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Network Oligopolies with Multiple Markets

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ABSTRACT

Cournot oligopolies with multiple markets are examined when transportation costs from each manufacturer to all markets are also included into the profit functions. Under general and realistic conditions the equilibrium always exists, and based on the Kuhn-Tucker conditions a computer method is introduced to compute it. The dynamic extension of the model is also introduced with gradient adjustment, and the asymptotic stability of the equilibrium is examined. In the case of independent markets and linear price and cost functions the solution algorithm is simplified and the global asymptotic stability of the equilibrium is proved.

RESUMEN

Son examinados oligopolios Cournot con mercados multiples cuando los costos de transporte de todas las manofacturas de todos los mercados son incluidos en la función de utilidad. Bajo condiciones generales y realistas siempre existe equilibrio, y basados en condiciones de Kuhn-Tucker un método computacional es introducido para calcularlo. La extensión dinámica del modelo es también presentada con adaptación gradiente, es examinada la estabilidad asintótica. En el caso de mercados independientes y funciones de precio y costo lineales el algoritmo de solución es simplificado y es provada la estabilidad asintótica global del equilibrio.

Key words and phrases: *Oligopoly, multiple market, equilibrium, stability.*

Math. Subj. Class.: *91A06, 91A80.*

1 Introduction

The classical oligopoly theory considers a homogeneous market and several firms offering a certains product to the market. This simple model has been examined and extended by many authors in the last few decades. Models with product differentiation, multi-product oligopolies, labor managed firms, rent-seeking games were analysed to mention only a few. A comprehensive summary of these models and their properties is given in Okuguchi (1976) and in Okuguchi and Szidarovszky (1999). Emition control, waste management, technology spill-over, secondary market competition were also included into oligopoly models, however very few attempts were made to examine the effect of multiple markets. They were mostly treated as multi-product oligopolies, where the same good sold in different markets was considered as different products. If we introduce transportation costs into the models, their structure becomes different because of the additional additive cost terms.

In this paper we make the first attempt to analyze the properties of oligopoly models including transportation costs to multiple markets. The paper is organized as follows. After the mathematical model is formulated, conditions are given for the existence of the noncooperative Nash equilibrium based on the theory of concave games. Then a computer method is introduced to find the equilibrium. In applying this method we have to find either feasible solutions for a system of usually nonlinear equalities and inequalities, or optimal solutions of a nonlinear programming problem. These methods can be significantly simplified in the cases of independent markets and linear cost and price functions. A similar algorithm will be then proposed to find the completely cooperative solution. A gradient adjustment process is then formulated as a dynamic extension of the model, and its asymptotical stability will be examined. In the case of independent markets and linear cost and price functions we will show that the equilibrium is always globally asymptotically stable.

2 Mathematical Model

Consider a network of N firms that produce the same product or offer the same service to M different markets. Let x_{ki} be the quantity offered by firm k to market i, then the supply on market i is given as $s_i = \sum_{k=1}^{N} x_{ki}$ and the output of firm k is $q_k = \sum_{i=1}^{M} x_{ki}$. The production cost $C_k(q_k)$ of firm k depends on q_k, and the price on each market i depends on the supplies on all markets, since consumers might have choices where they want to purchase the goods. So the price on market i is given as $p_i(s_1, \ldots, s_M)$.

Assume also that there is a transportation cost (including possible duties) for each firm for supplying its product on the different markets: $T_{k1}(x_{k1}) + \cdots + T_{kM}(x_{kM})$. Based on the above notation, the profit of firm k equals

$$\Pi_k = \sum_{i=1}^{M} x_{ki} p_i(s_1, \ldots, s_M) - C_k(q_k) - \sum_{i=1}^{M} T_{ki}(x_{ki}). \tag{1}$$

It is also assumed that each x_{ki} value is nonnegative and bounded from above: $x_{ki} \in [0, L_{ki}]$.

