl_{\mathrm{M}} \twoheadrightarrow M \tag{17}$$

on $S \subset Cl_{\mathrm{M}}$. Here, the Clifford bundle is the algebra bundle over (M, g_{M}) whose typical fiber is given by the Clifford algebra $Cl_{p,q}$ that is generated by the Minkowski space $\mathbb{R}^{p,q} \equiv (\mathbb{R}^4, \eta)$, where

$$\eta(\mathfrak{e}_\mu, \mathfrak{e}_\nu) : \begin{cases} \pm 1 & \text{for} \quad \mu = 0\,, \\ \mp 1 & \text{for} \quad 1 \leq \mu = \nu \leq 3\,, \\ 0 & \text{for} \quad 0 \leq \mu \neq \nu \leq 3 \end{cases} \tag{18}$$

and $\mathfrak{e}_0, \ldots, \mathfrak{e}_3 \in \mathbb{R}^4$ the standard basis.

Therefore, on a twisted spinor bundle the Clifford action γ is uniquely determined by the metric g_{M}. If γ denotes the Clifford action with respect to the metric g_{M}, then $i\gamma$ is the Clifford action with respect to the metric $-g_{\mathrm{M}}$. Likewise, if the Clifford action is "even", i.e. $\gamma(\omega)^t = -\gamma(\omega)$ for all $\omega \in T^*M$, then the Clifford action given by $i\gamma$ is "odd": $i\gamma(\omega)^t = i\gamma(\omega)$ and vice verse.

Apparently, the geometrical background of the equations (6– 9) seems quite different like the equations themselves. To summarize: The geometrical background of the Maxwell equations is given by the Grassmann bundle $\Lambda_{\mathrm{M}} \twoheadrightarrow M$ over a given orientable semi-Riemannian manifold (M, g_{M}). In contrast, the geometrical background of the Einstein equation is provided by so-called $SO(p,q)-$reductions of the frame bundle $F_{\mathrm{M}} \twoheadrightarrow M$. That is, the geometrical background is given by the fiber bundle

$$\mathcal{E}_{\mathrm{EH}} := F_{\mathrm{M}} \times_{\mathrm{Gl}(4)} GL(4)/SO(p,q) \longrightarrow M\,. \tag{19}$$

In fact, a section of this associated bundle with typical fiber $GL(4)/SO(p,q)$ is in one-to-one correspondence to a semi-Riemannian structure g_{M} of signature (p,q). We thus do not make a distinction between a section of the Einstein-Hilbert bundle (19) and the corresponding metric. We denote both by the same symbol. Finally, the geometrical background of the Dirac equation is provided by a Clifford module $(\mathcal{E}, \gamma)$ over a given orientable semi-Riemannian (spin-)manifold (M, g_{M}).

Apparently, Maxwell's equations, Einstein's equation and Dirac's equation are rather different equations. Nonetheless, one may ask for a common geometrical root of these three equations which play such a fundamental role in physics and mathematics.

An appropriate hint is provided by the gauge covariant Dirac equation (9) and the geometrical interpretation of the Maxwell equations (6–7) in terms of gauge theory. For this one may regard the electromagnetic field strength F as a section of the complexified Grassmann bundle

$$\Lambda_{\mathrm{M}} \otimes_{\mathbb{R}} \mathbb{C} \twoheadrightarrow M \tag{20}$$

which corresponds to the curvature of a $U(1)-$connection on $U(1) \hookrightarrow P \twoheadrightarrow M$. We emphasize that with respect to this geometrical interpretation of the electromagnetic field strength the Maxwell equation (6) becomes an identity (the "Bianci-identity"). If A denotes a local gauge potential of the curvature, then (7) is read as the $U(1)-$Yang-Mills equation:

$$\mathrm{d}_{\mathrm{A}} {*} F_{\mathrm{A}} = j\,. \tag{21}$$

Here, respectively, $F_{\rm A} = iF \in \Omega^2(M, i\mathbb{R})$ is, again, the curvature of a $U(1)-$connection and $\mathrm{d}_{\rm A}$ its gauge covariant exterior derivative, locally given by the first order differential operator $d + A$ and $A \in \Omega^1(M, i\mathbb{R})$. Clearly, the adjoint action is trivial, for $U(1)$ is abelian. Hence,

$$\begin{aligned} \mathrm{d}_{\rm A}{*}F_{\rm A} &\overset{\text{loc.}}{=} d{*}F_{\rm A} + [A, {*}F_{\rm A}] \\ &= d{*}F_{\rm A}\,. \end{aligned} \tag{22}$$

If $j \equiv ij_{\rm elm}$, then the Yang-Mills equation (21) is equivalent to (7).

Note that $F_{\rm A} = \mathrm{d}_{\rm A}^2 \overset{\text{loc.}}{=} dA$. That is, the square of the first order differential operator $\mathrm{d}_{\rm A}$ is a zero order differential operator taking values in $\Omega^2(M, i\mathbb{R})$.

Let $\delta_{\rm A}$ be the formal adjoint of $\mathrm{d}_{\rm A}$ with respect to the pairing $\int_M \alpha^{\rm c} \wedge {*}\beta$ for all compactly supported $\alpha, \beta \in \Omega(M, \mathbb{C}) \equiv \mathfrak{Sec}(M, \Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C})$. By $\alpha^{\rm c}$ we denote the complex conjugate of α with respect to the canonical real structure on $\Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C}$ that is given by $\alpha^{\rm c} := e^\mu \otimes \overline{\lambda}_\mu$ for $\alpha \overset{\text{loc.}}{=} e^\mu \otimes \lambda_\mu \in \Omega^1(M, \mathbb{C})$. It follows that $\delta_{\rm A} = \pm * \,\mathrm{d}_{\rm A}{*}$, where the sign depends on the signature of $g_{\rm M}$ and the degree of the form the operator acts on. Then, the equation (21) may be rewritten as

$$\delta_{\rm A} F_{\rm A} = \pm j \tag{23}$$

and thus the original Maxwell equations become equivalent to

$$(\,\mathrm{d}_{\rm A} + \delta_{\rm A})F_{\rm A} = \pm j\,. \tag{24}$$

The point to be stressed here is, that the (complexified) Grassmann bundle serves as a canonical Clifford module with respect to the Clifford action

$$\begin{aligned} \gamma : T^*M &\longrightarrow \mathrm{End}(\Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C}) \\ \omega &\mapsto \begin{cases} \Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C} \\ \alpha \mapsto -i(\mathrm{ext}_\omega(\alpha) - \mathrm{int}_\omega(\alpha))\,. \end{cases} \end{aligned} \tag{25}$$

Here, $\mathrm{ext}_\omega(\alpha) := \omega \wedge \alpha$ and $\mathrm{int}_\omega(\alpha) := \alpha(\omega^\sharp, \cdot)$ with $\omega^\sharp \in TM$ is the metric dual with respect to $g_{\rm M} : \beta(\omega^\sharp) := g_{\rm M}(\omega, \beta)$ for all $\beta \in T^*M$.

As consequence,

$$\mathrm{d}_{\rm A} + \delta_{\rm A} i\not{\partial}_{\rm A}\,, \tag{26}$$

with

$$i\not{\partial}_{\rm A} \overset{\text{loc.}}{=} i\gamma^\mu(\partial_\mu + \frac{1}{4}\omega^{\rm LC}_{\mu ab}\,[\gamma^a, \gamma^b] + A_\mu)\,. \tag{27}$$

The Maxwell equations for purely imaginary $F_{\rm A} \in \mathfrak{Sec}(M, \Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C})$ can thus be brought into a form analogous to the Dirac equation for $\psi \in \mathfrak{Sec}(M, \mathcal{E})$:

$$i\not{\partial}_{\rm A} F_{\rm A} = \pm j\,. \tag{28}$$

The similarity between the Dirac equation (9) and (28) can be made even more close by noting that $\Lambda_{\mathrm{M}} \otimes_{\mathbb{R}} \mathbb{C} \simeq Cl_{\mathrm{M}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{End}(S_{\mathbb{C}})$, where $S_{\mathbb{C}} \equiv S \otimes_{\mathbb{R}} \mathbb{C}$. Hence, $\Lambda_{\mathrm{M}} \otimes \mathbb{C} \simeq S_{\mathbb{C}} \otimes S^*_{\mathbb{C}}$ and the (complexified) spinor bundle $S_{\mathbb{C}} \twoheadrightarrow M$ (with respect to a chosen spin structure) can be regarded as a sub-vector bundle of the Grassmann bundle:

$$S_{\mathbb{C}} \hookrightarrow \Lambda_{\mathrm{M}} \otimes \mathbb{C} \twoheadrightarrow M\,. \tag{29}$$

Geometrically, the complexified Grassmann bundle $\Lambda_{\mathrm{M}} \otimes_{\mathbb{R}} \mathbb{C}$ is but a special *twisted Grassmann bundle*

$$\Lambda_{\mathrm{M}} \otimes L \longrightarrow M \tag{30}$$

with $L := M \times \mathbb{C} \to M$ being the trivial complex line bundle over M. The Hermitian Clifford module

$$\pi_{\Lambda,\,E} : \mathcal{E}_{\Lambda,\,E} \equiv \Lambda_{\mathrm{M}} \otimes E \longrightarrow M\,, \tag{31}$$

with $E := L \oplus W \twoheadrightarrow M$ being the Whitney sum of the two Hermitian vector bundles $L \twoheadrightarrow M$ and $E \twoheadrightarrow M$, actually provides a common geometrical setting for the Dirac and Maxwell equations.

Obviously, all of this can be immediately generalized to arbitrary twisted Grassmann bundles parameterized by arbitrary Hermitian vector bundles $E \twoheadrightarrow M$ over (M, g_{M}). In this case, one only has to replace the covariant derivative ∂^{S} that corresponds to a chosen spin structure on M by the covariant derivative ∂^{Λ} of the Levi-Civita connection on $\Lambda_{\mathrm{M}} \twoheadrightarrow M$ with respect to the induced metric $g_{\Lambda\mathrm{M}}$. Then, (12) is replaced by the *twisted Gauss-Bonnet* like operator

$$\begin{aligned} i\partial\!\!\!/_{\mathrm{A}} &= i\gamma \circ (\partial^{\Lambda} \otimes \mathrm{Id}_{\mathrm{E}} + \mathrm{Id}_{\Lambda} \otimes \partial^{\mathrm{E}}) \\ &= \mathrm{d}_{\mathrm{A}} + \delta_{\mathrm{A}}\,. \end{aligned} \tag{32}$$

Note that locally there is no distinction between the first order operators (32) and (12). This is, because the bundle of homomorphisms $\mathrm{End}(\mathcal{E}) \twoheadrightarrow M$ of any Clifford module $(\mathcal{E}, \gamma)$ over an even dimensional semi-Riemannian manifold (M, g_{M}) globally decomposes as

$$\mathrm{End}(\mathcal{E}) \simeq (Cl_{\mathrm{M}} \otimes_{\mathbb{R}} \mathbb{C}) \otimes \mathrm{End}_{\mathrm{Cl}}(\mathcal{E})\,. \tag{33}$$

Here, $\mathrm{End}_{\mathrm{Cl}}(\mathcal{E})$ denote the sub-algebra of $\gamma-$invariant endomorphisms on $\mathcal{E} \twoheadrightarrow M$. The fundamental isomorphism (33) can be inferred from the two Wedderburn Theorems about “invariant linear mappings” (c.f. [9] and [3]). In fact, the use of this global decomposition forces the dimension of M to be even such that $Cl_{p,q}$ is simple.

Finally, nothing basically chances even in the case the Maxwell equations are replaced by general Yang-Mills equations, i.e. the abelian structure group $U(1)$ is replaced by an arbitrary (semi-simple, real and compact) Lie group G. In this case, one only has to replace the (trivial) line bundle $L \twoheadrightarrow M$ by the adjoint bundle $\mathfrak{a}d(P) := P \times_{\mathrm{ad}} \mathrm{Lie}G \twoheadrightarrow M$.

Like in the particular case of a spinor bundle, the Clifford action (25) is uniquely determined by the metric g_{M}. Actually, both Clifford actions coincide on their common domain. Hence, with respect to twisted Grassmann bundles we may consider γ and $g_{\rm M}$ as being basically the same. Accordingly, the Einstein equation is seen to provide a physical constraint on the possible Clifford module structures to which (31) refers to. Note the change of the meaning of the metric when the Maxwell equations are written similar to the Dirac equation.

