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A Multiobjective Model of Oligopolies under Uncertainty

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ABSTRACT

It is assumed that in an n-firm single-product oligopoly without product differentiation the firms face an uncertain price function, which is considered random by the firms. At each time period each firm simultaneously maximizes its expected profit and minimizes the variance of the profit since it wants to receive as high as possible profit with the least possible uncertainty. It is assumed that the best response of each firm is obtained by the weighting method. We show the existence of a unique equilibrium, and investigate the local stability of the equilibrium.

RESUMEN

Es asumido que en un oligopolio de n-firmas "single-product" sin diferenciación producto firmas con función de precio variable, son consideradas randon por las firmas.

En todo período de tiempo todo firma simultaneamente maximiza la utilidad esperada y minimiza la variación de utilidad desde que estan quisen obtener la utilidad mas alta posible con la menor incertidumbre posible. Es asumido que la mejor respuesta de toda firma es obtenida por el metodo weighting. Mostramos la existencia de un equilibrio único y investigamos la estabilidad local del equilibrio.

Key words and phrases: *Uncertainty, n-person games, multiobjective optimization.*

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1 Introduction

The uncertainty in the inverse demand functions of oligopoly models has been previously examined by many researches. Cyert and DeGroot (1971, 1973) investigated mainly duopolies. Kirman (1975, 1983) examined the case of differentiated products and linear demand functions and analysed how the resulting equilibria depend on the way the firms misspecify and try to assess the demand function. Gates et al. (1982) also examined linear demand functions and differentiated products. Leonard and Nishimura (1999) assumed that the firms know the shape of the demand function but misspecify its scale, and investigated the asymptotic behavior of the equilibrium under discrete time scales. Chiarella and Szidarovszky (2001) have introduced the continuous counterpart of the Leonard-Nishimura model and in addition to equilibrium and local stability analysis, the destabilising effect of time delays, in obtaining and implementing information on the competitors' output, was analysed. Bischi et al. (2004) consider the situation in which the firms' reaction functions are unimodal and analyse the various equilibria that may arise and their complicated basins of attraction. All these earlier studies assumed that the firms maximized their misspecified or expected payoffs at each time period depending on the type (deterministic or stochastic) of the model being used.

In this paper we introduce a new approach. The uncertainty of the inverse demand function is treated here also with a stochastic model, but we assume that at each time period each firm maximize its expected profit and at the same time tends to reduce profit uncertainty by minimizing its variance. That is, at each time period the firms face a "Pareto-game", each with multiple payoffs (see for example Szidarovszky et al., 1986). In our model, in each time period each firm uses a multiobjective optimization approach to find its best response. Based on these best response functions a dynamic process develops. The subject of this paper is the properties of this dynamic process including equilibrium analysis and the investigation of the asymptotic behavior of the equilibrium.

2 The Mathematical Model and Equilibrium Analysis

Consider an n-firm single-product oligopoly without product differentiation. Let q_j be the output of firm j, and $C_j(q_j)$ the associated cost function. Assume that firm j believes that the true price function is $f(Q) + \eta_j$, where $Q = \sum_{j=1}^{n} q_j$ is the total output of the industry and η_j is a random variable such that

$$E(\eta_j) = 0 \text{ and } Var(\eta_j) = \sigma_j^2. \tag{2.1}$$

Using π_j to denote profit, the expected profit of firm j is given as

$$E(\pi_j) = q_j f(Q) - C_j(q_j), \tag{2.2}$$

and the variance of profit is

$$Var(\pi_j) = q_j^2 \sigma_j^2. \tag{2.3}$$

Assume that firm j wants to maximize its expected profit and at the same time to minimize the variance of the profit. That is, the firm tends to obtain as high a profit as possible with minimum uncertainty. It is also assumed that firm j uses the weighting method (see for example, Szidarovszky et al. 1986), therefore it maximizes a linear combination of the two objective functions, i.e.

$$\max \left[E(\pi_j) - \frac{\alpha_j}{2} Var(\pi_j) \right], \tag{2.4}$$

where α_j shows the relative importance of reducing uncertainty compared to the increase of the expected profit.

In oligopoly theory it is usually assumed that the functions f and C_j $(j = 1, 2, \cdots, n)$ are twice continuously differentiable, f is decreasing, C_j is increasing, furthermore

$$\begin{aligned} &(a) \quad f'(Q) + q_j f''(Q) \le 0, \\ &(b) \quad f'(Q) - C_j''(q_j) < 0, \end{aligned}$$

for all j and nonnegative q_j and Q.