In this way an N-person game is defined is which the N firms are the players, the strategy of each player is a vector $\underline{x}_k = (x_{k1}, \ldots, x_{kM}) \in [0, L_{k1}] \times \cdots \times [0, L_{kM}] = S_k$, and the payoff function of player k is its profit, Π_k.

The solution of this game is the noncooperative Nash equilibrium, which is a set of strategy vectors $\underline{x}_k^*$ $(1 \le k \le N)$ such that for all players k,

1. $\underline{x}_k^* \in S_k$;
2. $$\Pi_k(\underline{x}_1^*, \dots, \underline{x}_{k-1}^*, \underline{x}_k, \underline{x}_{k+1}^*, \dots, \underline{x}_N^*) \le \Pi_k(\underline{x}_1^*, \dots, \underline{x}_N^*) \tag{2}$$
   with all $\underline{x}_k \in S_k$.

Assume that all functions p_i, C_k and T_{ki} are twice continuously differentiable. Notice that

$$\begin{aligned} \frac{\partial \Pi_k}{\partial x_{ki}} &= p_i + x_{ki}\frac{\partial p_i}{\partial s_i} + \sum_{j \ne i} x_{kj}\frac{\partial p_j}{\partial s_i} - C_k' - T_{ki}' \\ &= p_i + \sum_{l=1}^{M} x_{kl}\frac{\partial p_l}{\partial s_i} - C_k' - T_{ki}', \end{aligned} \tag{3}$$

furthermore

$$\frac{\partial^2 \Pi_k}{\partial x_{ki}^2} = 2\frac{\partial p_i}{\partial s_i} + x_{ki}\frac{\partial^2 p_i}{\partial s_i^2} + \sum_{j \ne i} x_{kj}\frac{\partial^2 p_j}{\partial s_i^2} - C_k'' - T_{ki}'' \tag{4}$$

and

$$\frac{\partial^2 \Pi_k}{\partial x_{ki}\partial x_{kj}} = \frac{\partial p_i}{\partial s_j} + \frac{\partial p_j}{\partial s_i} + \sum_{l=1}^{M} x_{kl}\frac{\partial^2 p_l}{\partial s_i \partial s_j} - C_k''. \tag{5}$$

For the sake of convenient notation let $\underline{J}_p$ be the Jacobian of $\underline{p} = (p_1, \dots, p_M)$ and $\underline{H}_l$ the Hessian of p_l.

Then the Hessian matrix of Π_k with respect to vector $\underline{x}_k$ can be conveniently written as

$$\underline{J}_p + \underline{J}_p^T + \sum_{l=1}^{M} x_{kl}\underline{H}_l - C_k'' \cdot \underline{E} - \underline{D}_k, \tag{6}$$

where $\underline{E}$ is the $M \times M$ matrix with all unity elements, and

$$\underline{D}_k = \mathrm{diag}\,(T_{k1}'', \dots, T_{kM}'').$$

Now we make the following assumptions:

(A) $-\underline{p}$ is monotonic in the sense that

$$\left(\underline{s}^{(1)} - \underline{s}^{(2)}\right)^T \left(\underline{p}\left(\underline{s}^{(1)}\right) - \underline{p}\left(\underline{s}^{(2)}\right)\right) \le 0 \tag{7}$$

for all $\underline{s}^{(1)} = \left(s_1^{(1)}, \dots, s_M^{(1)}\right)$ and $\underline{s}^{(2)} = \left(s_1^{(2)}, \dots, s_M^{(2)}\right)$;

(B) p_l is concave for all l;

(C) $T''_{ki} \geq 0$ for all k and i;

(D) $C''_k \geq 0$.