Once we have established a common geometrical setup for the Dirac and the Maxwell (resp. Yang-Mills) equations we proceed to show that this common setup also provides an appropriate geometrical background for the Einstein equation. For this we remark that on the one hand side the Maxwell and Dirac equations make use of a given metric $g_{\rm M}$ (i.e. a fixed Clifford module structure of the underlying twisted Grassmann bundle). On the other hand, the Einstein equation are considered as differential equations determining $g_{\rm M}$. In particular, the (Levi-Civita) connection which fixes the first order operator ∂^Λ is fully determined by $g_{\rm M}$. In contrast to the Maxwell equations (resp. Yang-Mills equations), the gravitational gauge potential has thus an underlying geometrical structure given by the metric $g_{\rm M}$ from which the connection is derived. For this matter the Einstein-Hilbert functional, from which the Einstein equation can be derived as Euler-Lagrange equation, is *linear* in the curvature. In contrast, the Yang-Mills functional, which yields the (homogeneous) Maxwell equation (7) in the case $G = U(1)$, is *quadratic* in the curvature:

$$\mathcal{I}_{\rm EH}(g_{\rm M}) \;:=\; \lambda_{\rm grav}^{-1}\int_M scal(g_{\rm M})\, dvol_{\rm M}\,, \tag{34}$$

$$\mathcal{I}_{\rm YM}(g_{\rm M};A) \;:=\; \lambda_{\rm elm}^{-1}\int_M g_{\Lambda{\rm M}}(F_{\rm A},F_{\rm A})\, dvol_{\rm M}\,. \tag{35}$$

Note that the variation of $\mathcal{I}_{\rm YM}(g_{\rm M};A)$ with respect to the metric $g_{\rm M}$ gives rise to the energy-momentum current $\tau \in \mathfrak{Sec}(M,{\rm End}(TM))$ as a function of $F_{\rm A}$ as mentioned before.

To get a relation between these seemingly different functionals (34) and (35) we notice that in contrast to the square ${\rm d}_{\rm A}^2 = F_{\rm A}$ of the first order operator ${\rm d}_{\rm A}$, the square of the associated Dirac operator $i\partial\!\!\!/_{\rm A}$ has the well-known Lichnerowicz decomposition into a specific second order differential operator and a specific zero order operator:

$$\begin{aligned} i\partial\!\!\!/_{\rm A}^2 &= ({\rm d}_{\rm A}+\delta_{\rm A})^2 \overset{G=U(1)}{=} d\circ\delta+\delta\circ d \\ &= -\triangle_{\rm B}+V_{\rm D} \end{aligned} \tag{36}$$

with $\triangle_{\rm B} \overset{\rm loc.}{=} -g^{\mu\nu}(\nabla_\mu\circ\nabla_\nu-\Gamma^\sigma_{\mu\nu}\nabla_\sigma)$ being the *Bochner-Laplacian* and $\nabla_\mu \equiv \partial_\mu+\frac{1}{4}\omega^{\rm LC}_{\mu ab}\,[\gamma^a,\gamma^b]+A_\mu$. The local functions $\Gamma^\sigma_{\mu\nu}$ are the usual Christoffel symbols with respect to $g_{\rm M}$ and a chosen coordinate frame.

The *“Dirac potential”* has the specific form:

$$V_{\rm D} = \frac{1}{4}scal(g_{\rm M})+\gamma(F_{\rm A}) \in \mathfrak{Sec}(M,{\rm End}(\mathcal{E}_{\Lambda,\,E})) \tag{37}$$

where locally $\gamma(F_{\rm A})\frac{1}{2}\,\gamma^\mu\gamma^\nu\otimes F_{\mu\nu}$. The tensor product refers to the fundamental decomposition (33). As a consequence, the zero order operator $\gamma(F_{\rm A})$ is always a trace-less operator: ${\rm tr}_{\mathcal{E}}(\gamma(F_{\rm A}))\equiv 0$, where the trace is taken in ${\rm End}(\mathcal{E}_{\Lambda,\,E})$.

Therefore, the Einstein-Hilbert functional may be expressed in terms of $i\not{\partial}_{\rm A}$ as

$$\mathcal{I}_{\rm EH}(g_{\rm M}) \quad = \quad \lambda'^{-1}_{\rm grav} \int_M {\rm tr}_{\mathcal{E}} V_{\rm D}\, dvol_{\rm M}\,. \tag{38}$$

Note that the Dirac potential is uniquely determined by $i\not{\partial}_{\rm A}$.

We notice that the trace-less zero order operator $\gamma(F_{\rm A}) \in \mathfrak{S}ec(M, {\rm End}(\mathcal{E}_{\Lambda,\,E}))$ is indeed the "square root" of the Yang-Mills Lagrangian, for

$$\mathcal{I}_{\rm YM}(g_{\rm M}; A) \quad = \quad \lambda'^{-1}_{\rm elm} \int_M {\rm tr}_{\mathcal{E}}(\gamma(F_{\rm A})^2)\, dvol_{\rm M}\,. \tag{39}$$

However, also in this form the Yang-Mills action is still quadratic in the curvature in contrast to the Einstein-Hilbert action.

The question then is whether the Yang-Mills Lagrangian can be "linearized" such that it becomes most similar to the Einstein-Hilbert Lagrangian without violating the second order character of the Yang-Mills equations. Note that both the Einstein and the Yang-Mills equations are of second order. Hence, one cannot simply try to square the Einstein-Hilbert Lagrangian to bring it into a form similar to the Yang-Mills Lagrangian without obtaining higher order differential equations for $g_{\rm M}$.

In order to appropriately linearize the integrand of (39) one may take into account that $i\not{\partial}_{\rm A}$ also determines a specific curvature on the bundle (31), denoted by $curv(i\not{\partial}_{\rm A}) \in \Omega^2(M, {\rm End}(\mathcal{E}_{\Lambda,\,E}))$ (c.f. [22] and [23]). Explicitly it reads

$$\begin{aligned} curv(i\not{\partial}_{\rm A}) \quad &= \quad R_g \otimes {\rm Id}_{\rm E} + {\rm Id}_\Lambda \otimes F_{\rm A} \\ &\equiv \quad R_g + F_{\rm A}\,. \end{aligned} \tag{40}$$

Again, this is due to the fundamental decomposition (33). Here, $R_g \in \Omega^2(M, {\rm End}(\mathcal{E}_{\Lambda,\,E}))$ is the Riemannian curvature with respect to the induced metric $g_{\Lambda \rm M}$ on the Grassmann bundle over M. Locally, it reads: $R_g \overset{\rm loc.}{=} \frac{1}{2} e^\mu \wedge e^\nu \otimes \frac{1}{4}[\gamma^a, \gamma^b] R_{ab\mu\nu}$, where the local functions $R_{ab\mu\nu} \equiv g_{\rm M}(e_a, e_c)\, e^c({\rm Riem}(e_\mu, e_\nu)e_b)$ and ${\rm Riem} \in \Omega^2(M, {\rm End}(TM))$ denotes the (semi-)Riemann curvature tensor with respect to $g_{\rm M}$. Note again that Einstein's summation convention is employed in local formulae.

Therefore, the Yang-Mills curvature (especially the electromagnetic field strength) may be expressed in terms of $i\not{\partial}_{\rm A}$. In fact, it is but the *"relative curvature"* of the curvature of $i\not{\partial}_{\rm A}$ (again, neglecting the identity mappings):

$$F_{\rm A} = curv(i\not{\partial}_{\rm A}) - R_g \in \Omega^2(M, {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))\,. \tag{41}$$

This geometrical interpretation of $F_{\rm A}$ in terms of $i\not{\partial}_{\rm A}$ yields a different interpretation of the Yang-Mills (resp. of the Maxwell) equations. The latter are considered to yield a constraints for $i\not{\partial}_{\rm A}$. Of course, this simply means constraints for $\partial^{\rm E}$ and thus does not yield anything new in comparison with the usual description of Yang-Mills type gauge theories in terms of G principal bundles. However, the strength of the presented geometrical viewpoint of the Yang-Mills equations in terms of $i\partial\!\!\!/_{\rm A}$ has a powerful potential for a straightforward generalization. This is, because the geometrical point of view can be immediately generalized to arbitrary Dirac type first order differential operators. In other word, there are much more general Dirac type operators on (31) than those given by $i\partial\!\!\!/_{\rm A}$. In fact, the latter are only very specific Dirac type operators. They are fully characterized by the decomposition (37) and the fact that $F_{\rm A} \in \Omega^2(M, {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))$ is $\gamma-$invariant. This may provide a sufficient motivation to consider the form (38) of the Einstein-Hilbert function as more profound than the form (34). In fact, the former should be considered as a functional of a specific class of Dirac type operators on (31) and hence as a specific restriction of a much more general functional (c.f. our discussion in the next section).

As discussed in ([23]), the trace of the Dirac potential (37) can be recast into the geometrical form (neglecting boundary terms):

$$\mathrm{tr}_{\mathcal{E}} V_{\rm D} = \mathrm{tr}_\gamma curv(i\partial\!\!\!/_{\rm A})\,. \tag{42}$$

Therefore,

$$\begin{aligned}\mathcal{I}_{\rm EH}(g_{\rm M}) \equiv \mathcal{I}_{\rm EH}(i\partial\!\!\!/_{\rm A}) &= \lambda'^{-1}_{\rm grav}\int_M \mathrm{tr}_\gamma(curv(i\partial\!\!\!/_{\rm A}))\,dvol_{\rm M}\\ &\equiv \lambda'^{-1}_{\rm grav}\int_M \mathrm{tr}_\gamma(curv(\,\mathrm{d}_{\rm A}+\delta_{\rm A}))\,dvol_{\rm M}\end{aligned} \tag{43}$$

where $\mathrm{tr}_\gamma(curv(i\partial\!\!\!/_{\rm A})) \equiv \mathrm{tr}_{\mathcal{E}}[\gamma(curv(i\partial\!\!\!/_{\rm A}))] \in \mathcal{C}^\infty(M)$.

The form (43) of the Einstein-Hilbert action makes it most explicit how the metric $g_{\rm M}$ can be replaced by (a specific class of) Dirac type operators and hence how the Einstein-Hilbert functional determines a Clifford action γ on a twisted Grassmann bundle (31). Note that, despite of its appearance, (43) is actually independent of the connection on the twisting part $E \twoheadrightarrow M$ of (31). In other words, it is independent of the gauge potential A that (locally) determines the first order operator $\partial^{\rm E}$. The functional (43) thus yields a constraint only on how the vector bundle (31) can be actually regarded as a specific Clifford module. It thus determines γ as stated before. In fact, since $i\partial\!\!\!/_{\rm A}$ is fully characterized by (37), it is straightforward to prove that these Dirac type operators provide the biggest class of Dirac type first order differential operators on a twisted Grassmann bundle such that the universal Dirac action (1) is proportional to the Einstein-Hilbert action and thus only depends on $g_{\rm M}$. Note that there is only a canonical choice for $\partial^{\rm E}$ if the Hermitian vector bundle $E \twoheadrightarrow M$ equals the trivial bundle $M \times V \twoheadrightarrow M$. Only in this case, there is a natural choice for $i\partial\!\!\!/_{\rm A}$ given the Gauss-Bonnet like operator $d+\delta$. In the general case, however the latter operator is not gauge covariant. For this matter one has to chose some $\partial^{\rm E}$ to obtain an appropriate gauge covariant generalization $\mathrm{d}_{\rm A}+\delta_{\rm A}$ of $d+\delta$. Again, the functional (43) is independent of this arbitrary choice.

We are still left with the question whether it is possible to find a Dirac type operator $i\mathcal{D}\!\!\!/_{\rm A}$, say, on a certain twisted Grassmann bundle such that the Yang-Mills functional can be expressed in terms of the universal Dirac action (1).

The answer to this question turns out to be affirmative, actually, and has been discussed in some detail in [22] (c.f. also the appropriate references cited therein, in particular [2] in the case of a closed compact Riemannian manifold). In general, the Yang-Mills action may be written as

$$\begin{aligned}\mathcal{I}_{\mathrm{YM}}(g_{\mathrm{M}};A) &= \lambda'_{\mathrm{YM}}\left(\mathcal{I}_{\mathrm{D}}(i\not{D}_{\mathrm{A}})-\mathcal{I}_{\mathrm{D}}(i\not{\partial}_{\mathrm{A}})\right)\\ &= \lambda'_{\mathrm{YM}}\int_M \mathrm{tr}_\gamma(curv(i\not{D}_{\mathrm{A}})-curv(i\not{\partial}_{\mathrm{A}}))\,dvol_{\mathrm{M}}\,. \end{aligned}\tag{44}$$

where the corresponding Dirac type operator reads

$$i\not{D}_{\mathrm{A}} = i\not{\partial}_{\mathrm{A}} + \mathcal{I}\otimes\gamma(F_{\mathrm{A}})\,. \tag{45}$$

Here, $\mathcal{I}\equiv \mathrm{off}-\mathrm{diag}(-1,1)$ is an additional complex structure on the doubled twisted Grassmann bundle

$$2\mathcal{E}_{\Lambda,\,E}\equiv\mathcal{E}_{\Lambda,\,E}\oplus\mathcal{E}_{\Lambda,\,E}\Lambda_{\mathrm{M}}\otimes_{\mathbb{R}}(E\oplus E)\longrightarrow M\,. \tag{46}$$

The thus defined class of first order differential operators (45) are called Dirac operators of *"Pauli-type"*. The reason for this chosen terminology is that first order differential operators of the form

$$i\not{\partial}_{\mathrm{A}} + i\gamma(F_{\mathrm{A}}) \tag{47}$$

have been introduced in physics in order to describe the so-called "magnetic moment" of the proton long before it has been realized that the proton is a composite of more fundamental elementary particles (the "quarks"). In this context, the additional term $\gamma(F_{\mathrm{A}})=\frac{i}{2}\gamma^\mu\gamma^\nu\otimes F_{\mu\nu}$, with $F\in\Omega^2(M)$ being the electromagnetic field strength, is named "Pauli-term" after the famous physicist W. Pauli. Note that the first order operator (47), however, is not a Dirac type first order operator. This is because the Pauli-term $\gamma(F_{\mathrm{A}})$ is an even operator in the sense that it commutes with the canonical $\mathbb{Z}_2-$grading provided by the Riemannian volume form: $\gamma_{\mathrm{M}}=i\gamma(dvol_{\mathrm{M}})$ (called "γ_5" in the physics literature). Indeed, a first order differential operator is said to be of Dirac type provided it is odd with respect to a given $\mathbb{Z}_2-$grading of the underlying Clifford module and the principal symbol of its square is given by the underlying metric. Only the latter feature is shared by the first order operator (47). In contrast, the first order operator (45) is both odd and its square is a "generalized Laplacian". It is thus of Dirac type.