Under conditions (a) and (b) the composite objective function (2.4) is strictly concave in q_j with fixed value of $Q_j = \sum_{l \ne j} q_l$. The derivative of the objective function (2.4) with respect to q_j can be given as

$$q_j f'(Q) + f(Q) - C_j'(q_j) - \alpha_j q_j \sigma_j^2.$$

Notice that $f' \le 0$, $C' \ge 0$, $f(Q) \le f(0)$, therefore with positive α_j and σ_j, this derivative converges to $-\infty$ as $q_j \to \infty$ implying that there is a unique maximizing value, $q_j \ge 0$, with any fixed $Q_j \ge 0$. The best response of firm j, $R_j(Q_j)$, can be obtained in the following way. If

$$f(Q_j) - C_j'(0) \le 0, \tag{2.5}$$

then $R_j(Q_j) = 0$, otherwise it is the unique positive solution of the equation

$$q_j f'(q_j + Q_j) + f(q_j + Q_j) - C_j'(q_j) - \alpha_j q_j \sigma_j^2 = 0. \tag{2.6}$$

Since f is decreasing, from condition (2.5) we see that if $R_j(Q_j) = 0$ then for all $\bar{Q}_j > Q_j$, $R_j(\bar{Q}_j) = 0$. Assume next that $R_j(Q_j) > 0$. Then equation (2.6) is satisfied with $q_j = R_j(Q_j)$. By implicit differentiation we have[1]

$$R_j' f' + R_j f''(1 + R_j') + f'(1 + R_j') - C_j'' R_j' - \alpha_j R_j' \sigma_j^2 = 0,$$

from which we have

$$R_j' = -\frac{f' + R_j f''}{2f' + R_j f'' - C_j'' - \alpha_j \sigma_j^2}. \tag{2.7}$$

Conditions (a) and (b) imply that

$$-1 < R_j' \leq 0. \tag{2.8}$$

Therefore R_j is a decreasing function of Q_j. We can rewrite equation (2.6) as

$$q_j f'(q_j + Q_j) + f(q_j + Q_j) - C_j'(q_j) = \alpha_j q_j \sigma_j^2. \tag{2.9}$$

The left hand side strictly decreases in q_j, therefore the solution $q_j = R_j(Q_j)$ decreases if α_j and/or σ_j increases.

We can also consider q_j as a function of the total output level of the industry, $q_j = q_j(Q)$, which can be defined as follows. If

$$f(Q) - C_j'(0) \leq 0, \tag{2.10}$$

then $q_j(Q) = 0$, otherwise it is the unique positive solution of the equation

$$q_j f'(Q) + f(Q) - C_j'(q_j) - \alpha_j q_j \sigma_j^2 = 0. \tag{2.11}$$

With fixed values of Q, the left hand side is strictly decreasing in q_j, it has a positive value at $q_j = 0$ and converges to $-\infty$ as $q_j \to \infty$. Similarly to the previous case, condition (2.5) implies that if $q_j(Q) = 0$ then for all $\bar{Q} > Q$ we have $q_j(\bar{Q}) = 0$. If $q_j(Q) > 0$, then by letting $q_j' = \frac{d}{dQ} q_j(Q)$ we have

$$q_j' f' + q_j f'' + f' - C_j'' q_j' - \alpha_j q_j' \sigma_j^2 = 0,$$

implying that

$$q_j' = -\frac{f' + q_j f''}{f' - C_j'' - \alpha_j \sigma_j^2} \leq 0, \tag{2.12}$$

so $q_j(Q)$ is decreasing in Q. We can rewrite equation (2.11) as

$$q_j f'(Q) + f(Q) - C_j'(q_j) = \alpha_j q_j \sigma_j^2. \tag{2.13}$$

The left hand side is decreasing in q_j, therefore the solution $q_j(Q)$ decreases if α_j and/or σ_j increases.

[1] We use the notation $R_j' = \frac{d}{dQ_j} R_j(Q_j)$.

The total industry output Q at the equilibrium is the solution of

$$\sum_{j=1}^{n} q_j(Q) - Q = 0. \tag{2.14}$$

At $Q = 0$, all $q_j(Q) \geq 0$, so at $Q = 0$ the left hand side is nonnegative. Since

$$\sum_{j=1}^{n} q_j(Q) \leq \sum_{j=1}^{n} q_j(0),$$

it follows that the left hand side converges to $-\infty$ as $Q \to \infty$, furthermore it is strictly decreasing in Q. Consequently there is a unique nonnegative solution of equation (2.14) proving the existence and the uniqueness of the equilibrium. In summary, we have the following result.