Notice that (A) implies that $\underline{J}_p + \underline{J}_p^T$ is negative semidefinite (see for example Ortega and Rheinboldt, 1970). Condition (B) implies that all matrices $\underline{H}_l$ are negative semidefinite, and (C) implies that $\underline{D}_k$ is positive semidefinite. Consequently the Hessian of Π_k with respect to $\underline{x}_k$ is negative semidefinite, so Π_k is concave in $\underline{x}_k$. Therefore the Nikaido–Isoda theorem (Forgo et al., 1999) implies that there is at least one Nash equilibrium. Here we used the fact that $\underline{E}$ is positive semidefinite with eigenvalues 0 and M (see for example Okuguchi and Szidarovszky, 1999).

3 Computation of the Equilibrium

The equilibrium output of any firm k is the optimal solution of the problem

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{M} x_{ki} p_i(s_1, \ldots, s_M) - C_k(q_k) - \sum_{i=1}^{M} T_{ki}(x_{ki}) \\ \text{s.to} & 0 \leq x_{ki} \leq L_{ki} \text{ for all } i. \end{array} \tag{8}$$

The Kuhn–Tucker conditions are sufficient and necessary because of the concavity of Π_k. They can be written as follows. There exist constants $u_{k1}, \ldots, u_{kM}, v_{k1}, \ldots, v_{kM}$ such that

$$u_{ki} \geq 0, v_{ki} \geq 0 \text{ (all } i), \tag{9}$$

$$0 \leq x_{ki} \leq L_{ki} \text{ (all } i), \tag{10}$$

$$\left(\frac{\partial \Pi_k}{\partial x_{k1}}, \ldots, \frac{\partial \Pi_k}{\partial x_{kM}}\right) + (u_{k1}, v_{k1}, \ldots, u_{kM}, v_{kM}) \begin{pmatrix} 1 & & & \\ -1 & & & \\ & 1 & & \\ & -1 & & \\ & & \vdots & \\ & & & 1 \\ & & & -1 \end{pmatrix} = (0, \ldots, 0), \tag{11}$$

$$u_{ki} x_{ki} = 0 \text{ (all } i), \tag{12}$$

$$v_{ki} \left(L_{ki} - x_{ki}\right) = 0 \text{ (all } i). \tag{13}$$

Here we gave only the nonzero elements of the matrix and used the fact that the constraints of problem (8) have the form

$$x_{ki} \geq 0 \text{ and } L_{ki} - x_{ki} \geq 0.$$

The $(x_{k1}, \ldots, x_{kM})$ $(k = 1, \ldots, N)$ part of any feasible solution of this equality-inequality system provides Nash-equilibrium.

We can also rewrite this feasibility problem as a single objective optimization problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{N}\sum_{i=1}^{M}\left(u_{ki}x_{ki}+v_{ki}(L_{ki}-x_{ki})\right) \\ \text{subject to} \quad & u_{ki}\geq 0, v_{ki}\geq 0, 0\leq x_{ki}\leq L_{ki} \text{ (all } k \text{ and } i) \\ & \frac{\partial \Pi_k}{\partial x_{ki}}+u_{ki}-v_{ki}=0 \text{ (all } k \text{ and } i), \end{aligned} \tag{14}$$

which can be solved by routine methods.

The dimension of this problem can be reduced by introducing new variables

$$\alpha_{ki}=u_{ki}-v_{ki}. \tag{15}$$

From the last constraint we have

$$\alpha_{ki}=-\frac{\partial \Pi_k}{\partial x_{ki}} \tag{16}$$

and the nonnegativity of u_{ki} and v_{ki} implies that

$$v_{ki}=u_{ki}-\alpha_{ki}\geq -\alpha_{ki},$$

that is,

$$\alpha_{ki}+v_{ki}\geq 0$$

for all k and i. Substituting (15) and (16) into the objective function we have a simplified model:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{N}\sum_{i=1}^{M}\left(-\frac{\partial \Pi_k}{\partial x_{ki}}x_{ki}+v_{ki}L_{ki}\right) \\ \text{subject to} \quad & v_{ki}\geq 0 \\ & v_{ki}\geq \frac{\partial \Pi_{ki}}{\partial x_{ki}} \text{ (all } k \text{ and } i) \\ & 0\leq x_{ki}\leq L_{ki}. \end{aligned} \tag{17}$$