Note that Dirac's original first order operator (or its gauge covariant generalization)

$$i\not{\partial}_{\mathrm{A}} - m \tag{48}$$

is also not of Dirac type for exactly the same reason as (47) is not of Dirac type. We shall come back to this in our third section where we discuss a specific class of Hermitian Clifford modules and the most general Dirac type operators thereof.

Concerning Dirac type operators of the form (45) the "square root" of the Yang-Mills Lagrangian becomes most obvious. It is not simply given by the traceless zero order operator $\gamma(F_{\rm A})$ itself but, instead, by Dirac operators of Pauli type. Basically, this is because of the additional grading one obtains from the doubling of (31). This additional grading also allows to express the "fermionic part" of the universal Dirac action (1) as

$$\langle \Psi,\, i\not{D}_{\rm A}\, \Psi\rangle_{2\mathcal{E}} = 2\langle \psi, i\not{\partial}_{\rm A}\, \psi\rangle_{\mathcal{E}}\,. \tag{49}$$

At least, this holds true for those sections $\Psi \in \mathfrak{S}ec(M, 2\mathcal{E}_{\Lambda,\,E})$ that are given by $\Psi = (\psi, \psi)$ and hence are determined by a section $\psi \in \mathfrak{S}ec(M, \mathcal{E}_{\Lambda,\,E})$. In other words, the "Pauli-term" does not contribute to the fermionic action but only to the bosonic action. This is a very desirable feature of this class of Pauli type Dirac operators, for it is well-known that the Pauli-term in the fermionic action yields a generalized Dirac equation that is not compatible with "quantization". We shall return to the Pauli type Dirac operators when considering a specific class of Hermitian Clifford modules and the corresponding most general Dirac type operators thereof. The underlying structure of this class of Clifford modules is basically motivated by our fourth equation: the *Majorana equation*:

$$i\not{\partial}\psi = m\psi^c \tag{50}$$

where ψ^c denotes the "charge conjugate" of ψ (c.f. below).

We call in mind that the Einstein-Hilbert functional may be expressed in terms of Dirac type operators of the form (32) with an arbitrary choice of $\partial^{\rm E}$. In contrast, when restricted to Pauli type Dirac operators $i\not{D}_{\rm A}$, the universal Dirac action (38) yields the combined Einstein-Hilbert-Yang-Mills functional. It reduces to the pure Yang-Mills functional only if $(M, g_{\rm M})$ is fixed to be (Ricci) flat. This is consistent with the Einstein equation, however, only with respect to the physical approximation that the gravitational field produced by the energy-momentum of the Yang-Mills field can be neglected to some extend. In general, however, (1) yields the coupled Einstein-Yang-Mills-Weyl equations as the corresponding Euler-Lagrange equations if (1) is restricted to Pauli type Dirac operators (45). In this case, the right-hand side of the Yang-Mills equation is similar to (10) and the energy-momentum current is a well-determined function of $(g_{\rm M}, F_{\rm A}, \psi)$.

We stress that the Pauli type Dirac operators are more general than those given by $i\not{\partial}_{\rm A}$. In particular, the relative curvature of $i\not{D}_{\rm A}$:

$$F_{\rm D} := curv(i\not{D}_{\rm A}) - R_g \tag{51}$$

is not $\gamma-$invariant, i.e.

$$F_{\rm D} \notin \Omega^2(M, {\rm End}_{\rm Cl}(2\mathcal{E}_{\Lambda,\,E}))\,. \tag{52}$$

For that matter, $\gamma(F_{\rm D}) \in \Omega^0(M, {\rm End}(\mathcal{E}))$ is not a traceless operator.

We close our motivation for the universal Dirac action (1) with the remark that the underlying invariance group of this functional is provided by the full diffeomorphism group $Diff(\mathcal{E}_{\Lambda,\,E})$ of (31).

This (infinite) gauge group decompose into the semi-direct product (c.f. [22]):

$$Diff(\mathcal{E}_{\Lambda,\,E})Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E}) \ltimes Diff(M) \tag{53}$$

with $Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})$ consisting of all (bundle) automorphisms of (31) over the identity mapping on the base manifold M. Moreover, this group decomposes further into the direct sum of to sub-groups:

$$Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})Aut_{\mathrm{EH}}(\mathcal{E}_{\Lambda,\,E}) \times Aut_{\mathrm{YM}}(\mathcal{E}_{\Lambda,\,E})\,. \tag{54}$$

Here, the "Yang-Mills" sub-group $Aut_{\mathrm{YM}}(\mathcal{E}_{\Lambda,\,E}) \subset Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})$ consists of all automorphisms of (31) being isomorphic to the gauge transformations on the frame bundle that is induced by the vector bundle (31). It is thus a normal sub-group of $Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})$ and

$$Aut_{\mathrm{EH}}(\mathcal{E}_{\Lambda,\,E}) : Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})/Aut_{\mathrm{YM}}(\mathcal{E}_{\Lambda,\,E})\,. \tag{55}$$

Locally, the "Einstein-Hilbert" sub-group $Aut_{\mathrm{EH}}(\mathcal{E}_{\Lambda,\,E})$ consists of all $SO(p,q)$ rotations of orthonormal frames of $TM \twoheadrightarrow M$ and $Aut_{\mathrm{YM}}(\mathcal{E}_{\Lambda,\,E})$ consists of all ordinary gauge transformations encountered in the usual geometrical description of Yang-Mills gauge theories in terms of principal G-bundles $G \hookrightarrow P \twoheadrightarrow M$. Thus, the universal Dirac action (1) contains all the physical symmetries which are usually imposed on physical field theories. To enlarge this symmetry to "super-symmetry" transformations, however, is still an open issue.

Having presented a detailed discussion of the motivation for the universal Dirac action (1) and how it is related to well-known field equations, we may proceed with a discussion in what sense the Dirac functional is more general than the ordinary Yang-Mills functional. In other words, in the following section we want to discuss the precise domain of dependence of the universal Dirac functional.

3 The Dirac action as a functional of "super fields"

In this section we discuss in more detail the domain of dependence of the (bosonic part of the) universal Dirac action:

$$\mathcal{I}_{\mathrm{D,bos}} : \int_M \mathrm{tr}_\gamma curv(\not{D})\, dvol_{\mathrm{M}}\,. \tag{56}$$

In the foregoing section we have shown how this functional covers both the Einstein-Hilbert and the Yang-Mills functional. In fact, the Dirac functional may be considered as a natural generalization of the Einstein-Hilbert functional of the form (43). Hence, when restricted to certain "sub-domains" on the "set of all Dirac type operators" (c.f. below), the universal functional (56) becomes a functional on (an appropriate subset of) $\mathfrak{Sec}(M, \mathcal{E}_{\mathrm{EH}})$ in the case of the Einstein-Hilbert action, or a functional on the affine manifold of all linear connections $\mathcal{A}(E)$ in the case of the Yang-Mills action. Accordingly, when restricted to Pauli type Dirac operators the (bosonic part of the) universal Dirac action becomes a functional on the smooth manifold $\mathfrak{Sec}(M,\mathcal{E}_{\rm EH})\times\mathcal{A}(E)$. In general, (1) is considered as a functional on the smooth manifold

$$\mathcal{D}(\mathcal{E}_{\Lambda,\,E})\times\mathfrak{Sec}(M,\mathcal{E}_{\Lambda,\,E}) \tag{57}$$

of all Dirac type first order operators on a twisted Grassmann bundle (31) and the module of smooth sections therein. Note that the "fermionic part" of (1)

$$\mathcal{I}_{\rm D,ferm}:\int_M\langle\psi,\not{D}\psi\rangle_{\mathcal{E}}\,dvol_{\rm M}\,, \tag{58}$$

is viewed simply as a quadratic form on $\mathfrak{Sec}(M,\mathcal{E}_{\Lambda,\,E})$. This quadratic form is fully determined by (symmetric) elements of $\mathcal{D}(\mathcal{E}_{\Lambda,\,E})$. For this reason, it suffices to focus on the affine manifold of all Dirac type operators on a given Grassmann bundle.

The aim of this section is to make this more precise and to show how the above two cases of the Einstein-Hilbert and the Yang-Mills functional are special cases of the more general functional (56). Basically, the reason is provided by the following (highly non-canonical) isomorphisms:

$$\mathcal{D}(\mathcal{E}_{\Lambda,\,E})\simeq\Omega^0(M,\mathrm{End}(\mathcal{E}_{\Lambda,\,E}))\simeq\Omega^*(M,\mathrm{End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))\,, \tag{59}$$

which holds true for a fixed Clifford module structure on (31) (i.e. metric on M). The second isomorphism of (59) is implied by (33), where the abbreviation

$$\Omega^*(M,\mathrm{End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))\;\equiv\;\bigoplus_{p\in\mathbb{Z}}\Omega^p(M,\mathrm{End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E})) \tag{60}$$

has been used.

Consequently, any Dirac type operator on a Clifford module is determined by differential forms of all degrees. This is in strong contrast to connections on a vector bundle which are determined by one-forms, only.

To make this more precise, let again M be a smooth orientable manifold of even dimension $n=2k\geq 2$. Also, let again $E\twoheadrightarrow M$ be a smooth Hermitian vector bundle over M and $\mathcal{E}_{\Lambda,\,E}\twoheadrightarrow M$ the corresponding twisted Grassmann bundle. We call the smooth fiber bundle

$$\mathcal{E}_{\rm D}:\mathcal{E}_{\rm EH}\times\mathrm{End}(\mathcal{E}_{\Lambda,\,E})\longrightarrow M \tag{61}$$

the "Dirac bundle" associated to the twisted Grassmann bundle. So far (31) is considered as a vector bundle over M. There is no given Clifford structure at all, for M is not yet supposed to be endowed with a metric. We call in mind that a metric on M is in one-to-one correspondence with a section of the Dirac bundle given by

$$\begin{array}{rccl}\sigma_{\rm D}: M & \longrightarrow & \mathcal{E}_{\rm D} & \\ x & \mapsto & (g_{\rm M}(x),0) & \end{array} \tag{62}$$

where $g_{\rm M}\in\mathfrak{Sec}(M,\mathcal{E}_{\rm EH})$.

We consider the following equivalence relation on the manifold of smooth sections $\mathfrak{Sec}(M,\mathcal{E}_{\mathrm{D}})$:

$$\sigma'_{\mathrm{D}} \equiv (g',\Phi') \sim \sigma_{\mathrm{D}} \equiv (g,\Phi) \in \mathfrak{Sec}(M,\mathcal{E}_{\mathrm{D}}) \tag{63}$$

iff $g' = g \in \mathfrak{Sec}(M,\mathcal{E}_{\mathrm{EH}})$ and there exists an $\alpha \in \Omega^1(M,\mathrm{End}(E)) \hookrightarrow \Omega^1(M,\mathrm{End}_{\mathrm{Cl}}(\mathcal{E}_{\Lambda,\,E}))$, such that $\Phi' = \Phi + \gamma(\alpha) \in \Omega^0(M,\mathrm{End}(\mathcal{E}_{\Lambda,\,E}))$. We put

$$\mathfrak{S}_{\mathrm{D}} := \mathfrak{Sec}(M,\mathcal{E}_{\mathrm{D}})/\sim\ . \tag{64}$$

There are various equivalent definitions available for Dirac type first order differential operators, depending on the appropriate focus (see, for example, [1], [5], [3], [4]). We present a different one which is most adopted to our purpose.

Definition 1. *Let $\mathcal{D}(\mathcal{E}_{\Lambda,\,E})$ be the set of all first order differential operators acting on $\mathfrak{Sec}(M,\mathcal{E}_{\Lambda,\,E})$, such that for $\not{D} \in \mathcal{D}(\mathcal{E}_{\Lambda,\,E})$ there exists a section $g_M \in \mathfrak{Sec}(M,\mathcal{E}_{EH})$ with*

$$\begin{array}{ccc} T^*M & \xrightarrow{\gamma} & \mathrm{End}(\mathcal{E}_{\Lambda,\,E}) \\ df & \mapsto & [\not{D}, f]\,. \end{array} \tag{65}$$

Here, the g_M-induced Clifford action γ is defined by (25).

