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Cournot Models: Dynamics, Uncertainty and Learning

Ferenc Szidarovszky
Systems & Industrial Engineering Department,
The University of Arizona, Tucson,
Arizona, 85721-0020, USA
email: `szidar@sie.arizona.edu`
and
Vernon L. Smith and Steven Rassenti
Interdisciplinary Center for Economic Science,
George Mason University, 3330 Washington Boulevard
Arlington, VA 22201
emails: `vsmith2@gmu.edu, srassent@gmu.edu`

ABSTRACT

This chapter gives an overview of the recent developments in the theory of dynamic oligopolies including some new results. We will discuss the Cournot classical model and its extensions to product differentiation, multiproduct models, price adjusting oligopolies, labor managed and rent seeking games. The dynamic process based on these models will be analyzed. From the theoretical point of view we will investigate models with and without full information, with partial cooperation among the firms, and under the assumption that the information about the production levels of the rivals has time delay. We will also introduce and discuss special learning procedures based on repeated price information. We will also briefly discuss investigations based on laboratory experiments, in which more realistic cases can be examined.

RESUMEN

Damos una descripción general de los recientes desarrollos de la teoria de dinámica de oligopolios incluyendo algunos resultados novos. Discutimos el modelo clásico de Cournot y sus extensiones a la diferenciación producto, modelos multiproducto, ajuste de precios de oligopolios, labores dirigidas y juegos de rent seeking. La dinámica de procesos basados en estos modelos será analizada. Desde el punto de vista teórico investigaremos modelos con y sin información completa, con cooperación parcial entre las firmas, y bajo la suposición que la información al respecto de los niveles de producción de los rivales tiene un tiempo de retrazo. Introduzimos y discutimos procesos especiales de aprendizaje basados en información de precios repetidos. También brevemente discutimos investigaciones sobre experimentos de laboratórios, en los cuales casos mas realisticos son examinados.

Key words and phrases: *n-person game, oligopoly, dynamic systems, stability learning.*

Math. Subj. Class.: *91A20, 91A80.*

1 Introduction

Since the pioneering work of [1] oligopoly models are the most frequently discussed topics in the literature of mathematical economics. They describe the interaction of manufacturers and service suppliers through some market demand structure. Most of the authors consider this problem as a game in which the supplied quantities (or selected prices) are the strategies and the payoff functions are defined as the profits of the firms.

In a competitive environment the Nash equilibrium is the solution of the game, in which none of the firms can increase its profit by changing its production level alone. In an N-person oligopoly there is no guarantee in general for the existence of such equilibrium, and even if equilibrium exists, there is the possibility of the existence of multiple equilibria ([5]).

Several versions of the Cournot model has been developed in the literature including oligopolies without and with product differentiation, multiproduct oligopolies, labor managed and rent seeking models as well as oligopsonies, in which the firms also compete on the factor market. A comprehensive summary of the different model variants is presented in [6].

If the state of an oligopoly is a Nash equilibrium at a certain time, then no firm has the interest to move away from the equilibrium, therefore the state will remain at the equilibrium for all future times. However in a disequibilium state at least one of the firms is able to increase its profit by changing strategy, so the state of the system will change. If the new state is an equilibrium, then it will remain the state of the system for all future times. Otherwise another change of the state will occur, and so, a dynamic process will develop. The dynamics of this process depend on the desires of the firms, and also on the accuracy and timing of the available information. During this process the firms are able to monitor repeated information on the demand structure and the actions of the competitors, which rises the possibility of some learning mechanisms.

The research on dynamic behavior of the firms can be done in two fundamentally different ways. The mathematical models are always based on certain assumptions on the objectives of the firms, on the types and analytical properties of the functions involved, and on the information structure. Those assumptions very often differ from the economic reality. In more realistic cases it is very often impossible to obtain nice mathematical results, so simulation is used. The applied simulation methodology still depends on the basic assumptions of the model, it simple generates experimental results in the absence of analytical tools.

Another methodology is based on actual laboratory experiments, when real people make the decisions under computer generated environment, and by monitoring their repeated actions we are able to gain a detailed understanding of their priorities, information usage, and decision making mechanism.

In this paper we will give a brief overviews of the most important problems arising in examining dynamic oligopolies, and in addition we will offer some new results on this topic. This chapter is developed as follows. Static oligopoly models will be first introduced in Session 2 and the existence and uniqueness of the equilibrium will be discussed briefly. Dynamic models with full information on the demand structure will be then examined, and Session 4 will be devoted to partial cooperation among the firms. Session 5 will deal with the effect of uncertainty in the information on the demand structure. Information delays will be considered in Session 6 when we will discuss how the stability of the equilibrium is lost through Hopf bifurcation giving the possibility of the birth of limit cycles. Some special learning schemes will be introduced in Session 7. The fundamentals of experimental economics will be outlined in Session 8. Finally, some conclusions will be drawn.