Theorem 1 *Under conditions (a) and (b) there is a unique equilibrium of the modified n-person oligopoly with payoff functions (2.4).*

The unique equilibrium of theorem 1 is usually different from the Cournot-Nash equilibrium of the n-firm oligopoly under the assumption that all firms know the true price function f. Let $Q^*(\underline{\alpha}, \underline{\sigma})$ denote the total industry output at the equilibrium with given parameters $\underline{\alpha} = (\alpha_1, \cdots, \alpha_n)$ and $\underline{\sigma} = (\sigma_1, \cdots, \sigma_n)$. We will prove the following result:

Theorem 2 *The value of $Q^*(\underline{\alpha}, \underline{\sigma})$ decreases if any α_j or σ_j increases.*

Proof: Assume that $\alpha_j < \bar{\alpha}_j$ with all other α_i and all σ_i values unchanged. Let q_i and $\bar{q}_i$ denote the corresponding equilibrium outputs and let $Q = \sum_i q_i$ and $\bar{Q} = \sum_i \bar{q}_i$. Contrary to the assertion assume that $Q < \bar{Q}$. From the monotonicity of the functions $q_i(Q)$ we have for all $i \neq j$,

$$q_i(Q) \geq q_i(\bar{Q}) = \bar{q}_i(\bar{Q}), \tag{2.15}$$

since α_i and σ_i do not change. However

$$q_j(Q) \geq \bar{q}_j(Q) \geq \bar{q}_j(\bar{Q}), \tag{2.16}$$

where we have used the monotonicity of the function $q_j(Q)$ in α_j. Hence

$$Q = \sum_{i=1}^{n} q_i(Q) \geq \sum_{i=1}^{n} \bar{q}_i(\bar{Q}) = \bar{Q}, \tag{2.17}$$

which is an obvious contradiction. ∎

The assertion of theorem 2 can be reformulated as follows: If any firm increases its weight α_j of uncertainty and, or assumes larger level σ_j^2 of uncertainty of the price function, then the total industry output decreases at the equilibrium.

3 Dynamic Models and Local Stability Analysis

We recall from the previous section that $R_j(Q_j)$ denotes the best response of firm j. In this section we consider dynamic processes with the firms' adjustment of output based on their best responses.

Considering continuous time scales first and assuming that each firm adjusts its output in the direction toward its best response, we obtain the system of ordinary differential equations

$$\dot{q}_j = K_j(R_j(Q_j) - q_j), \quad (j = 1, 2, \cdots, n), \tag{3.1}$$

where K_j is a sign preserving function, i.e.

$$K_j(\Delta) \begin{cases} = 0, & \text{if} \quad \Delta = 0, \\ > 0, & \text{if} \quad \Delta > 0, \\ < 0, & \text{if} \quad \Delta < 0. \end{cases}$$

Theorem 3 *The equilibrium is locally asymptotically stable under conditions (a) and (b) and by assuming that $K_j'(0) > 0$ for all j.*

Proof: The Jacobian of system (3.1) at the equilibrium has the special form,

$$\underline{J}_C = \begin{pmatrix} -K_1' & K_1'R_1' & \cdots & K_1'R_1' \\ K_2'R_2' & -K_2' & \cdots & K_2'R_2' \\ \vdots & \vdots & & \vdots \\ K_n'R_n' & K_n'R_n' & \cdots & -K_n' \end{pmatrix}, \tag{3.2}$$

where $K_j' = K_j'(0)$, and R_j' is the derivative of R_j at the equilibrium.

The Jacobian (3.2) may be represented as

$$\underline{J}_C = \underline{K} + \underline{a} \cdot \underline{1}^T, \tag{3.3}$$

with

$$\underline{K} = diag(-K_1'(1 + R_1'), \cdots, -K_n'(1 + R_n')), \quad \underline{1}^T = (1, \cdots, 1)$$

and

$$\underline{a} = (K_1'R_1', K_2'R_2', \cdots, K_n'R_n')^T.$$

The characteristic polynomial of $\underline{J}_C$ is given by

$$\begin{aligned} &det(\underline{K} + \underline{a} \cdot \underline{1}^T - \lambda \underline{I}) = det(\underline{K} - \lambda \underline{I}) det(\underline{I} + (\underline{K} - \lambda \underline{I})^{-1} \underline{a} \cdot \underline{1}^T) \\ =&\Pi_{i=1}^n (-K_i'(1 + R_i') - \lambda_i) \left[1 + \sum_{i=1}^n \frac{K_i'R_i'}{-K_i'(1 + R_i') - \lambda} \right]. \end{aligned} \tag{3.4}$$

Notice that relation (2.8) implies that $1 + R_j' > 0$ for all j, so all roots of the first product are real and negative. It is therefore sufficient to show that all roots of the equation

$$\sum_{i=1}^n \frac{K_i'R_i'}{-K_i'(1 + R_i') - \lambda} = -1, \tag{3.5}$$

are also real and negative. We might assume that the denominators are different, otherwise the sum of terms with identical denominators can be represented as a single term where the numerators are added. Equation (3.5) is equivalent to a polynomial equation of degree n, so there are n real or complex roots. Let $g(\lambda)$ denote the left hand side, then clearly

$$\lim_{\lambda\to\pm\infty} g(\lambda)=0, \qquad \lim_{\lambda\to -K_i'(1+R_i')^{\pm 0}} g(\lambda)=\pm\infty,$$

and

$$g'(\lambda)=\sum_{i=1}^{n}\frac{K_i'R_i'}{(-K_i'(1+R_i')-\lambda)^2}<0,$$

so g is strictly decreasing locally. The graph of the function g is shown in figure 1. There are n negative poles at $\lambda=-K_j'(1+R_j')$ $(j=1,2,\cdots,n)$, there is a root between each pair of consecutive poles and there is an additional root before the first pole. We have demonstrated that there are n real negative roots, consequently all roots are real and negative. ∎