A first order differential operator $\not{D} \in \mathcal{D}(\mathcal{E}_{\Lambda,\,E})$ is called a "Dirac type operator" provided it is odd with respect to the $\mathbb{Z}_2-$grading that is given by an involution $\tau_{\mathcal{E}} := \gamma_M \otimes \tau_E$.

The set of all Dirac type operators on $\mathcal{E}_{\Lambda,\,E}$ carries a natural action of the translational group

$$\mathfrak{T}_{\mathrm{E}} \equiv \Omega^1(M,\mathrm{End}^+(E)) \hookrightarrow \Omega^1(M,\mathrm{End}^+(\mathcal{E}_{\Lambda,\,E})) \tag{66}$$

that is given by

$$\begin{array}{ccc} \mathcal{D}(\mathcal{E}_{\Lambda,\,E}) \times \mathfrak{T}_{\mathrm{E}} & \xrightarrow{\mu} & \mathcal{D}(\mathcal{E}_{\Lambda,\,E}) \\ (\not{D},\alpha) & \mapsto & \not{D} + \gamma(\alpha)\,. \end{array} \tag{67}$$

Clearly, this action is free and the corresponding orbit space $\mathcal{D}(\mathcal{E}_{\Lambda,\,E})/\mu$ can be identified with $\mathfrak{S}_{\mathrm{D}}$. Furthermore, with respect to this identification

$$\begin{array}{rccc} \pi_{\mathrm{D}}: & \mathcal{D}(\mathcal{E}_{\Lambda,\,E}) & \longrightarrow & \mathfrak{S}_{\mathrm{D}} \\ & \not{D} & \mapsto & \mathfrak{s} \equiv [(g_{\mathrm{M}},\Phi)] \end{array} \tag{68}$$

is a principal fibering with structure group $\mathfrak{T}_{\mathrm{E}}$.

This principal fibering is actually trivial. However, every bijection

$$\begin{array}{rccc} \chi_{\mathrm{A}}: & \mathcal{D}(\mathcal{E}_{\Lambda,\,E}) & \xrightarrow{\simeq} & \mathfrak{S}_{\mathrm{D}} \times \mathfrak{T}_{\mathrm{E}} \\ & \not{D} & \mapsto & (\mathfrak{s},\alpha) \end{array} \tag{69}$$

strongly depends on the choice of ∂^{E}. This holds true unless $E \twoheadrightarrow M$ is trivial.

Indeed, for every choice of $\partial^{\rm E}$ one may define

$$\mathcal{D}(\mathcal{E}_{\Lambda,\,E}) \ni \not{D} \equiv \chi_{\rm A}^{-1}(\mathfrak{s},\alpha) \;:=\; \not{\partial}_{\rm A} + \hat{\Phi}_{\rm A} + \gamma(\alpha) \;\equiv\; \not{\partial}_{\rm A} + \Phi_{\rm A}\,. \tag{70}$$

Here, $\not{\partial}_{\rm A} \in \mathcal{D}(\mathcal{E}_{\Lambda,\,E})$ is given by (32) and

$$\hat{\Phi}_{\rm A} \in \mathfrak{Sec}(M, {\rm End}(\mathcal{E}_{\Lambda,\,E})) \simeq \mathfrak{Sec}(M, \Lambda_{\rm M} \otimes {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))\,, \tag{71}$$

which does not contain a one-form part. Note that $\hat{\Phi}_{\rm A}$ has to have odd total degree. It is uniquely defined as follows: every $\not{D} \in \mathcal{D}(\mathcal{E}_{\Lambda,\,E})$ can be decomposed (in a highly non-unique way) as $\not{D} = \not{\partial}_{\rm A} + \Phi_{\rm A}$ with $\Phi_{\rm A} \equiv \not{D} - \not{\partial}_{\rm A}$. Then, $\Phi_{\rm A} =: \hat{\Phi}_{\rm A} + \gamma(\alpha)$ and $\not{\partial}_{\rm A} + \hat{\Phi}_{\rm A} + \gamma(\alpha)$ is equivalent to $\not{\partial}_{\rm A} + \hat{\Phi}_{\rm A}$. It follows that $\pi_{\rm D}(\chi_{\rm A}^{-1}(\mathfrak{s},\alpha)) = {\rm pr}_1(\mathfrak{s},\alpha) = \mathfrak{s}$, if and only if $[\not{D}] \in \mathcal{D}(\mathcal{E})/\mu$ corresponds to $\mathfrak{s} \in \mathfrak{S}_{\rm D}$.

Proposition 1. *Let $\mathcal{E}_D \twoheadrightarrow M$ be the Dirac bundle associated with a twisted Grassmann bundle $\mathcal{E}_{\Lambda,\,E} \twoheadrightarrow M$. The functional (56) can be considered as a canonical functional on $\mathfrak{Sec}(M,\mathcal{E}_D)$:*

Proof: Since the value of the integral

$$\int_M {\rm tr}_\gamma curv(i\not{\partial}_{\rm A})\, dvol_{\rm M} \tag{72}$$

is independent of the choice of $\partial^{\rm E}$, it follows that (56) is constant along the fibers of (68). Hence, it descents to a well-defined functional on $\mathfrak{S}_{\rm D}$. For this matter $\mathcal{I}_{\rm D,bos}$ can be considered as a functional of $(g_{\rm M},\Phi)$ that constitutes a general section of the Dirac bundle (61). □

As a consequence

$$\mathcal{I}_{\rm D} = \mathcal{I}\,(g_{\rm M},\Phi,\psi) \tag{73}$$

with

$$\Phi \in \mathfrak{Sec}(M, \Lambda_{\rm M} \otimes {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E})) \bigoplus_{0\le l\le n} \mathfrak{Sec}(M, \Lambda^l T^*M \otimes {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))\,. \tag{74}$$

being a "super-field" of odd total degree that takes values in the $\gamma-$invariant endomorphisms on $\mathcal{E}_{\Lambda,\,E}\mathcal{E}^+_{\Lambda,\,E} \oplus \mathcal{E}^-_{\Lambda,\,E} \twoheadrightarrow M$.

Especially, for $\Phi = 0$, the action (1) reduces to the sum of the usual (massless) Dirac functional and the Einstein-Hilbert functional:

$$\int_M [\langle\psi,\, i\not{\partial}_{\rm A}\psi\rangle_{\mathcal{E}} + {\rm tr}_\gamma curv(i\not{\partial}_{\rm A})]\, dvol_{\rm M}\,. \tag{75}$$

In this case, the appropriate Euler-Lagrange equations are given by the combined Einstein-Weyl equations with the energy-momentum current τ being defined by $\psi \in \mathfrak{Sec}(M, \mathcal{E}_{\Lambda,\,E})$.

Likewise, to obtain the combined Einstein-Yang-Mills-Weyl equations one considers $\Phi = \mathcal{F}_{\rm A}$, with the requirement that $\mathcal{F}_{\rm A} \in \mathfrak{Sec}(M, \Lambda^2 T^*M \otimes \mathrm{End}_{\rm Cl}(2\mathcal{E}_{\Lambda,\,E}))$ being defined by the curvature $F_{\rm A} \in \Omega^2(M, \mathrm{End}(E))$ with respect to the chosen $\partial^{\rm E}$. In other words, one restrict the right-hand side of (1) to Pauli type Dirac operators on $2\mathcal{E}_{\Lambda,\,E} \twoheadrightarrow M$. In this case, the energy-momentum current τ is fully determined as a function of $(\psi, F_{\rm A})$, whereas the electromagnetic current is given by (10) (or an appropriate generalization thereof if $G \neq U(1)$). Of course, this reduces to pure Yang-Mills theory when one restricts to $\psi = 0$ and $g_{\rm M}$ (Ricci) flat.

Eventually, one can also recover the full action functional of the so-called Standard Model of elementary particles including the famous Higgs potential. For this one has to consider even more general super fields Φ for appropriate twisted Grassmann bundles. Interestingly, the structure of this bundle is determined by the topology of M and the choice of the "ground-state" of the still to find "Higgs boson" (c.f. [22] and the corresponding References cited therein).

The above mentioned examples may suffice to exhibit the generality of the Dirac action (1) and how it covers important classes of coupled partial differential equations as Euler-Lagrange equations of a natural generalization of the Einstein-Hilbert Lagrangian of Einstein's theory of gravity (43). Once one has the universal functional (1) one may ask for the corresponding form of the Euler-Lagrange equations. This, however, depends on the choice of the (twisting part of the) underlying twisted Grassmann bundle and is still a major challenge to exhibit in full generality. In the case, where the bundle is fixed and endowed with sufficient structure one may determine the most general Dirac type operator that is compatible with the endowed structure. Basically, this amounts to determine the most general super-field that is compatible with the given structure and then rewriting the universal Dirac action in terms of this super-field. As a specific example, this will be demonstrated in the next section in terms of a specific class of "real, Hermitian Clifford modules", called "particle-anti-particle modules".

Before, however, we want to briefly comment on "spin versus non-spin manifolds". So far, we concentrated on twisted Grassmann bundles and one may ask what does it give more than twisted spinor bundles. Also, one may ask how the latter fits with the geometrical frame of twisted Grassmann bundles.

First of all, if M is a spin-manifold (i.e. it has vanishing second Stiefel-Whitney classes) and $S \twoheadrightarrow M$ is the (complexified) spinor bundle with respect to a chosen spin-structure, then

$$\Lambda_{\rm M} \otimes \mathbb{C} \simeq S \otimes S^* \longrightarrow M \tag{76}$$

and hence

$$\mathcal{E}_{\Lambda,\,E} \simeq S \otimes W \longrightarrow M \tag{77}$$

where $W \equiv S^* \otimes E \twoheadrightarrow M$.

Moreover, if $\mathcal{S} \simeq Cl_{\rm M}\mathfrak{e} \equiv \{\mathfrak{a}\mathfrak{e} \in Cl_{\rm M} \,|\, \mathfrak{a} \in Cl_{\rm M}\}$ with $\mathfrak{e} \in \mathfrak{Sec}(M, Cl_{\rm M})$ being an appropriately global primitive idempotent and $S \simeq \mathcal{S} \otimes \mathbb{C}$, then

$$\begin{array}{rcl} S \otimes E & \hookrightarrow & \mathcal{E}_{\Lambda,\,E} \\ s \otimes y & \mapsto & s \otimes \mathfrak{e}^* \otimes y \end{array} \tag{78}$$

yields a canonical inclusion of the twisted spinor bundle

$$\mathcal{E}_{\rm E} := S \otimes E \longrightarrow M \tag{79}$$

into the twisted Grassmann bundle (31). Here, $\mathfrak{e}^*$ is the idempotent that yields the dual spinor module $\mathcal{S}^* := \mathfrak{e}\, Cl_{\rm M}$ and $S^* \simeq \mathcal{S}^* \otimes \mathbb{C}$. Note that we only consider complex modules.

In this way, the slightly more general situation of a twisted Grassmann bundles also covers the geometrical situation where M is supposed to be a spin-manifold. On the other hand, by a famous Theorem due to R. Geroch, a non-compact four-dimensional Lorentzian manifold possesses a spin-structure if and only if its frame bundle is trivial (c.f. [7]). Apparently, to propose that M is a spin-manifold is thus a very strong assumption about the topology of M. Note that locally every Clifford module looks like a twisted spinor bundle according to the fundamental decomposition (33).

Therefore, the geometrical setup of twisted Grassmann bundles is slightly more general than twisted spinor modules and much less restrictive (actually, twisted Grassmann bundles always exist). On the other hand, to consider arbitrary Clifford modules seems far too general. In particular, the metric $g_{\rm M}$ does not fix the Clifford module structure γ, in general, like (25) does in the case of a twisted Grassmann bundle. For that matter it becomes difficult to fix the domain of the universal Dirac action for general Clifford modules. Only in the case of twisted Grassmann bundles, the Einstein-Hilbert functional may interpret to provide restrictions also with respect to the module structure of the vector bundle (31).

We close this section by two remarks: First, one obtains for $(M, g_{\rm M})$ denoting a closed compact and orientable Riemannian manifold of even dimension that there exists real constants α, β such that

$$\int_M {\rm tr}_{\mathcal{E}} V_{\rm D}\, dvol_{\rm M} = \alpha\, \mathcal{I}_{\rm EH}(i\partial\!\!\!/_{\rm A}) + \beta\, \mathcal{W}res\left(D\!\!\!\!/^{\,2-2k}\right) \tag{80}$$

independent of the chosen $\partial^{\rm E}$. Here, "$\mathcal{W}res$" is the "Wodzicki residue", i.e. *the* trace functional on the algebra of classical pseudo-differential operators acting on $\mathfrak{Sec}(M, \mathcal{E}_{\Lambda,\,E})$ (see, for example, [20] and the given References therein; also see [2] and [21]).

Therefore, in the case of $\dim M = 4$, the universal Dirac action is basically equal (up to a shift and the quadratic term in (4)) to the trace of the "propagator" (i.e. the Greens operator) of $D\!\!\!\!/^{\,2}$. This may demonstrate once again how natural the functional (1) actually is.