In the objective function $L_{ki}>0$, so the objective is minimal if and only if v_{ki} is minimal. From the first two constraints of (17) we see that the minimal value of v_{ki} is the larger of 0 and $\frac{\partial \Pi_k}{\partial x_{ki}}$. Hence a more simple version of the optimization model is the following:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{N}\sum_{i=1}^{M}\left(-\frac{\partial \Pi_k}{\partial x_{ki}}x_{ki}+L_{ki}\cdot\max\left\{0;\frac{\partial \Pi_k}{\partial x_{ki}}\right\}\right) \\ \text{subject to} \quad & 0\leq x_{ki}\leq L_{ki}. \end{aligned} \tag{18}$$

4 The Case of Independent Markets

In this section we assume that the market price p_i depends on only the supply s_i on market i, so $p_i=p_i(s_i)$. In this special case

$$\Pi_k=\sum_{i=1}^{M}x_{ki}p_i(s_i)-C_k(q_k)-\sum_{i=1}^{M}T_{ki}(x_{ki}) \tag{19}$$

and therefore

$$\frac{\partial \Pi_k}{\partial x_{ki}} = p_i + x_{ki} p_i' - C_k' - T_{ki}', \tag{20}$$

furthermore

$$\frac{\partial^2 \Pi_k}{\partial x_{ki}^2} = 2p_i' + x_{ki} p_i'' - C_k'' - T_{ki}'' \tag{21}$$

and with $j \neq i$,

$$\frac{\partial^2 \Pi_k}{\partial x_{ki} \partial x_{kj}} = -C_k''. \tag{22}$$

Therefore the Hessian of Π_k has the more special diagonal form

$$\begin{pmatrix} 2p_1' - T_{k1}'' + x_{k1}p_1'' & & & \\ & 2p_2' - T_{k2}'' + x_{k2}p_2'' & & \\ & & \ddots & \\ & & & 2p_M' - T_{kM}'' + x_{kM}p_M'' \end{pmatrix} - C_k'' \cdot \underline{E}. \tag{23}$$

Recall that $\underline{E}$ is positive semidefinite, so this matrix is negative semidefinite if for all i,

(A') $p_i' < 0$;

(B') $p_i'' \leq 0$

in addition to assumptions (C) and (D). The Nikaido–Isoda theorem implies that under conditions (A'), (B'), (C) and (D) there is at least one Nash-equilibrium.

In this special case the optimization problem (17) reduces to the following:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{N} \sum_{i=1}^{M} \left(x_{ki} \left(-p_i - x_{ki} p_i' + C_k' + T_{ki}' \right) + v_{ki} L_{ki} \right) \\ \text{subject to} \quad & v_{ki} \geq 0 \\ & v_{ki} \geq p_i + x_{ki} p_i' - C_k' - T_{ki}' \\ & 0 \leq x_{ki} \leq L_{ki}. \end{aligned} \tag{24}$$

Assume next that all function p_i, C_k and T_{ki} are linear $p_i(s_i) = A_i - B_i s_i, C_k(q_k) = a_k + b_k q_k$ and $T_{ki} = \alpha_{ki} + \beta_{ki} x_{ki}$. Then

$$p_i' = -B_i, C_k' = b_k, T_{ki}' = \beta_{ki},$$

so we have a simple optimization problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{N} \sum_{i=1}^{M} \left(x_{ki} \left(-A_i + B_i s_i + x_{ki} B_i + b_k + \beta_{ki} \right) + v_{ki} L_{ki} \right) \\ \text{subject to} \quad & v_{ki} \geq 0 \\ & v_{ki} - A_i + B_i s_i + x_{ki} B_i + b_k + \beta_{ki} \geq 0 \\ & 0 \leq x_{ki} \leq L_{ki}. \end{aligned} \tag{25}$$