2 The Cournot Model and Its Extensions

Consider an economy of N firms that produce the same product or offer the same service to a homogeneous market. Let $k = 1, 2, \ldots, N$ denote the firms and x_k the produced or offered quantity of firm k. The total output of the industry is $Q = \sum_{k=1}^{N} x_k$, and we assume that the market price f is a function of Q. If $C_k(x_k)$ is the cost of firm k, then its profit is given as

$$\varphi_k(x_1, \ldots, x_N) = x_k f(Q) - C_k(x_k). \tag{2.1}$$

Let L_k denote the capacity limit of firm k. Then an N-person noncooperative game is defined, in which the firms are the players, interval $[0, L_k]$ is the set of strategies of player k, and φ_k is its payoff function. This model is known as the <u>classical Cournot model</u>, or the <u>single-product quantity adjusting oligopoly without product differentiation</u>. In the literature of oligopoly theory it is usually assumed that functions f and C_k $(k = 1, 2, \ldots, N)$ are twice continuously differentiable and for all $x_k \in [0, L_k]$ and $Q \in [0, \sum_{k=1}^{N} L_k]$,

(A) $f'(Q) < 0$;

(B) $x_k f''(Q) + f'(Q) < 0$;

(C) $f'(Q) - C_k''(x_k) < 0$.

Notice that the payoff φ_k of player k does not depend explicitly on the outputs of each individual competitor, it depends on only the output $Q_k = \sum_{l \neq k} x_l$ of the rest of the industry. For each feasible value of Q_k we can easily determine the best strategy choice of firm k, which is called its <u>best reply</u>. Under assumptions (A)-(C), function φ_k is strictly concave in x_k with any fixed value of Q_k, and simple differentiation shows that

$$\arg\max_{x_k} \{x_k f(x_k + Q_k) - C_k(x_k)\} = \begin{cases} 0, & \text{if } f(Q_k) - C_k'(0) < 0; \\ L_k, & \text{if } f(Q_k + L_k) + L_k f'(Q_k + L_k) \\ & -C_k'(L_k) > 0; \\ z_k, & \text{otherwise.} \end{cases} \tag{2.2}$$

where z_k is the unique solution of the strictly monotonic equation

$$f(Q_k + x_k) + x_k f'(Q_k + x_k) - C_k'(x_k) = 0 \tag{2.3}$$

in $[0, L_k]$. Since the value of z_k depends on Q_k, we may write $z_k = R_k(Q_k)$.

In our future analysis we will need the derivative of the best response function. In the neighborhood of an interior optimum, R_k is differentiable as the consequence of the implicit function theorem. By differentiating equation (2.3) implicitly with respect to Q_k, we have

$$f' \cdot (1 + R_k') + R_k' \cdot f' + x_k f'' \cdot (1 + R_k') - C_k'' \cdot R_k' = 0$$

implying that

$$R_k'(Q_k) = -\frac{f'(Q) + x_k f''(Q)}{2f'(Q) + x_k f''(Q) - C_k''(x_k)}.$$

Assumptions (B) and (C) imply that

$$-1 < R_k'(Q_k) < 0.$$

Clearly a strategy vector $\underline{x}^* = (x_1^*, \ldots, x_N^*)$ is a Nash-equilibrium if and only if for all k,

(i) $x_k^* \in [0, L_k]$;

(ii) $x_k^* = R_k\left(\sum_{l \neq k} x_l^*\right)$.

It is well known (see for example, [5]) that under conditions (A)-(C) there is a unique Nash equilibrium.

Assume next that the firms produce different but related goods. Let $x_1, \ldots, x_N$ denote again the produced quantities and let $f_k(x_1, \ldots, x_N)$ denote the price of the product of firm k. Then the profit of this firm is given as

$$\varphi_k(x_1, \ldots, x_N) = x_k f_k(x_1, \ldots, x_N) - C_k(x_k). \tag{2.4}$$

This model is known as a <u>single product oligopoly with product differentiation</u>.

In <u>Bertrand</u> (or price adjusting) <u>oligopolies</u> we consider a single product oligopoly with product differentiation in which each firm selects its price. If $P_1, \ldots, P_N$ are the selected prices and $d_k(P_1, \ldots, P_N)$ is the demand function of the product of firm k, then the profit of this firm can be obtained again by equation (2.4), where for all k, $f_k = P_k$ and

$$x_k = d_k(P_1, \ldots, P_N),$$

so the profit functions now depend on only the price selections.