Second, the Dirac-like form (28) has been studied since from the beginning of the last century, c.f. [19], [13], [15], [14], [16], [10], [12], [11] and [17]. Apparently, this form of the Maxwell equations has a natural generalization:

$$\not{D}F_{\mathrm{D}} = 0 \tag{81}$$

where, again, $F_{\mathrm{D}} := curv(\not{D}) - R_g$ is the relative curvature with respect to $\not{D} \in \mathcal{D}(\mathcal{E}_{\Lambda,\,E})$. Accordingly, solutions of this generalized Maxwell equation like, for example, (anti-) self dual solutions may provide interesting restrictions to $\mathcal{D}(\mathcal{E}_{\Lambda,\,E})$ and hence to the Dirac action (1). Note that, when expressed in terms of the super field $\Phi \in \mathfrak{S}ec(M, \Lambda_{\mathrm{M}} \otimes \mathrm{End}_{\mathrm{Cl}}(\mathcal{E}_{\Lambda,\,E}))$ the generalized Maxwell equation (81) actually becomes a system of nonhomogeneous partial differential equations.

We finally mention that generalizations of the Dirac type operator $i\not{\partial}_{\mathrm{A}}$ also play a fundamental role in A. Connes's noncommutative geometry (c.f., for example, [6]) and in the case of the proof of the family index theorem, (c.f., for example, in [18], [4]).

4 Particle-anti-particle modules and Dirac operators of Pauli type

In this section we discuss another specific class of Clifford modules. These modules are mainly motivated by the structure that underpins the Majorana equation (50). These "particle-anti-particle" modules will also provide us with a better geometrical understanding of Pauli type Dirac operators. In particular, these modules will yield a geometrical motivation for the restriction of "diagonal sections" $\Psi = (\psi, \psi)$, such that the Pauli term appears in the bosonic part of the universal Dirac action (1) but drops out in fermionic part (58) of (1).

To get started let, again, (M, g_{M}) be a given orientable, semi-Riemannian manifold of even dimension $n = 2k \geq 2$. Also, let $\tau_{\mathrm{Cl}} \equiv (Cl_{\mathrm{M}}, M)$ collect the data of the Clifford bundle $Cl_{\mathrm{M}} \twoheadrightarrow M$ associated with (M, g_{M}).

Definition 2. *By a "real Hermitian Clifford module (bundle)" we understand a collection of data*

$$(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, J_{\mathcal{E}}, \gamma_{\mathcal{E}}) \tag{82}$$

where, respectively, $\mathcal{E}$ is the total space of a complex vector bundle $\xi_{\mathcal{E}} \equiv (\mathcal{E}, M, \pi_{\mathcal{E}})$ over M, $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ a fiber-wise Hermitian product turning $\xi_{\mathcal{E}}$ into a Hermitian vector bundle over M, $\tau_{\mathcal{E}} \in \mathrm{End}(\mathcal{E})$ is an involution giving rise to a $\mathbb{Z}_2-$grading of $\xi_{\mathcal{E}}$ and $J_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ denotes a real structure, i.e. an anti-linear involution on $\mathcal{E}$ that allows to identify $\xi_{\mathcal{E}}$ with its conjugate complex vector bundle $\xi_{\bar{\mathcal{E}}} := \bar{\xi}_{\mathcal{E}} \equiv (\bar{\mathcal{E}}, M, \bar{\pi}_{\mathcal{E}})$ over M. Finally,

$$\begin{array}{rccc} \gamma_{\mathcal{E}} : & T^*M & \longrightarrow & \mathrm{End}(\mathcal{E}) \\ & \nu & \mapsto & \gamma_{\mathcal{E}}(\nu) \end{array} \tag{83}$$

is a Clifford mapping such that all mappings are "quasi-Hermitian" (i.e. either Hermitian or skew-Hermitian) and $\tau_{\mathcal{E}}$ and $\gamma_{\mathcal{E}}$ are "quasi real" (i.e. either real or purely imaginary with respect to

$J_{\mathcal{E}})$:

$$\begin{aligned} J_{\mathcal{E}} \circ \tau_{\mathcal{E}} \circ J_{\mathcal{E}} &= \pm\tau_{\mathcal{E}} \\ J_{\mathcal{E}} \circ \gamma_{\mathcal{E}} \circ J_{\mathcal{E}} &= \pm\gamma_{\mathcal{E}} \,. \end{aligned} \tag{84}$$

Here, a real structure is called "quasi Hermitian" provided it fulfills

$$\langle J_{\mathcal{E}}(z), J_{\mathcal{E}}(w)\rangle_{\mathcal{E}} \pm \langle w, z\rangle_{\mathcal{E}} \tag{85}$$

for all $z, w \in \mathcal{E}$. *Similar to complex linear mappings this is denoted by* $J_{\mathcal{E}}^{\mathrm{t}} = \pm\, J_{\mathcal{E}}$, *where, in general, "* $^{\mathrm{t}}$ *" means Hermitian transpose with respect to* $\langle\cdot,\cdot\rangle_{\mathcal{E}}$. *If* $J_{\mathcal{E}}^{\mathrm{t}} = +\, J_{\mathcal{E}}$, *the real structure is also called an "anti-unitary involution".*

In the following we are interested in a specific class of real Hermitian Clifford modules, called "particle modules".

Definition 3. *A real Hermitian Clifford module over* τ_{Cl} *is called a "particle module" if*

1. *The involution is skew-Hermitian and purely imaginary;*
2. *The Clifford mapping is skew-Hermitian and real.*

The corresponding conjugate complex module is called an "anti-particle module" over τ_{Cl}.

We denote a particle module by

$$\xi_{\mathrm{P}} \equiv (\mathcal{P}, \langle\cdot,\cdot\rangle_{\mathrm{P}}, \tau_{\mathrm{P}}, J_{\mathrm{P}}, \gamma_{\mathrm{P}})\,. \tag{86}$$

A *particle-anti particle module* over M is a real Hermitian Clifford module (bundle) over τ_{Cl}

$$\xi_{\mathrm{P\bar{P}}} \equiv (\mathcal{P}\bar{\mathcal{P}}, \langle\cdot,\cdot\rangle_{\mathrm{P\bar{P}}}, \tau_{\mathrm{P\bar{P}}}, J_{\mathrm{P\bar{P}}}, \gamma_{\mathrm{P\bar{P}}}) \tag{87}$$

where, respectively

1. $\mathcal{P}\bar{\mathcal{P}} := \mathcal{P} \oplus_{\mathrm{M}} \bar{\mathcal{P}}$;
2. $\langle(z_1, w_1), (z_2, w_2)\rangle_{\mathrm{P\bar{P}}} := \frac{1}{2}\left(\langle z_1, z_2\rangle_{\mathrm{P}} + \langle w_1, w_2\rangle_{\mathrm{P}}\right)$;
3. $\tau_{\mathrm{P\bar{P}}}(z, w) : (\tau_{\mathrm{P}}(z), -\tau_{\mathrm{P}}(w))$;
4. $J_{\mathrm{P\bar{P}}}(z, w) : (J_{\mathrm{P}}(w), J_{\mathrm{P}}(z))$;
5. $\gamma_{\mathrm{P\bar{P}}}(z, w) : (\gamma_{\mathrm{P}}(z), \gamma_{\mathrm{P}}(w))$

for all $z, w, \ldots, w_2 \in \mathcal{P}$.

It follows that for all $\nu \in T^*M$:

1. $J^{\rm t}_{\rm P\bar{P}} = \pm\, J_{\rm P\bar{P}} \quad \Leftrightarrow \quad J^{\rm t}_{\rm P} = \pm\, J_{\rm P}\,;$

2. $\tau^{\rm t}_{\rm P\bar{P}} = \pm\, \tau_{\rm P\bar{P}} \quad \Leftrightarrow \quad \tau^{\rm t}_{\rm P} = \pm\, \tau_{\rm P}\,;$

3. $\gamma^{\rm t}_{\rm P\bar{P}}(\nu) = \pm\, \gamma_{\rm P\bar{P}}(\nu) \quad \Leftrightarrow \quad \gamma^{\rm t}_{\rm P}(\nu) = \pm\, \gamma_{\rm P}(\nu)\,;$

4. $J_{\rm P\bar{P}} \circ \tau_{\rm P\bar{P}} = \pm\, \tau_{\rm P\bar{P}} \circ J_{\rm P\bar{P}} \quad \Leftrightarrow \quad J_{\rm P} \circ \tau_{\rm P} = \mp\, \tau_{\rm P} \circ J_{\rm P}$

5. $J_{\rm P\bar{P}} \circ \gamma_{\rm P\bar{P}}(\nu) = \pm\, \gamma_{\rm P\bar{P}}(\nu) \circ J_{\rm P\bar{P}} \quad \Leftrightarrow \quad J_{\rm P} \circ \gamma_{\rm P}(\nu) = \pm\, \gamma_{\rm P}(\nu) \circ J_{\rm P}\,;$

6. $\tau_{\rm P\bar{P}} \circ \gamma_{\rm P\bar{P}}(\nu) = \pm\, \gamma_{\rm P\bar{P}}(\nu) \circ \tau_{\rm P\bar{P}} \quad \Leftrightarrow \quad \tau_{\rm P} \circ \gamma_{\rm P}(\nu) = \pm\, \gamma_{\rm P}(\nu) \circ \tau_{\rm P}\,.$

Theorem 1. *The most general real Dirac type operator on a particle-anti-particle module $\xi_{\rm P\bar{P}}$ reads*

$$\not{D}_{\rm P\bar{P}} \begin{pmatrix} \not{D}_{P} & J_P \circ \Phi_P \circ J_P \\ \Phi_P & J_P \circ \not{D}_{P} \circ J_P \end{pmatrix} \equiv \begin{pmatrix} \not{D}_{P} & \Phi^{\rm c}_P \\ \Phi_P & \not{D}^{\rm c}_{P} \end{pmatrix}, \tag{88}$$

with

$$\not{D}_{P} :\, \mathfrak{S}ec(M,\mathcal{P}) \longrightarrow \mathfrak{S}ec(M,\mathcal{P}) \tag{89}$$

being a general Dirac type operator on the underlying particle module ξ_P and $\Phi_P \in \mathfrak{S}ec(M, {\rm End}(\mathcal{P}))$ being a zero order operator that is even with respect to the $\mathbb{Z}_2-$grading on $\mathcal{P}$

Proof: To prove the statement we mention that an odd first order differential operator on a $\mathbb{Z}_2-$graded vector bundle $\xi_{\rm W} \equiv (\mathcal{W}, M, \pi_{\rm W})$ over a (semi-)Riemannian manifold $(M, g_{\rm M})$ is of Dirac type if and only if for all $f \in \mathcal{C}^\infty(M)$ the mapping

$$\begin{aligned} \gamma_{\rm W} :\, T^*M &\longrightarrow {\rm End}(\mathcal{W}) \\ df &\mapsto [\not{D}, f] \equiv \not{D} \circ f - f \circ \not{D} \end{aligned} \tag{90}$$

yields a Clifford action on $\xi_{\rm W}$. Here, the ring $\mathcal{C}^\infty(M)$ acts multiplicatively on the (sheave) of sections $\mathfrak{S}ec(M, \mathcal{W})$.

Likewise, if $(\xi_{\rm W}, \gamma_{\rm W})$ denotes a Clifford module, then an odd first order differential operator

$$\not{D} : \mathfrak{S}ec(M,\mathcal{W}) \longrightarrow \mathfrak{S}ec(M,\mathcal{W}) \tag{91}$$

is of Dirac type (that is compatible with the given module structure) if and only if

$$[\not{D}, f] = \gamma_{\rm W}(df)\,. \tag{92}$$

Let $\xi_{\rm P\bar{P}}$ be a particle-anti-particle module and

$$\not{D} := \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \tag{93}$$

be a general first order differential operator acting on $\mathfrak{S}ec(M, \mathcal{P}\bar{\mathcal{P}})$:

$$D_k : \mathfrak{S}ec(M, \mathcal{P}) \longrightarrow \mathfrak{S}ec(M, \mathcal{P}) \tag{94}$$

for $k = 1, \ldots, 4$.

The operator $\not{D}$ is odd with respect to $\tau_{\mathrm{P}\bar{\mathrm{P}}}$ if and only if

$$\tau_{\mathrm{P}\bar{\mathrm{P}}} \circ D_k - D_k \circ \tau_{\mathrm{P}\bar{\mathrm{P}}} \tag{95}$$

for $k = 1, 4$ and

$$\tau_{\mathrm{P}\bar{\mathrm{P}}} \circ D_k + D_k \circ \tau_{\mathrm{P}\bar{\mathrm{P}}} \tag{96}$$

for $k = 2, 4$.

Then,

$$[\not{D}, f] = \gamma_{\mathrm{P}\bar{\mathrm{P}}}(df) \tag{97}$$

for all $f \in \mathcal{C}^\infty(M)$ if and only if

$$[D_k, f] = \gamma_{\mathrm{P}}(df) \tag{98}$$

for $k = 1, 4$ and

$$[D_k, f] = 0 \tag{99}$$

for $k = 2, 3$.