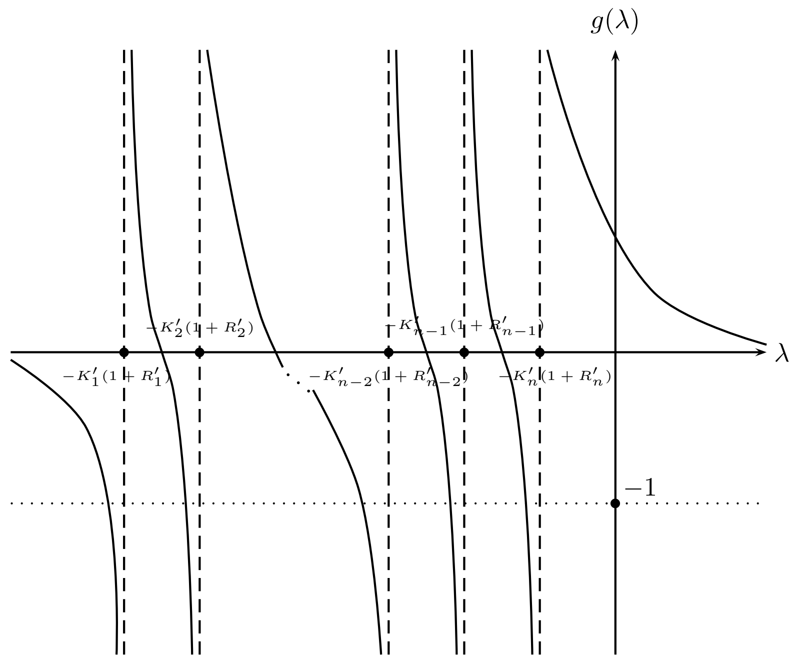

Figure 1: Graph of the function $g(\lambda)$

Considering discrete time scales next, we assume again that the firms adjust their outputs into the direction toward their best responses, and so the outputs adjust according to

$$q_j(t+1)=q_j(t)+K_j(R_j(Q_j)-q_j), \quad (j=1,2,\cdots,n), \tag{3.6}$$

where K_j is a sign-preserving function for all j.

Theorem 4 *Assume that conditions (a) and (b) hold, furthermore $K_j'(0) > 0$ for all j. The equilibrium is locally asymptotically stable if*

$$K_j'(0) < \frac{2}{1 + R_j'(Q_j^*)}, \tag{3.7}$$

for all j, where $\underline{q}^* = (q_j^*)$ is the equilibrium and $Q_j^* = \sum_{i \neq j} q_i^*$, and

$$\sum_{j=1}^{n} \frac{K_j'(0) R_j'(Q_j^*)}{2 - K_j'(0)(1 + R_j'(Q_j^*))} > -1. \tag{3.8}$$

If any of these conditions is violated with strict inequality, then the equilibrium is unstable.

Proof: The Jacobian of system (3.6) at the equilibrium can be written as

$$\underline{J}_D = \underline{I} + \underline{J}_C, \tag{3.9}$$

where $\underline{I}$ is the $n \times n$ identity matrix. All eigenvalues of $\underline{J}_D$ are inside the unit circle if and only if all eigenvalues of $\underline{J}_C$ are inside the interval $(-2, 0)$. From the proof of theorem 3 we know that all eigenvalues of $\underline{J}_C$ are negative, so the eigenvalues are larger than -2 if and only if

$$-K_j'(1 + R_j') > -2, \tag{3.10}$$

for all j, and

$$g(-2) > -1. \tag{3.11}$$

Notice that inequality (3.10) can be rewritten as (3.7), and inequality (3.11) can also be rewritten as (3.8).

If either (3.10) or (3.11) is violated with strict inequality, then at least one eigenvalue of $\underline{J}_C$ becomes less than -2, so at least one eigenvalue of $\underline{J}_D$ is below -1, demonstrating the instability of the equilibrium in this case. ■

Notice that all denominators of inequality (3.7) are positive because of the relation (2.8). Condition (3.7) implies that all denominators on the left hand side of (3.8) are also positive. Therefore condition (3.8) can be interpreted as stating that all derivatives $K_j'(0)$ must be sufficiently small in order to guarantee the local asymptotical stability of the equilibrium.

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