Multiproduct oligopolies are obtained by assuming that the firms produce M different items. Let $x_k^{(m)}$ denote the output of firm k of product m, then the firm's production can be characterized by its production vector $\underline{x}_k = (x_k^{(1)}, \ldots, x_k^{(M)})$. The vector $\underline{Q} = \sum_{k=1}^{N} \underline{x}_k$ shows the total production vectors of the industry. If $C_k(\underline{x}_k)$ is the production cost of firm k, then its profit is given as

$$\varphi_k(\underline{x}_1, \ldots, \underline{x}_N) = \underline{x}_k^T \underline{f}(\underline{Q}) - C_k(\underline{x}_k). \tag{2.5}$$

Here $\underline{f}(\underline{Q}) = \big(f_1(\underline{Q}), \ldots, f_M(\underline{Q})\big)$ with $f_m(\underline{Q})$ $(1 \le m \le M)$ being the price function of product m.

Consider again the classical Cournot model (2.1) and let $l_k(x_k)$ denote the labor force needed by firm k to produce output x_k. Then the profit per labor of this firm can be given as

$$\Psi_k(x_1, \ldots, x_N) = \frac{\varphi_k(x_1, \ldots, x_N)}{l_k(x_k)} = \frac{x_k f_k(Q) - C_k(x_k)}{l_k(x_k)}. \tag{2.6}$$

If the firms maximize profits per labor instead of their total profits, then the oligopoly is said to be labor managed.

Let N denote the number of agents involved in rent seeking activity. Let x_k be the expenditure of agent k and $f_k(x_k)$ its production function for lotteries, then the probability that agent k will win the rent is

$$P_k = \frac{f_k(x_k)}{\sum_{l=1}^{N} f_l(x_l)}.$$

If the rent is normalized to 1, then the expected net rent of agent k is given as

$$\Pi_k = \frac{f_k(x_k)}{\sum_{l=1}^{N} f_l(x_l)} - x_k. \tag{2.7}$$

This model is known as a rent-seeking game. By introducing the new variables $y_k = f_k(x_k)$ and $C_k = f_k^{-1}$ it is clear that the payoff function (2.7) has the new form

$$\Pi_k = \frac{y_k}{\sum_{l=1}^{N} y_l} - C_k(y_k). \tag{2.8}$$

Notice that function form (2.8) reduces to (2.1) by selecting $f(Q) = 1/Q$, so rent-seeking games are usually considered as special oligopolies.

The equilibria of the above extensions can be defined in the same way as it has been presented for the classical Cournot model. Existence and uniqueness results are represented in [6].

3 Dynamic Models with Full Information

In this session we consider only the classical Cournot model, and assume that the firms know the true price function and the simultaneous output values of all competitors. In this case for firm k, the best output choice is $R_k(Q_k)$ as it was shown in Session 2.2. If x_k is the current output of the firm and $R_k(Q_k)$ is its desired output, then under continuous time scales the firm will change its output in the direction toward the desired output, since it cannot "jump" with the output value instantaneously. Therefore the output change of the firms can be described by the system of ordinary differential equations

$$\dot{x}_k(t) = K_k\left(R_k(Q_k(t)) - x_k(t)\right) \quad (k = 1, 2, \ldots, N) \tag{3.1}$$

where $K_k > 0$ is a constant being called the speed of adjustment of firm k.

Here we assume that the firms know their best response functions, and instantaneous information is available about the market price. If $P(t)$ denotes the price at time period t, then

$$P(t) = f\left(x_k(t) + Q_k(t)\right)$$

from which firm k is able to calculate the output if the rest of the industry:

$$Q_k(t) = f^{-1}\left(P(t)\right) - x_k(t).$$

In this way the firms have all necessary information to proceed with output adjustments (3.1).

An output vector $\underline{x}^* = (x_1^*, \ldots, x_N^*)$ is a steady state of system (3.1) if and only if for all k,

$$x_k^* = R_k\left(\sum_{l \neq k} x_l^*\right),$$

that is, when $\underline{x}^*$ is a Nash equilibrium.