Therefore, the first order differential operators $D_1 \equiv \not{D}_1$ and $D_4 \equiv \not{D}_2$ are of Dirac type on the underlying particle module ξ_{P}. In contrast, the operators $D_2 \equiv \Phi_2$ and $D_3 \equiv \Phi_1$ are of zero order.

Next, we consider the conditions on the Dirac type operator

$$\not{D} := \begin{pmatrix} \not{D}_1 & \Phi_2 \\ \Phi_1 & \not{D}_2 \end{pmatrix} \tag{100}$$

such that $\not{D}$ is real with respect to $J_{\mathrm{P}\bar{\mathrm{P}}}$.

It follows that

$$J_{\mathrm{P}\bar{\mathrm{P}}} \circ \not{D} \circ J_{\mathrm{P}\bar{\mathrm{P}}} \not{D} \quad \Leftrightarrow \quad \begin{cases} \not{D}_2 & = & J_{\mathrm{P}} \circ \not{D}_1 \circ J_{\mathrm{P}} \,, \\ \Phi_2 & = & J_{\mathrm{P}} \circ \Phi_1 \circ J_{\mathrm{P}} \,. \end{cases} \tag{101}$$

This finally proves the statement. □

Note that neither $\not{D}_{\mathrm{P}}$, nor Φ_{P} are supposed to be real, in general.

Let

$$\mathcal{M}_{\mathrm{P\bar{P}}} : \{(z, z^{\mathrm{c}}) \in \mathcal{P}\bar{\mathcal{P}} \,|\, z, z^{\mathrm{c}} \equiv J_{\mathrm{P}}(z) \in \mathcal{P}\} \tag{102}$$

be the real subspace defined by $J_{\mathrm{P\bar{P}}}$ such that

$$\mathcal{P}\bar{\mathcal{P}} = \mathcal{M}_{\mathrm{P\bar{P}}} \otimes \mathbb{C}\,. \tag{103}$$

The corresponding real vector bundle is denoted by $\xi_{\mathcal{M}} \equiv (\mathcal{M}_{\mathrm{P\bar{P}}}, M, \pi_{\mathcal{M}})$ with the projection $\pi_{\mathcal{M}}$ being given by the restriction of $\pi_{\mathrm{P\bar{P}}}$ to $\mathcal{M}_{\mathrm{P\bar{P}}} \subset \mathcal{P}\bar{\mathcal{P}}$. Note that $\xi_{\mathcal{M}} \subset \xi_{\mathrm{P\bar{P}}}$ is a real τ_{Cl} submodule. Clearly, the latter itself contains a distinguished real sub-module given by $z \in \mathcal{P}$ fulfilling $z^{\mathrm{c}} = z$. That is, it is given by the real sub (bundle) space

$$\mathcal{M}_{\mathrm{P}} \oplus \mathcal{M}_{\mathrm{P}} : \{(z, z) \in \mathcal{P}\bar{\mathcal{P}} \,|\, J_{\mathrm{P}}(z) = z \in \mathcal{P}\} \subset \mathcal{M}_{\mathrm{P\bar{P}}}\,, \tag{104}$$

where $\mathcal{M}_{\mathrm{P}} := \{z \in \mathcal{P} \,|\, z = J_{\mathrm{P}}(z)\} \subset \mathcal{P}$, such that $\mathcal{P} = \mathcal{M}_{\mathrm{P}} \otimes \mathbb{C}$.

On a particle-anti-particle module, the first order differential operator (88) is the most general real Dirac type operator. Hence, one may restrict the universal Dirac action (1) to this type of Dirac operators:

$$\mathcal{I}_{\mathrm{D,real}} \;:\; \tfrac{1}{2}\int_M [\langle \Psi_{\mathrm{P\bar{P}}},\, \not{D}_{\mathrm{P\bar{P}}} \Psi_{\mathrm{P\bar{P}}}\rangle_{\mathrm{P\bar{P}}} + \mathrm{tr}_\gamma curv(\not{D}_{\mathrm{P\bar{P}}})]\; dvol_{\mathrm{M}} \tag{105}$$

with $\Psi_{\mathrm{P\bar{P}}} = (\Psi_{\mathrm{P}}, \Psi^{\mathrm{c}}_{\mathrm{P}}) \in \mathfrak{S}ec(M, \mathcal{M}_{\mathrm{P\bar{P}}})$ and $\not{D}_{\mathrm{P\bar{P}}}$ any real Dirac operator on the particle-anti-particle module $\xi_{\mathrm{P\bar{P}}}$.

Proposition 2. *When boundary terms are neglected, the Dirac action (105) decomposes as follows:*

$$\mathcal{I}_{D,real}\mathcal{I}_{D,ferm}(\not{D}_{\mathrm{P\bar{P}}}) + \mathcal{I}_{D,bos}(\not{D}_{\mathrm{P\bar{P}}}) \tag{106}$$

where

$$\begin{aligned} 2\mathcal{I}_{D,ferm}(\not{D}_{\mathrm{P\bar{P}}}) \;\equiv\; & \int_M [\langle \Psi_P, \not{D}_P \Psi_P\rangle_P + \langle \Psi^{\mathrm{c}}_P, \not{D}^{\mathrm{c}}_P \Psi^{\mathrm{c}}_P\rangle_P \\ & + \langle \Psi_P, \Phi^{\mathrm{c}}_P \Psi^{\mathrm{c}}_P\rangle_P + \langle \Psi^{\mathrm{c}}_P, \Phi_P \Psi_P\rangle_P]\, dvol_M\,; \end{aligned} \tag{107}$$

$$\begin{aligned} 2\mathcal{I}_{D,bos}(\not{D}_{\mathrm{P\bar{P}}}) \;\equiv\; & \int_M [\mathrm{tr}_\gamma curv(\not{D}_P) + \mathrm{tr}_\gamma curv(\not{D}^{c}_P) + 2\,\mathrm{tr}(\Phi^{\mathrm{c}}_P \circ \Phi_P) \\ & + 8\,(\mathrm{tr} \circ \mathrm{ev}_g)(\alpha^{\mathrm{c}}_P \circ \alpha_P) + 2\,(\mathrm{tr} \circ \mathrm{ev}_g)(\beta^{\mathrm{c}}_P \circ \beta_P)]\, dvol_M \end{aligned} \tag{108}$$

where $2\alpha_P(v) : \Phi_P \circ \gamma_P(v^\flat) \in \mathrm{End}(\mathcal{P})$ *and* $v^\flat(u) = g_M(v,u)$ *for all* $u, v \in TM$*. Accordingly,* $2\alpha^{\mathrm{c}}_P(v) : J_P \circ 2\alpha_P(v) \circ J_P^{-1}\Phi^{c}_P \circ \gamma_P(v^\flat) \in \mathrm{End}(\mathcal{P})$*. Furthermore,* $\beta_P : \mathrm{ext}_\Theta(\Phi_P - 2\,\not{\alpha}_P)$ *with* $\Theta \in \Omega^1(M, \mathrm{End}(\mathcal{P}))$ *being the canonical one-form that exists on every Clifford module (c.f. [22]) and* $\beta^{\mathrm{c}}_P : J_P \circ \beta_P \circ J_P^{-1}\mathrm{ext}_\Theta(\Phi^{\mathrm{c}}_P - 2\,\not{\alpha}^{c}_P)$*.*

In the sequel, we make use of the common "dagger" abbreviation: $\not\alpha \equiv \gamma(\alpha) \in \Omega^0(M, \mathrm{End}(\mathcal{P}))$ for any $\alpha \in \Omega^*(M, \mathrm{End}(\mathcal{P}))$. For example, for $\alpha = e^k \otimes \lambda_k$ one has $\not\alpha = \gamma^k \otimes \lambda_k$, etc.

Proof: The fermionic part is straightforward to prove. The bosonic part of the Dirac action is proved in several steps.

First, we prove the following

Lemma 1. *Let $\not{D}_1$ and $\not{D}_2$ be two Dirac type first order differential operators on an arbitrary Clifford module $(\xi_\mathcal{E}, \gamma_\mathcal{E}) \equiv (\mathcal{E}, M, \pi_\mathcal{E}, \gamma_\mathcal{E})$ with $\pi_\mathcal{E} : \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \to M$ being a $\mathbb{Z}_2$−graded (Hermitian) vector bundle over (M, g_M). The zero-order term V_H of the generalized Laplacian*

$$\begin{array}{rcl} H : \mathfrak{S}ec(M, \mathcal{E}) & \longrightarrow & \mathfrak{S}ec(M, \mathcal{E}) \\ \Psi & \mapsto & \not{D}_1(\not{D}_2\Psi) \end{array} \tag{109}$$

has the explicit form:

$$V_H = V_D + \Phi \circ \not\omega_D + \mathrm{ev}_g(\alpha_H^2) + U\,. \tag{110}$$

Here, respectively, V_D and ω_D are the Dirac potential and Dirac form of $\not{D} \equiv \not{D}_2$ (c.f. [23]). Moreover, $\Phi := \not{D}_1 - \not{D}_2 \in \mathfrak{S}ec(M, \mathrm{End}^-(\mathcal{E}))$ and

$$U := \mathrm{ev}_g\left(\nabla_H^{T^*M\otimes\mathrm{End}(\mathcal{E})}\alpha_H\right) \tag{111}$$

with $\nabla_H^\mathcal{E}$ being the covariant derivative that defines the connection Laplacian of H :

$$\triangle_H : -\mathrm{ev}_g\left(\nabla_H^{T^*M\otimes\mathcal{E}} \circ \nabla_H^\mathcal{E}\right) \tag{112}$$

and $\alpha_H \in \Omega^1(M, \mathrm{End}_M(\mathcal{E}))$ is given by

$$\begin{array}{rcl} 2\,\alpha_H(\mathrm{grad}_g f) & := & [\Phi \circ \not{D}, f] \\ & = & \Phi \circ \gamma_\mathcal{E}(df) \end{array} \tag{113}$$

for all $f \in \mathcal{C}^\infty(M)$.

Here and henceforth we make use of the following notation: "ev_g" means "evaluation/contraction" with respect to g_M. For instance, $\mathrm{ev}_g(\alpha^2) \overset{\mathrm{loc.}}{=} \mathrm{ev}_g(e^\mu \otimes e^\nu \otimes \alpha_\mu \circ \alpha_\nu) : g_\mathrm{M}(e^\mu, e^\nu)\, \alpha_\mu \circ \alpha_\nu \in \mathrm{End}(\mathcal{E})$ for all $\alpha \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$ etc.

Proof: With $\not{D}_1 = \not{D}_2 + \Phi \equiv \not{D} + \Phi$ it becomes sufficient to consider Laplace type operators of the form

$$H = \not{D}^2 + \Phi \circ \not{D}\,. \tag{114}$$

Every generalized Laplacian H decomposes as (see, for instance, in [3])

$$H = -\triangle_{\mathrm{H}} + V_{\mathrm{H}} \tag{115}$$

with $\nabla^{\varepsilon}_{\mathrm{H}}$ being given by

$$2\,\mathrm{ev}(f_0\,\mathrm{grad} f_1, \nabla^{\varepsilon}_{\mathrm{H}}\Psi) := f_0\left([H, f_1] + \triangle_g f_1\right)\Psi \tag{116}$$

for all $f_0, f_1 \in \mathcal{C}^\infty(M)$ and $\Psi \in \mathfrak{S}ec(M, \mathcal{E})$. Here, $\triangle_g$ denotes the usual Laplace-Beltrami operator restricted to zero-forms on M.

It follows that

$$\nabla^{\varepsilon}_{\mathrm{H}}\nabla^{\varepsilon}_{\mathrm{D}} + \alpha_{\mathrm{H}} \tag{117}$$

with $\nabla^{\varepsilon}_{\mathrm{D}}$ being the covariant derivative that defines the Bochner-Laplacian of $\not{D}^2$.