In this problem all constraints are linear (notice that $s_i = \sum_{k=1}^{N} x_{ki}$), and the objective function can be rewritten as follows:

$$\begin{aligned}
&\sum_{k=1}^{N}\sum_{i=1}^{M} v_{ki}L_{ki} + \sum_{k=1}^{N}\sum_{i=1}^{M} x_{ki}(-A_i + B_i s_i + x_{ki}B_i + b_k + \beta_{ki}) \\
&= \sum_{k=1}^{N}\sum_{i=1}^{M} \left\{ v_{ki}L_{ki} + x_{ki}\left(\beta_{ki} + b_k - A_i\right) + B_i x_{ki}\left(2x_{ki} + \sum_{l\neq k} x_{li}\right)\right\} \qquad (26)
\end{aligned}$$

which is a quadratic function of its unknows. For the actual solution of this problem standard software is available.

5 Cooperative Solutions

The most commonly applied cooperative solutions (such as the Shapley value, see for example, Forgó et al., 1999) maximizes the total profit of all firms, and then this maximal profit is distributed among the firms in a well defined, concept-dependent way. The total profit equals

$$\Pi = \sum_{k=1}^{N} \Pi_k = \sum_{j=1}^{M} s_j p_j\left(s_1, \ldots, s_M\right) - \sum_{k=1}^{N} C_k(q_k) - \sum_{k=1}^{N}\sum_{j=1}^{M} T_{kj}(x_{kj}). \qquad (27)$$

The maximal solution of this function can be obtained in the following way. Notice first that

$$\frac{\partial \Pi}{\partial x_{ki}} = p_i + s_i \frac{\partial p_i}{\partial s_i} + \sum_{j\neq i} s_j \frac{\partial p_j}{\partial s_i} - C_k' - T_{ki}' \qquad (28)$$

and the constraints can be rewritten as

$$\begin{aligned}
x_{ki} &\geq 0 \\
L_{ki} - x_{ki} &\geq 0,
\end{aligned}$$

so the Kuhn–Tucker conditions guarantee the existence of nonnegative constants u_{ki} and v_{ki} such that

$$\begin{aligned}
u_{ki} \geq 0, v_{ki} \geq 0 \ (\text{all } k \text{ and } i) \\
0 \leq x_{ki} \leq L_{ki}
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{\partial \Pi}{\partial x_{11}}, \ldots, \frac{\partial \Pi}{\partial x_{1M}}, \ldots, \frac{\partial \Pi}{\partial x_{N1}}, \ldots, \frac{\partial \Pi}{\partial x_{NM}},\right) \\
&+ (u_{11}, v_{11}, \ldots, u_{NM}, v_{NM}) \begin{pmatrix} 1 & & & \\ -1 & & & \\ & 1 & & \\ & -1 & & \\ & & \ddots & \\ & & & 1 \\ & & & -1 \end{pmatrix} \\
&= (0, \ldots, 0) \qquad (29)
\end{aligned}$$

$$\begin{aligned} u_{ki}x_{ki} &= 0 \\ v_{ki}(L_{ki} - x_{ki}) &= 0. \end{aligned}$$

From the third condition we see that for all k and i,

$$\frac{\partial \Pi}{\partial x_{ki}} = u_{ki} - v_{ki}. \tag{30}$$

So the set of feasible solutions of this system of equalities and inequalities contains the optimal solutions. If Π is concave as an NM-variable function, then the Kuhn-Tucker conditions are sufficient and necessary, so every feasible solution of system (29) is optimal for problem (27). Another way is the direct optimization of the objective function Π subject to the constraints $0 \le x_{ki} \le L_{ki}$ for all k and i.