Example 1 *Assume linear price and cost functions:*

$$f(Q) = B - AQ, \quad C_k(x_k) = a_k x_k + b_k \quad (k = 1, 2, \ldots, N).$$

In this case equation (2.3) can be written as

$$B - A(Q_k + x_k) + x_k(-A) - a_k = 0$$

implying that

$$x_k = -\frac{Q_k}{2} + \frac{B - a_k}{2A},$$

so the best response of firm k is

$$R_k(Q_k) = -\frac{Q_k}{2} + \frac{B - a_k}{2A}$$

by assuming interior optimum. Therefore the dynamic system (3.1) can be rewritten as follows:

$$\dot{x}_k = K_k \left(-\frac{1}{2} \sum_{l \neq k} x_l - x_k + \frac{B - a_k}{2A} \right) \tag{3.2}$$

for $k = 1, 2, \ldots, N$*.*

∇

Let now $\underline{x}^* = (x_1^*, \ldots, x_N^*)$ denote an interior equilibrium of the classical Cournot model. We can examine the local asymptotic stability of this equilibrium with respect to the dynamic process (3.1). The Jacobian $\underline{J}^C$ of the system at the equilibrium has the special structure

$$\begin{pmatrix} -K_1 & K_1 r_1 & \cdots & K_1 r_1 \\ K_2 r_2 & -K_2 & \cdots & K_2 r_2 \\ \vdots & \vdots & \ddots & \vdots \\ K_N r_N & K_N r_N & \cdots & -K_N \end{pmatrix} \tag{3.3}$$

where $r_k = R_k'(Q_k^*)$ with $Q_k^* = \sum_{l \neq k} x_l^*$. We have seen earlier that $-1 < r_k < 0$. The characteristic polynomial of $\underline{J}^C$ can be written as

$$\varphi(\lambda) = \det(\underline{D} + \underline{a}\underline{b}^T - \lambda \underline{I})$$

with $\underline{D} = \text{diag}(-K_1(1 + r_1), \ldots, -K_N(1 + r_N))$, $\underline{a} = (K_1 r_1, \ldots, K_N r_N)^T$, and $\underline{b}^T = (1, \ldots, 1)$. In obtaining a closed form representation of $\varphi(\lambda)$ we can use the well-known fact that with any N-element vectors $\underline{a}$ and $\underline{b}$, $\det(\underline{I} + \underline{a}\underline{b}^T) = 1 + \underline{a}^T \underline{b}$. Therefore

$$\begin{aligned} \varphi(\lambda) &= \det(\underline{D} - \lambda \underline{I}) \cdot \det(\underline{I} + (\underline{D} - \lambda \underline{I})^{-1} \underline{a}\underline{b}^T) \\ &= \prod_{k=1}^{N} (-K_k(1 + r_k) - \lambda) \cdot \left[1 + \sum_{k=1}^{N} \frac{K_k r_k}{-K_k(1 + r_k) - \lambda} \right]. \end{aligned} \tag{3.4}$$

The main result of this session can be formulated as

Theorem 1 *All eigenvalues of* $\underline{J}^C$ *have negative real parts implying the local asymptotic stability of the equilibrium.*

Proof The roots of function (3.4) are $\lambda = -K_k(1 + r_k)$, which are all negative, and the roots of the bracketed factor. This equation clearly can be rewritten as

$$\sum_{i=1}^{s} \frac{\alpha_i}{\beta_i - \lambda} + 1 = 0 \tag{3.5}$$

where $\alpha_i < 0$, $\beta_i < 0$ and the β_i values are different. This equation is equivalent to a polynomial equation of degree s, so there are s real or complex roots. Let $g(\lambda)$ denote the left hand side, then

$$\begin{aligned} \lim_{\lambda \to \pm\infty} g(\lambda) &= 1, \quad \lim_{\lambda \to \pm\beta_i} g(\lambda) = \pm\infty, \\ g'(\lambda) &= \sum_{i=1}^{s} \frac{\alpha_i}{(\beta_i - \lambda)^2} < 0. \end{aligned}$$

The graph of function g is shown in Figure 1. Clearly there is a root before β_1 and one root between β_i and β_{i+1} for $i = 1, 2, \ldots, s-1$. Since $\beta_i < 0$ for all i, we found s real negative roots. Hence all eigenvalues of $\underline{J}^C$ are real and negative, which completes the proof.

□

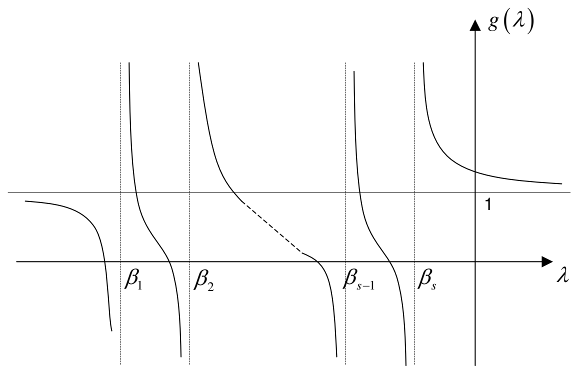

Figure 1: The graph of function g.