As a consequence, the connection Laplacian of H may be expressed in terms of the Bochner-Laplacian[1] of $\not{D}^2$:

$$\triangle_{\mathrm{H}} = \triangle_{\mathrm{D}} + 2\,\mathrm{ev}_g\left(\alpha_{\mathrm{H}} \circ \nabla^{\varepsilon}_{\mathrm{D}}\right) + \mathrm{ev}_g(\alpha^2_{\mathrm{H}}) + U\,. \tag{118}$$

The statement then follows by comparison of the general Lichnerowicz decomposition (115), taking into account that $\not{D}^2 = -\triangle_{\mathrm{D}} + V_{\mathrm{D}}$ and

$$2\,\mathrm{ev}_g(\alpha_{\mathrm{H}} \circ \nabla^{\varepsilon}_{\mathrm{D}}) = \Phi \circ \not{\nabla}^{\varepsilon}_{\mathrm{D}}\,. \tag{119}$$

□

Note that

$$\mathrm{tr}_{\mathcal{E}} U = \mathrm{div}_g\,\xi_{\mathrm{H}} \tag{120}$$

with

$$\xi_{\mathrm{H}} := (\mathrm{tr}_{\mathcal{E}}\alpha_{\mathrm{H}})^\sharp \in \mathfrak{S}ec(M, TM)\,. \tag{121}$$

Hence,

$$\mathrm{tr}_{\mathcal{E}} V_{\mathrm{H}}\, dvol_{\mathrm{M}} \left[\mathrm{tr}_{\mathcal{E}} V_{\mathrm{D}} + \mathrm{tr}_{\mathcal{E}}(\Phi \circ \not{\phi}_{\mathrm{D}}) + (\mathrm{tr}_{\mathcal{E}} \circ \mathrm{ev}_g)(\alpha^2_{\mathrm{H}})\right]\, dvol_{\mathrm{M}} + \pounds_{\xi_{\mathrm{H}}} dvol_{\mathrm{M}} \tag{122}$$

which demonstrates that

$$[*\mathrm{tr}_{\mathcal{E}} V_{\mathrm{H}}][*\mathrm{tr}_{\mathcal{E}}(V_{\mathrm{D}} + \Phi \circ \not{\phi}_{\mathrm{D}} + \mathrm{ev}_g(\alpha^2_{\mathrm{H}}))] \in \mathrm{H}^{\mathrm{n}}_{\mathrm{deR}}(M) \tag{123}$$

with "$*$" being the Hodge map induced by g_{M} and the orientation defined by $dvol_{\mathrm{M}}$.

[1] I would like to thank M. Schneider for appropriate comments.

Clearly, for $\not{D}_1 = \not{D}_2 = \not{D}$ one has

$$V_{\rm H} = V_{\rm D}\,. \tag{124}$$

Next, we present a Bochner-Lichnerowicz-Weizenböck type formula for a slightly more general Laplace type second order differential operator H'.

Corollary 2. *Let again $(\xi_\mathcal{E}, \gamma_\mathcal{E})$ be a Clifford module over (M, g_M). Also, let $D_k = \not{D}_k + \Phi_k$ $(k = 1, 2)$ be two first order differential operators acting on $\mathfrak{Sec}(M, \mathcal{E})$. The zero-order term $V_{\rm H'}$ of the generalized Laplacian*

$$\begin{array}{rcl} H' : \mathfrak{Sec}(M,\mathcal{E}) & \longrightarrow & \mathfrak{Sec}(M,\mathcal{E}) \\ \Psi & \mapsto & D_1(D_2\Psi) \end{array} \tag{125}$$

reads:

$$V_{\rm H'} = V_H + V \tag{126}$$

where V_H is given by (110) with the replacement

$$\Phi \;:=\; D_1 - D_2 + 2\Phi_2 \tag{127}$$

and

$$V := (\Phi - \Phi_2)\circ\Phi_2 + [\not{D}_2, \Phi_2]\,. \tag{128}$$

Proof: We put $D_1 = \not{D}_2 + \Phi_0 + \Phi_1 \equiv \not{D} + \Phi_{01}$ and rewrite H' as

$$\begin{array}{rcl} H' & = & \not{D}^2 + \Phi\circ\not{D} + V \\ & = & H + V\,. \end{array} \tag{129}$$

Hence, the connection Laplacian $\triangle_{\rm H'}$ of H' is the same as the connection Laplacian $\triangle_{\rm H}$ of H. One may thus apply the former Lemma 1 to prove the statement. □

As a consequence, one obtains explicitly (neglecting boundary terms):

$${\rm tr}_\mathcal{E} V_{\rm H'} {\rm tr}_\mathcal{E} V_{\rm D} + {\rm tr}_\mathcal{E}(\Phi\circ\Phi_2 - \Phi_2^2) + {\rm tr}_\mathcal{E}[\not{D}_2, \Phi_2] + {\rm tr}_\mathcal{E}(\Phi\circ\not{\omega}_{\rm D}) + ({\rm tr}_\mathcal{E}\circ{\rm ev}_g)(\alpha_{\rm H}^2)\,. \tag{130}$$

We are now in the position to prove the bosonic part of Proposition 2. In fact, this will be an immediate consequence of the following more general

Proposition 3. *Let $(\xi_{\mathcal{E}}, \gamma_{\mathcal{E}})$ be a Clifford module over (M, g_M). Also, let $(\xi_{2\mathcal{E}}, \gamma_{2\mathcal{E}})$ be the Clifford module that is defined by the corresponding Whitney sum:*

$$\xi_{2\mathcal{E}} : \xi_{\mathcal{E}} \oplus \xi_{\mathcal{E}}\,, \quad \tau_{2\mathcal{E}} : \tau_{\mathcal{E}} \oplus (-\tau_{\mathcal{E}})\,, \quad \gamma_{2\mathcal{E}} : \gamma_{\mathcal{E}} \oplus \gamma_{\mathcal{E}} \tag{131}$$

with $\tau_{\mathcal{E}}$ being the grading involution on $\xi_{\mathcal{E}}$.

The zero order term $V_{\mathcal{D}} \in \mathfrak{Sec}(M, \mathrm{End}(2\mathcal{E}))$ associated with the (square of the) most general Dirac type first order differential operator

$$\mathcal{D\!\!\!/} \equiv \begin{pmatrix} D\!\!\!/_1 & \Phi_2 \\ \Phi_1 & D\!\!\!/_2 \end{pmatrix} : \begin{array}{c} \mathfrak{Sec}(M,\mathcal{E}) \\ \oplus \\ \mathfrak{Sec}(M,\mathcal{E}) \end{array} \longrightarrow \begin{array}{c} \mathfrak{Sec}(M,\mathcal{E}) \\ \oplus \\ \mathfrak{Sec}(M,\mathcal{E}) \end{array} \tag{132}$$

reads:

$$V_{\mathcal{D}} \;=$$

$$\begin{pmatrix} V_1 + \Phi_2 \circ \Phi_1 + 4\,\mathrm{ev}_g(\alpha_2 \circ \alpha_1) & [D\!\!\!/_1\,, \Phi_2] + \Phi_2 \circ (D\!\!\!/_1 - D\!\!\!/_2) + 2\,\Phi_2 \circ \omega\!\!\!/_2 \\ [D\!\!\!/_2\,, \Phi_1] + \Phi_1 \circ (D\!\!\!/_2 - D\!\!\!/_1) + 2\,\Phi_1 \circ \omega\!\!\!/_1 & V_2 + \Phi_1 \circ \Phi_2 + 4\,\mathrm{ev}_g(\alpha_1 \circ \alpha_2) \end{pmatrix} \tag{133}$$

where, respectively, V_k and ω_k denote the Dirac potential and the Dirac form of $D\!\!\!/_k$ $(k = 1, 2)$ and α_k is defined in terms of $D\!\!\!/_k^2$, similar to α_H of Lemma (1).

Proof: We may write

$$\mathcal{D\!\!\!/} = D\!\!\!/ + \Phi \tag{134}$$

with

$$D\!\!\!/ : \begin{pmatrix} D\!\!\!/_1 & 0 \\ 0 & D\!\!\!/_2 \end{pmatrix}, \qquad \Phi : \begin{pmatrix} 0 & \Phi_2 \\ \Phi_1 & 0 \end{pmatrix}. \tag{135}$$

Then, we simply make use of the preceding Corollary 2 and apply the corresponding Bochner-Lichnerowicz-Weizenböck type formula to $H' := \mathcal{D\!\!\!/}^2$. Note that the Bochner-Laplacian of $\mathcal{D\!\!\!/}$ is given by

$$\nabla^{2\mathcal{E}}_{\mathcal{D}} \nabla^{2\mathcal{E}}_{\mathrm{D}} + \beta_{\mathcal{D}} \tag{136}$$

with $\beta_{\mathcal{D}} \in \Omega^1(M, \mathrm{End}(2\mathcal{E}))$ being

$$\beta_{\mathcal{D}} : \begin{pmatrix} 0 & 2\alpha_2 \\ 2\alpha_1 & 0 \end{pmatrix}. \tag{137}$$

Here, again

$$\begin{aligned} 2\,\alpha_k(\mathrm{grad}_g f) \;&:=\; [\Phi_k \circ D\!\!\!/_k\,, f] \\ &=\; \Phi_k \circ \gamma_{\mathcal{E}}(df) \end{aligned} \tag{138}$$

for all $f \in \mathcal{C}^\infty(M)$ and $k = 1, 2$.

Then, similar to the results presented before

$$V_{\mathcal{D}} = V_{\rm D} + U + 2\,\Phi \circ \not{\omega}_{\rm D} + {\rm ev}_g(\beta^2_{\mathcal{D}}) \tag{139}$$

where

$$\not{\omega}_{\rm D} : \gamma_{2\mathcal{E}}(\omega_{\rm D}) \equiv \begin{pmatrix} \not{\omega}_1 & 0 \\ 0 & \not{\omega}_2 \end{pmatrix}, \qquad U := {\rm ev}_g\left(\nabla^{T^*M\otimes{\rm End}(2\mathcal{E})}_{\mathcal{D}}\beta_{\mathcal{D}}\right) . \tag{140}$$

□

Therefore,

$$\mathrm{tr}_{2\mathcal{E}} V_{\mathcal{D}} \mathrm{tr}_{\mathcal{E}} V_1 + \mathrm{tr}_{\mathcal{E}} V_2 + 2\,\mathrm{tr}_{\mathcal{E}}(\Phi_1 \circ \Phi_2) + 8\,(\mathrm{tr}_{\mathcal{E}} \circ {\rm ev}_g)(\alpha_1 \circ \alpha_2) + div_g \xi_{\mathcal{D}} \tag{141}$$

with

$$\xi_{\mathcal{D}} : (\mathrm{tr}_{2\mathcal{E}}\beta_{\mathcal{D}})^\sharp\,. \tag{142}$$

The bosonic part of the Proposition (2) is then implied by (again, omitting all boundary terms):

$$\begin{aligned} \mathrm{tr}_\gamma(curv(\not{\mathcal{D}})) &= \mathrm{tr}_{2\mathcal{E}} V_{\mathcal{D}} + (\mathrm{tr}_{2\mathcal{E}} \circ {\rm ev}_g)(\omega^2_{\mathcal{D}}) \\ &= \mathrm{tr}_\gamma(curv(\not{D}_1)) + \mathrm{tr}_\gamma(curv(\not{D}_2)) + \\ &\quad + 2\,\mathrm{tr}_{\mathcal{E}}(\Phi_1 \circ \Phi_2) + 8\,(\mathrm{tr}_{\mathcal{E}} \circ {\rm ev}_g)(\alpha_1 \circ \alpha_2) + \\ &\quad + 2\,(\mathrm{tr}_{\mathcal{E}} \circ {\rm ev}_g)(\sigma_1 \circ \sigma_2)\,, \end{aligned} \tag{143}$$

where $\sigma_k := {\rm ext}_\Theta(\Phi_k - 2\,\not{\alpha}_k) \in \Omega^1(M, {\rm End}(\mathcal{E}))$ and $2\alpha_k(v)\Phi_k \circ \gamma_{\mathcal{E}}(v^\flat)$ for all $v \in TM$ and $k = 1, 2$.

This finally ends the proof of Proposition (2). □

The functionals (107–108) may look complicated at first glance. However, they yield a straightforward generalization of the usual action of the Standard Model of particle physics as discussed in [22], which allows to also include Majorana mass terms. The latter feature will be discussed in detail elsewhere.

Proposition 4. *Let J_P be anti-unitary and $\Psi_{\rm P\bar{P}}(\Psi_P, \Psi_P) \in \mathfrak{S}ec(M, \mathcal{M}_P \oplus \mathcal{M}_P)$. Also, let $\not{D}_P$ be real with respect to J_P and both $\not{D}_P$ and the real part of Φ_P be formally self-adjoint. Then, the Dirac action*

$$\mathcal{I}_{D,real}\mathcal{I}_{D,ferm}(\not{D}_{\rm P\bar{P}}) + \mathcal{I}_{D,bos}(\not{D}_{\rm P\bar{P}}) \tag{144}$$

reads:

$$\begin{aligned}\mathcal{I}_{D,ferm}(\not{D}_{\mathrm{P}\bar{\mathrm{P}}}) &= \int_M [\langle \Psi_P,\, (\not{D}_P + \mathcal{Y}_P)\Psi_P\rangle_P]\, dvol_M\,,\\ \mathcal{I}_{D,bos}(\not{D}_{\mathrm{P}\bar{\mathrm{P}}}) &= \int_M \left[\mathrm{tr}_\gamma curv(\not{D}_P) + (\mathrm{tr}\circ \mathrm{ev}_{g'})(Y_P^2) - (\mathrm{tr}\circ \mathrm{ev}_{g'})(F_P^2)\right] dvol_M \qquad (145)\end{aligned}$$

where $Y_P,\ F_P \in \Omega^*(M, \mathrm{End}_{Cl}(\mathcal{P}))$.