6 Dynamic Extensions

Assuming continuous time scales – as usual is the theory of dynamic economics – it is assumed that the firms adjust their production levels proportionally to their marginal profits. This concept can be mathematically modeled as

$$\dot{x_{ki}} = K_{ki}\frac{\partial \Pi_k}{\partial x_{ki}} = K_{ki}\left(p_i + \sum_{j=1}^{M} x_{kj}\frac{\partial p_j}{\partial s_i} - C_k' - T_{ki}'\right) \tag{31}$$

for all k and i, where $K_{ki} > 0$ is a constant known as the speed of adjustment of firm k. The motivation of this construct is the following. If $\frac{\partial \Pi_k}{\partial x_{ki}} > 0$, then the profit of firm k increases by the increasing value of x_{ki}. If $\frac{\partial \Pi_k}{\partial x_{ki}} < 0$, then the profit increases by decreasing value of x_{ki}. If this partial derivative is zero, then the current production level is at a stationary point, so the firm does not want changes.

Notice first that the interior equilibria are the steady states of this system. Using linearization, the local asymptotic stability of the equilibrium is guaranteed if all eigenvalues of the Jacobian of the right hand side functions are in the left half of the complex plane. Let g_{ki} denote the right hand side of equation (k, i). Simple differentiation shows that

$$\frac{\partial g_{ki}}{\partial x_{ki}} = K_{ki}\left(2 \cdot \frac{\partial p_i}{\partial s_i} + x_{ki}\frac{\partial^2 p_i}{\partial s_i^2} + \sum_{j\neq i} x_{kj}\frac{\partial^2 p_j}{\partial s_i^2} - C_k'' - T_{ki}''\right) \tag{32}$$

$$\frac{\partial g_{ki}}{\partial x_{kj}} = K_{ki}\left(\frac{\partial p_i}{\partial s_j} + \frac{\partial p_j}{\partial s_i} + \sum_{l=1}^{M} x_{kl}\frac{\partial^2 p_l}{\partial s_i \partial s_j} - C_k''\right) \tag{33}$$

and if $l \neq k$, then

$$\frac{\partial g_{ki}}{\partial x_{lj}} = K_{ki}\left(\frac{\partial p_i}{\partial s_j} + \sum_{r=1}^{M} x_{kr}\frac{\partial^2 p_r}{\partial s_i \partial s_j}\right) \tag{34}$$

for all j. Based on these derivatives, the Jacobian can be represented in a block form $(\underline{J}_{kl})$ where $\underline{J}_{kl}$ is an $M \times M$ matrix:

$$\underline{J}_{kl} = \begin{cases} \underline{K}_k\left(\underline{J}_p + \underline{J}_p^T + \sum_j x_{kj}\underline{H}_j - C_k'' \cdot \underline{E} - \underline{D}_k\right) & \text{if } l = k \\ \underline{K}_k\left(\underline{J}_p + \sum_j x_{kj}\underline{H}_j\right) & \text{if } l \neq k \end{cases} \tag{35}$$

with $\underline{K}_k = \mathrm{diag}(K_{k1}, K_{k2}, \ldots, K_{kM})$ and the notations of (6). The equilibrium is locally asymptotically stable if all eigenvalues of the Jacobian have negative real parts. The structure of the Jacobian is complicated, so we will only illustrate this condition in the linear case, when $p_i(s_i) = A_i - B_i s_i$, $C_k(q_k) = a_k + b_k q_k$ and $T_{ki} = \alpha_{ki} + \beta_{ki} x_{ki}$ (as in constructing the optimization problem (25)). Then with the notation $\underline{P} = \mathrm{diag}(-B_1, -B_2, \ldots, -B_M)$, the Jacobian is

$$\begin{pmatrix} 2\underline{K}_1\underline{P} & \underline{K}_1\underline{P} & \ldots & \underline{K}_1\underline{P} \\ \underline{K}_2\underline{P} & 2\underline{K}_2\underline{P} & \ldots & \underline{K}_2\underline{P} \\ \vdots & \vdots & & \vdots \\ \underline{K}_N\underline{P} & \underline{K}_N\underline{P} & \ldots & 2\underline{K}_N\underline{P} \end{pmatrix}$$