Assume next that the time scales are discrete. If $x_k(t)$ denote the production level of firm k at time period t, then its best choice with given $x_l(t)$ $(l \neq k)$ values is $R_k(Q_k(t))$, where $Q_k(t) = \sum_{l \neq k} x_l(t)$. In many industries the firms are unable to make large changes in their production levels during a single time period, therefore they select levels in the direction toward their best choices. This dynamism can be conveniently modelled as

$$x_k(t+1) = \alpha_k x_k(t) + (1 - \alpha_k) R_k(Q_k(t)) \tag{3.6}$$

for $k = 1, 2, \ldots, N$, where $0 \leq \alpha_k < 1$ is a given constant for all k. In order to examine the asymptotic behavior of the equilibrium notice first that the Jacobian of this system at the equilibrium can be given as follows:

$$\underline{J}^D = \begin{pmatrix} \alpha_1 & (1-\alpha_1)r_1 & \cdots & (1-\alpha_1)r_1 \\ (1-\alpha_2)r_2 & \alpha_2 & \cdots & (1-\alpha_2)r_2 \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha_N)r_N & (1-\alpha_N)r_N & \cdots & \alpha_N \end{pmatrix}. \tag{3.7}$$

We can rewrite this matrix similarly to the continuous case:

$$\underline{J}^D = \underline{D} + \underline{a}\underline{b}^T$$

with

$$\begin{aligned} \underline{D} &= \operatorname{diag}\left((\alpha_1 - 1)r_1 + \alpha_1, \dots, (\alpha_N - 1)r_N + \alpha_N\right), \\ \underline{a} &= \left((1-\alpha_1)r_1, \dots, (1-\alpha_N)r_N\right)^T, \quad \text{and } \underline{b}^T = (1, \dots, 1). \end{aligned}$$

Therefore the characteristic polynomial of $\underline{J}^D$ is given as

$$\begin{aligned} \varphi(\lambda) &= \det(\underline{D} + \underline{a}\underline{b}^T - \lambda\underline{I}) = \det(\underline{D} - \lambda\underline{I}) \det(\underline{I} + (\underline{D} - \lambda\underline{I})^{-1}\underline{a}\underline{b}^T) \\ &= \prod_{k=1}^{N} \left((\alpha_k - 1)r_k + \alpha_k - \lambda\right) \cdot \left[1 + \sum_{k=1}^{N} \frac{(1-\alpha_k)r_k}{(\alpha_k - 1)r_k + \alpha_k - \lambda}\right]. \end{aligned} \tag{3.8}$$

In this case Theorem 1 can be modified as follows:

Theorem 2 *All eigenvalues of $\underline{J}^D$ are inside the unit circle if and only if*

$$\sum_{k=1}^{N} \frac{(1-\alpha_k)r_k}{(\alpha_k - 1)r_k + \alpha_k + 1} > -1. \tag{3.9}$$

In this case the equilibrium is locally asymptotically stable. If the left hand side is smaller than -1, then the equilibrium is unstable.

Proof The eigenvalues are $\lambda = (\alpha_k - 1)r_k + \alpha_k$, which are all positive, and the roots of the bracketed factor. First we show that the roots $(\alpha_k - 1)r_k + \alpha_k$ are inside the unit circle. Since they are positive it is sufficient to show that

$$(\alpha_k - 1)r_k + \alpha_k < 1,$$

which is obviously true, since it can be rewritten as

$$(\alpha_k - 1)(-r_k - 1) > 0,$$

where both factors are negative. The other roots are the solutions of equation (3.5) where $\alpha_i < 0$ and $0 < \beta_i < 1$ for all i. The graph of function g is the same as it is shown in Figure 1 with the only difference that all β_i values are between 0 and 1. All roots between the pairs β_i and β_{i+1} are inside the unit circle, and the smallest root is also inside the unit circle if and only if $g(-1) > 0$. It is easy to see that this inequality is equivalent to condition (3.9).