Proof: This is a simple application of Proposition 2 taking into account that the (endomorphism valued) one-forms $\alpha_{\mathrm{P}},\, \beta_{\mathrm{P}} \in \Omega^1(M, \mathrm{End}(\mathcal{P}))$ are linearly determined by the zero order operator $\Phi_{\mathrm{P}} \in \mathrm{Sec}(M, \mathrm{End}(\mathcal{P}))$. Furthermore, due to the fundamental decomposition (33) every zero order operator locally reads:

$$\Phi_{\mathrm{P}}\gamma^{\mathrm{I}} \otimes \phi_{\mathrm{I}} \qquad (146)$$

where $\mathrm{I} = (i_1, i_2, \ldots, i_l)$ is a multi-index ($1 \le i_k \le n$ for $k = 0, 1, \ldots, n$), $\gamma^{\mathrm{I}} \equiv \gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_l}$ and ϕ_{I} are local sections of $\mathrm{Sec}(M, \mathrm{End}_{\mathrm{Cl}}(\mathcal{P}))$ which are totally antisymmetric with respect to the multi index I. To avoid double counting the summation is thus take only for the ordered indices: $i_1 < i_2 < \ldots < i_l\,,\ l = 0, 1, , \ldots, n$. In other words, the zero order section $\Phi_{\mathrm{P}} \in \Omega(M, \mathrm{End}(\mathcal{P}))$ is in one-to-one correspondence with a general section $\phi_{\mathrm{P}} \in \Omega^*(M, \mathrm{End}_{\mathrm{Cl}}(\mathcal{P}))$. Then, $\phi_{\mathrm{P}} = e^{\mathrm{I}} \otimes \phi_{\mathrm{I}}$ with $\{e^{\mathrm{I}} \equiv e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_l} \,|\, l = 0, 1, \ldots, n\}$ being a local basis of $\Lambda_{\mathrm{M}} \twoheadrightarrow M$.

Hence, the bosonic part of (105) reduces to

$$2\mathcal{I}_{\mathrm{D,bos}}(\not{D}_{\mathrm{P}\bar{\mathrm{P}}}) \equiv \int_M [\mathrm{tr}_\gamma curv(\not{D}_{\mathrm{P}}) + \mathrm{tr}_\gamma curv(\not{D}^{\mathrm{c}}_{\mathrm{P}}) + 2(\mathrm{tr}\circ \mathrm{ev}_{g'})(\phi^{\mathrm{c}}_{\mathrm{P}} \circ \phi_{\mathrm{P}})]\, dvol_{\mathrm{M}} \qquad (147)$$

where the evaluation map on the right hand side refers to the re-scaled (fiber) metric $g'_{\Lambda\mathrm{M}}$ on the Grassmann bundle $\Lambda_{\mathrm{M}} \twoheadrightarrow M$ that is defined by

$$\begin{aligned}g'^{\mathrm{IJ}} \equiv \lambda'\, g_{\Lambda\mathrm{M}}(e^{\mathrm{I}}, e^{\mathrm{J}}) \ &:= \ \mathrm{tr}\gamma^{\mathrm{I}}\gamma^{\mathrm{J}} + \tfrac{1}{4}\, g_{ij}\, \mathrm{tr}\gamma^{\mathrm{I}}\gamma^i\gamma^{\mathrm{J}}\gamma^j + \\ &\quad + \tfrac{1}{n^2}\, g_{ij}\, \mathrm{tr}\big(\gamma^i\gamma^{\mathrm{I}} - g_{ab}\,\gamma^i\gamma^a\gamma^{\mathrm{I}}\gamma^b\big)\big(\gamma^j\gamma^{\mathrm{J}} - g_{cd}\,\gamma^j\gamma^c\gamma^{\mathrm{J}}\gamma^d\big)\,. \qquad (148)\end{aligned}$$

Indeed, one explicitly has

$$\begin{aligned}\Phi_{\mathrm{P}} \circ \Phi^{\mathrm{c}}_{\mathrm{P}} &= \gamma^{\mathrm{I}}\gamma^{\mathrm{J}} \otimes \phi_{\mathrm{I}} \circ \phi^{\mathrm{c}}_{\mathrm{J}}\,,\\ \mathrm{ev}_g(\alpha_{\mathrm{P}} \circ \alpha^{\mathrm{c}}_{\mathrm{P}}) &= \tfrac{1}{4}\, g_{ij}\, \gamma^{\mathrm{I}}\gamma^i\gamma^{\mathrm{J}}\gamma^j \otimes \phi_{\mathrm{I}} \circ \phi^{\mathrm{c}}_{\mathrm{J}}\,,\\ \mathrm{ev}_g(\beta_{\mathrm{P}} \circ \beta^{\mathrm{c}}_{\mathrm{P}}) &= \tfrac{1}{n^2}\, g_{ij}\, \big(\gamma^i\gamma^{\mathrm{I}} - g_{ab}\,\gamma^i\gamma^a\gamma^{\mathrm{I}}\gamma^b\big)\big(\gamma^j\gamma^{\mathrm{J}} - g_{cd}\,\gamma^j\gamma^c\gamma^{\mathrm{J}}\gamma^d\big) \otimes \phi_{\mathrm{I}} \circ \phi^{\mathrm{c}}_{\mathrm{J}} \qquad (149)\end{aligned}$$

where $g_{ij} \equiv g_{\mathrm{M}}(e_i, e_j)$ etc., $\mathrm{J} = (j_1, j_2, \ldots, j_k)$ is again a multi-index and Einstein's summation convention is applied. Let us call in mind that we do not distinguish between the metric on the tangent and the co-tangent space of M.

As a consequence,

$$\mathrm{tr}(\Phi^{\mathrm{c}}_{\mathrm{P}} \circ \Phi_{\mathrm{P}}) + 4\,(\mathrm{tr}\circ \mathrm{ev}_g)(\alpha^{\mathrm{c}}_{\mathrm{P}} \circ \alpha_{\mathrm{P}}) + (\mathrm{tr}\circ \mathrm{ev}_g)(\beta^{\mathrm{c}}_{\mathrm{P}} \circ \beta_{\mathrm{P}})(\mathrm{tr}\circ \mathrm{ev}_{g'})(\phi^{\mathrm{c}}_{\mathrm{P}} \circ \phi_{\mathrm{P}})\,. \tag{150}$$

Furthermore, every real Dirac type first order differential operator on a particle-anti-particle module may be rewritten as

$$\not{D}_{\mathrm{P}\bar{\mathrm{P}}} \begin{pmatrix} \not{D}_{\mathrm{P}} & \mathcal{Y}_{\mathrm{P}} - \mathcal{F}_{\mathrm{P}} \\ \mathcal{Y}_{\mathrm{P}} + \mathcal{F}_{\mathrm{P}} & \not{D}^{\mathrm{c}}_{\mathrm{P}} \end{pmatrix}. \tag{151}$$

Here,

$$\begin{aligned} \mathcal{Y}_{\mathrm{P}} &:= \tfrac{1}{2}(\Phi_{\mathrm{P}} + \Phi^{\mathrm{c}}_{\mathrm{P}}) \equiv \mathrm{Re}_{\mathrm{J}}\Phi_{\mathrm{P}}\,, & (152)\\ \mathcal{F}_{\mathrm{P}} &:= \tfrac{1}{2}(\Phi_{\mathrm{P}} - \Phi^{\mathrm{c}}_{\mathrm{P}}) \equiv i\mathrm{Im}_{\mathrm{J}}\Phi_{\mathrm{P}}\,, & (153) \end{aligned}$$

such that $\mathcal{Y}_{\mathrm{P}}$ is the real and $-i\mathcal{F}_{\mathrm{P}}$ is the imaginary part of the zero order operator Φ_{P} with respect to the real structure J_{P}.

According to the general case, we put

$$\begin{aligned} \mathcal{Y}_{\mathrm{P}} &= \gamma^{\mathrm{I}} \otimes Y_{\mathrm{I}}\,,\\ \mathcal{F}_{\mathrm{P}} &= \gamma^{\mathrm{J}} \otimes F_{\mathrm{J}}\,. \end{aligned} \tag{154}$$

The statement then follows from

$$\mathrm{tr}(\Phi^{\mathrm{c}}_{\mathrm{P}} \circ \Phi_{\mathrm{P}})\mathrm{tr}(\mathcal{Y}^2_{\mathrm{P}}) - \mathrm{tr}(\mathcal{F}^2_{\mathrm{P}})\,, \tag{155}$$

which is analogous to the case of complex numbers (remember that $\mathcal{F}^{\mathrm{c}}_{\mathrm{P}} - \mathcal{F}_{\mathrm{P}}$). □

We note that $\not{D}_{\mathrm{P}\bar{\mathrm{P}}}$ leaves the real submodule $\mathfrak{S}ec(M, \mathcal{M}_{\mathrm{P}} \oplus \mathcal{M}_{\mathrm{P}}) \subset \mathfrak{S}ec(M, \mathcal{M}_{\mathrm{P}\bar{\mathrm{P}}})$ invariant if and only if $\not{D}^{\mathrm{c}}_{\mathrm{P}} = \not{D}_{\mathrm{P}}$ and $\mathcal{F}_{\mathrm{P}} = 0$.

Clearly, for $\mathcal{Y}_{\mathrm{P}} = 0$ and $\mathcal{F}_{\mathrm{P}}$ the curvature of $\not{D}_{\mathrm{P}} := i\not{\partial}_{\mathrm{A}}$ we get back the Pauli type Dirac operators as specific real Dirac type operators on the real Hermitian Clifford module $\xi_{\mathrm{P}\bar{\mathrm{P}}}$. Moreover, the “diagonal sections” are motivated by the distinguished real submodule

$$\mathfrak{S}ec(M, \mathcal{M}_{\mathrm{P}} \oplus \mathcal{M}_{\mathrm{P}}) \subset \mathfrak{S}ec(M, \mathcal{P}\bar{\mathcal{P}}) \tag{156}$$

of $\xi_{\mathrm{P}\bar{\mathrm{P}}}$. Note that it is the doubling of $\mathcal{M}_{\mathrm{P}}$ (which we may identify with our former twisted Grassmann bundle $\mathcal{E}_{\Lambda,\,E}$) which allows to add the Pauli-term to $i\not{\partial}_{\mathrm{A}}$ such that the resulting first order operator is still of Dirac type. Also note that the additional complex structure encountered in the definition of (45) corresponds to the assumption that the zero order part of (88) is purely imaginary. For the same matter it has to drop out in the fermionic part of the universal Dirac action since it would yield a non-real contribution.

5 Outlook

We presented a detailed motivation of "Dirac type gauge theories" which are gauge theories that are based on the universal Dirac action (1). In particular, we have exhibit how the Dirac action covers well-known differential equations, like the Maxwell and the Einstein equation. Indeed, the Dirac action turns out to be a natural generalization of the Einstein-Hilbert functional. To also obtain the Yang-Mills functional, one has to introduce a specific class of Dirac type operators and we discussed their geometrical origin in terms of real Hermitian Clifford modules. We also discussed the domain of the Dirac action from a geometrical point of view. We thereby proved several Lichnerowicz type formulae for decomposable Laplace type operators which generalize the corresponding result presented in [23].

It is well-known that there is a one-to-one correspondence between Dirac type operators on a Clifford module and Clifford super-connections (see, for instance, in [3]). For this reason, the domain of dependence of the Dirac action may not come as a surprise, especially because of the isomorphisms (59). However, the latter hold true only when the module structure is fixed from the outset. This, of course, does not permit interpreting the Einstein-Hilbert functional as a constraint on the module structure. Moreover, our discussion clearly demonstrates that there exists a natural functional on the Dirac bundle provided by the Dirac action.

The presented results clearly exhibit in what sense Dirac type operators and Clifford modules provide a more general geometrical setting to describe gauge theories than connections and principal bundles. Indeed, Dirac type gauge theories allow to describe different types of gauge theories, like Yang-Mills theory, Einstein's theory of gravity and spontaneously broken Yang-Mills gauge theories, in a geometrically unified setting based on the same universal Dirac functional. This is independent of whether the base manifold ("space-time") M is supposed to be spin or not.

In order to gain more insight, however, one has to deal with the "moduli space of Dirac operators"

$$\mathfrak{M}(\mathcal{E}) \equiv \mathcal{D}(\mathcal{E})/Diff(\mathcal{E}) \tag{157}$$

on which the Dirac functional descents. Of course, this set is probably far too wild and thus has to be restricted to appropriate subsets like the solutions of

$$*F_{\rm D} = \pm F_{\rm D}\,, \tag{158}$$

similar to the moduli space of (anti-) self-dual solutions of the ordinary Yang-Mills equation: $*F_{\rm A} \pm F_{\rm A}$. For this, however, the domain of the Dirac functional has to be discussed more seriously, in particular from an analytical point of view.

In contrast to the ordinary Yang-Mills equations one obtains still another reasonable constrains to Dirac type operators, similar to the (anti-) self-duality condition as, for instance, the "unimodularity" condition:

$$\pounds_{\xi_{\rm D}} dvol_{\rm M} = 0\,. \tag{159}$$

Finally, one may pose the question to what extent is there a relation between the stationary points of the Dirac action (1) and the generalized Maxwell equation

$$\not{D} F_{\mathrm{D}} = 0\,. \tag{160}$$

Again, in full generality this seems a hopeless task. However, it might be reasonable to discuss this question using appropriate simple geometrical settings. This will be done in a forthcoming work.

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