where each block is diagonal. By interchanging the rows and columns, this matrix can be rearranged as a blockdiagonal matrix

$$\mathrm{diag}(\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_M)$$

with $N \times N$ blocks,

$$\underline{A}_i = \begin{pmatrix} -2K_{1i}B_i & -K_{1i}B_i & \ldots & -K_{1i}B_i \\ -K_{2i}B_i & -2K_{2i}B_i & \ldots & -K_{2i}B_i \\ \vdots & \vdots & & \vdots \\ -K_{Ni}B_i & -K_{Ni}B_i & \ldots & -2K_{Ni}B_i \end{pmatrix}$$

so the equilibrium is locally asymptotically stable if the eigenvalues of each block have negative real parts. This is guaranteed if the eigenvalues of matrix

$$\begin{pmatrix} 2K_{1i} & K_{1i} & \ldots & K_{1i} \\ K_{2i} & 2K_{2i} & \ldots & K_{2i} \\ \vdots & \vdots & & \vdots \\ K_{Ni} & K_{Ni} & \ldots & -2K_{Ni} \end{pmatrix}$$

have positive real parts for all i. This matrix can be rewritten as $\underline{D} + \underline{k} \cdot \underline{1}^T$ with

$$\underline{D}_i = \mathrm{diag}(K_{1i}, K_{2i}, \ldots, K_{Ni}), \underline{k}_i = (K_{1i}, K_{2i}, \ldots, K_{Ni})^T, \text{ and } \underline{1}^T = (1, 1, \ldots, 1).$$

The characteristic polynomial has the form

$$\begin{aligned} \varphi_i(\lambda) &= \det\left(\underline{D} + \underline{k} \cdot \underline{1}^T - \lambda \underline{I}\right) = \det\left(\underline{D} - \lambda \underline{I}\right) \det\left(\underline{I} + (\underline{D} - \lambda \underline{I})^{-1} \underline{k} \cdot \underline{1}^T\right) \\ &= \prod_{l=1}^{N} (K_{li} - \lambda) \left[1 + \sum_{l=1}^{N} \frac{K_{li}}{K_{li} - \lambda}\right]. \end{aligned}$$

The roots of the first factor are all positive. Let $h(\lambda)$ denote the bracketed factor. Clearly

$$\lim_{\lambda \to \pm\infty} h(\lambda) = 1, \lim_{\lambda \to K_{li} \pm 0} h(\lambda) = \mp\infty,$$

$$h'(\lambda) = \sum_{l=1}^{N} \frac{K_{li}}{(K_{li} - \lambda)^2} > 0.$$

So $h(\lambda)$ locally strictly increasing. Therefore there is one root between each consecutive pair of poles (which are the positive K_{ki} numbers) and an additional root after the largest pole. We found this way N real positive roots. Equating the bracketed term with zero leads to a polynomial equation of degree N, so we found all roots, they are real and positive. So the equilibrium is always locally asymptotically stable. Notice that in the linear case system (31) is also linear, so local stability implies global asymptotical stability, therefore independently of the initial conditions the trajectories of the system converge to the Nash-equilibrium.

7 Conclusions

In this paper Cournot oligopolies were examined with multiple markets, when additional transportation costs were included into the profit functions of the firms. After the mathematical model was formulated sufficient conditions were presented for the existence of the Nash equilibrium, and computer method were presented for determining the equilibrium based on the Kuhn–Tucker conditions. This method was simplified in the case of independent markets, and was further modified to find the completely cooperative solution. A gradient adjustment dynamic model was finally introduced and its asymptotical stability examined. In the case of independent markets and linear price and cost functions the equilibrium is always globally asymptitocally stable with respect to gradient adjustments.

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