□

If all firms select the best response, then $\alpha_k = 0$ for all k, and in this case condition (3.9) can be rewritten as

$$\sum_{k=1}^{N} \frac{r_k}{1 - r_k} > -1, \tag{3.10}$$

which certainly holds if all values r_k are sufficiently close to zero. In the case of symmetric oligopolies the R_k best responses are identical, so $r_k \equiv r$. In this further special case relation (3.10) simplifies to the following:

$$\frac{Nr}{1 - r} > -1$$

which can be rewritten as

$$r > \frac{-1}{N - 1}. \tag{3.11}$$

Example 2 *Consider again the linear case given in the previous example. Since $R_k'(Q_k) = -\frac{1}{2}$ for all k, relation (3.11) holds only for $N = 2$, so the equilibrium is asymptotically stable for only duopolies with $\alpha_k = 0$ $(k = 1, 2, \ldots, N)$.*

In this case relation(3.9) simplifies as

$$\sum_{k=1}^{N} \frac{\alpha_k - 1}{\alpha_k + 3} > -1 \tag{3.12}$$

which certainly holds if the α_k values are sufficiently close to 1.

∇

In this session we have focused on the classical Cournot model, however its extensions and all model variants can be examined similarly to this case.

4 Models with Partial Cooperation

A usual way how the firms might increase their profits is to seek certain cooperation with the rivals. A common way to model such "partial" cooperation can be given as follows. Let φ_k denote the profit of firm k $(1 \leq k \leq N)$, and let γ_{kl} denote a nonnegative constant for all k and l, which shows the cooperation level of firm k toward firm l. Then firm k maximizes $\varphi_k + \sum_{l\neq k} \gamma_{kl}\varphi_l$ that is, it takes a certain portion of the profits of the competitors into account in addition to its own profit. Thus the payoff function of firm k becomes

$$\Psi_k(x_1, \ldots, x_N) = (x_k f(Q) - C_k(x_k)) + \sum_{l\neq k} \gamma_{kl}(x_l f(Q) - C_l(x_l)) \tag{4.1}$$

in the case of the classical Cournot model. For the sake of mathematical convenience assume that $\gamma_{kl} \equiv \gamma_k$ for all k and l, that is, each firm has identical cooperation levels toward its rivals. In this special case

$$\Psi_k(x_1, \ldots, x_N) = (x_k + \gamma_k Q_k) f(Q) - C_k(x_k) - \gamma_k \sum_{l\neq k} C_l(x_l). \tag{4.2}$$

With given value of Q_k, the best response of firm k can be obtained by simple differentiation. Assuming interior optimum, then at the optimum

$$f(x_k + Q_k) + (x_k + \gamma_k Q_k) f'(x_k + Q_k) - C_k'(x_k) = 0. \tag{4.3}$$

The derivative of the left hand side with respect to x_k is the following:

$$2f'(Q) + (x_k + \gamma_k Q_k) f''(Q) - C_k''(x_k).$$

Assume that conditions (A) and (C) introduced in Session 2.2 are satisfied with a modified version of condition (B):

(B') $(x_k + \gamma_k Q_k) f''(Q) + f'(Q) < 0$

for all $x_k \in [0, L_k]$, $Q_k \in [0, \sum_{l\neq k} L_l]$ and $Q = x_k + Q_k$. Then Ψ_k is strictly concave in x_k, so the payoff maximizing x_k value is unique. If it is interior, then it can be obtained as the unique solution of the monotonic equation (4.3). Let $R_k(Q_k)$ denote the solution as before. By differentiating equation (4.3) implicitly with respect to Q_k it is easy to see that

$$R_k'(Q_k) = -\frac{(1+\gamma_k) f'(Q) + (x_k + \gamma_k Q_k) f''(Q)}{2f'(Q) + (x_k + \gamma_k Q_k) f''(Q) - C_k''(x_k)}. \tag{4.4}$$

Clearly both the numerator and denominator are negative, so $R_k'(Q_k)$ is always negative. In addition, $R_k'(Q_k) > -1$ if the following stronger version of condition (C) is satisfied:

(C') $(1 - \gamma_k) f'(Q) - C_k''(x_k) < 0$

for all $0 \leq x_k \leq L_k$ and $0 \leq Q \leq \sum_{l=1}^{N} L_l$.

Under conditions (A), (B') and (C') all results of the previous session remain true for this case with the only difference that $R_k'(Q_k)$ is now given by equation (4.4).

5 Models with Uncertain Price Function

In this session we will examine again the classical Cournot model, however the methodology and the results to be discussed here can be extended to other model variants.

Assume now that the firms do not know the true price function f, but they have certain estimates of it. Let f_k denote the estimated price function by firm k. Then this firm believes that its profit is

$$\overline{\varphi}_k(x_1, \dots, x_N) = x_k \overline{f}_k(Q) - C_k(x_k). \tag{5.1}$$

Assume that conditions (A)-(C) are satisfied with f being replaced by $\overline{f}_k$. Then with all fixed Q_k, firm k has a believed profit maximizing output $\overline{R}_k(Q_k)$, which is usually different than the "true" best response of the firm.

Consider first continuous time scales and assume that similarly to the full information case, each firm adjusts its production level into the direction toward its believed best reply. Then we have a modified version of system (3.1):

$$\dot{x}_k(t) = K_k\left(\overline{R}_k\left(\overline{Q}_k(t)\right) - x_k(t)\right) \quad (k = 1, 2, \dots, N) \tag{5.2}$$

where $\overline{Q}_k(t)$ is the estimate of $Q_k(t)$ by firm k. Notice that the true price function is not known by the firm, so it cannot compute the true value of Q_k. Instead the following method is used. The true market price, which the firm observes is $f(x_k + Q_k)$. On the other hand, firm k believes that it equals $\overline{f}_k(x_k + \overline{Q}_k)$, so in fact, firm k solves the equation

$$\overline{f}_k(x_k + \overline{Q}_k) = f(x_k + Q_k)$$

to get its estimate

$$\overline{Q}_k = (\overline{f}_k^{-1} \circ f)(x_k + Q_k) - x_k. \tag{5.3}$$

For the sake of simplicity introduce the function $G_k = \overline{f}_k^{-1} \circ f$. Then system (5.2) can be rewritten as

$$\dot{x}_k(t) = K_k\left(\overline{R}_k\left(G_k\left(x_k(t) + Q_k(t)\right) - x_k(t)\right) - x_k(t)\right) \tag{5.4}$$

for $k = 1, 2, \dots, N$. Notice that the steady state of this system (if exists) is usually different than the Nash equilibrium, since $\overline{R}_k$ usually differs from the true best response function R_k. We can refer to this steady state as the "believed" equilibrium, which will be denoted as $\underline{x}^* = (x_1^*, \dots, x_N^*)$. The Jacobian $\overline{\underline{J}}^C$ of the system has a similar form to matrix (3.3):

$$\begin{pmatrix} K_1\left(r_1(g_1-1)-1\right) & K_1 r_1 g_1 & \cdots & K_1 r_1 g_1 \\ K_2 r_2 g_2 & K_2\left(r_2(g_2-1)-1\right) & \cdots & K_2 r_2 g_2 \\ \vdots & \vdots & \ddots & \vdots \\ K_N r_N g_N & K_N r_N g_N & \cdots & K_N\left(r_N(g_N-1)-1\right) \end{pmatrix} \tag{5.5}$$

with $g_k = G'_k(x^*_k + Q^*_k)$.

In the full information case, $f_k \equiv f$ for all k, therefore G_k is the identity function with $g_k = 1$. In this case $\underline{\overline{J}}^C$ reduces to matrix (3.3). Otherwise f_k is only an estimate if f, and if this estimate is sufficiently good, then G_k is close to the identity function with $g_k \approx 1$. The location of the eigenvalues of $\underline{\overline{J}}^C$ can be similarly examined as it was shown in the full information case. Observe that

$$\underline{\overline{J}}^C = \underline{D} + \underline{a}\underline{b}^T$$

where in this case $\underline{D} = \operatorname{diag}(-K_1(1+r_1), \ldots, -K_N(1+r_N))$, $\underline{a} = (K_1 r_1 g_1, \ldots, K_N r_N g_N)^T$ and $\underline{b}^T = (1, \ldots, 1)$. The characteristic polynomial of this matrix is also similar to (3.4):

$$\prod_{k=1}^{N} (-K_k(1+r_k) - \lambda) \cdot \left[1 + \sum_{k=1}^{N} \frac{K_k r_k g_k}{-K_k(1+r_k) - \lambda} \right]. \tag{5.6}$$

Since f is strictly decreasing and we may assume that all estimates f_k are also strictly decreasing (otherwise the firms' estimates are irrealistic), G_k is strictly increasing with nonnegative derivative g_k. Therefore the proof of Theorem 1 can be used without any changes to show that the "believed" equilibrium is locally asymptotically stable in this case as well.

Assume next that the time scales are discrete. In this case the discrete dynamic system (3.6) is modified as

$$x_k(t+1) = \alpha_k x_k(t) + (1-\alpha_k)\overline{R}_k \left(G_k \left(x_k(t) + Q_k(t) \right) - x_k(t) \right) \tag{5.7}$$

for $k = 1, 2, \ldots, N$, where we used equation (5.3) again. The steady state of this system (if exists) is usually different than the Nash equilibrium, so we might refer to the steady state as a "believed" equilibrium similarly to the continuous case.

The Jacobian $\underline{\overline{J}}^C$ of system (5.7) has the special structure

$$\begin{pmatrix} \alpha_1 + (1-\alpha_1)r_1(g_1 - 1) & (1-\alpha_1)r_1 g_1 & \cdots & (1-\alpha_1)r_1 g_1 \\ (1-\alpha_2)r_2 g_2 & \alpha_2 + (1-\alpha_2)r_2(g_2-1) & \cdots & (1-\alpha_2)r_2 g_2 \\ \vdots & \vdots & \ddots & \vdots \\ (1-\alpha_N)r_N g_N & (1-\alpha_N)r_N g_N & \cdots & \alpha_N + (1-\alpha_N)r_N(g_N - 1) \end{pmatrix} \tag{5.8}$$

where r_k and g_k are as before. In the full information case $g_k = 1$, so $\underline{\overline{J}}^C$ reduces to matrix (3.7). Otherwise f_k is only an estimate of f, and if this estimate is sufficiently good, then G_k is close to the identity function with $g_k \approx 1$. This matrix can also be rewritten as $\underline{\overline{J}}^D = \underline{D} + \underline{a}\underline{b}^T$ with $\underline{D} = \operatorname{diag}((\alpha_1 - 1)r_1 + \alpha_1, \ldots, (\alpha_N - 1)r_N + \alpha_N)$, $\underline{a} = ((1-\alpha_1)r_1 g_1, \ldots, (1-\alpha_N)r_N g_N)^T$ and $\underline{b}^T = (1, \ldots, 1)$. The characteristic polynomial of $\underline{\overline{J}}^D$ has a similar form to the previously discussed cases:

$$\prod_{k=1}^{N} ((\alpha_k - 1)r_k + \alpha_k - \lambda) \cdot \left[1 + \sum_{k=1}^{N} \frac{(1-\alpha_k)r_k g_k}{(\alpha_k - 1)r_k + \alpha_k - \lambda} \right]. \tag{5.9}$$

It is easy to see that Theorem 2 remains valid in this case with the only difference that condition (3.9) has to be modified as

$$\sum_{k=1}^{N} \frac{(1-\alpha_k) r_k g_k}{(\alpha_k - 1) r_k + \alpha_k + 1} > -1. \tag{5.10}$$

6 The Effect of Information Delay

In this session we will examine how delayed information affects the asymptotic behavior of the equilibrium. For the sake of simplicity we will consider only the classical Cournot model.

Assume that at time period t firm k obtains a delayed information on the production level of the rest of the industry, $Q_k(s_k)$, where $t - s_k$ is the delay. If firm k uses this latest information to form its best response, then the dynamic system becomes

$$\dot{x}_k(t) = K_k \left(R_k \left(Q_k(s_k) \right) - x_k(t) \right) \quad (k = 1, 2, \ldots, N). \tag{6.1}$$

If the delay is known and it is denoted by $d_k(t)$, then $s_k = t - d_k(t)$, and so equation (6.1) becomes a difference-differential equation. However the delay is uncertain in real economies, therefore a convenient modelling way is offered by considering it as a random variable and replacing the random right hand sides of equation (6.1) by their expected values. In this way a Volterra-type integro-differential equation is obtained:

$$\dot{x}_k(t) = K_k \left(\int_0^t w(t-s, T_k, m_k) R_k \left(Q_k(s) \right) ds - x_k(t) \right). \tag{6.2}$$

The weighting function w is defined as

$$w(t-s, T, m) = \begin{cases} \frac{1}{T} e^{-\frac{t-s}{T}}, & \text{if } m = 0; \\ \frac{1}{m!} \left(\frac{m}{T} \right)^{m+1} (t-s)^m e^{-\frac{m(t-s)}{T}}, & \text{if } m \geq 1, \end{cases} \tag{6.3}$$

where $T > 0$ is a real and $m \geq 0$ is an integer parameter. This weighting function has the following properties:

(a) $\int_0^\infty w(s, T, m) ds = 1$;

(b) If $m = 0$, then weights are exponentially decreasing with the largest weight given to the most current data. If $m \geq 1$, then the most current data has zero weight, the weight is increasing to a maximal value at $t - s = T$, and decreases thereafter.

(c) With increasing value of m, the weighting function becomes more peaked around $t - s = T$, as $m \to \infty$ the weighting function converges to the Dirac delta function centered at $t - s = T$.

(d) If $T \to 0$, then the weighting function converges to the Dirac delta function centered at zero.

Theorem 3 *System (6.2) is equivalent to a system of ordinary differential equations by introducing additional unknown functions.*

Proof Assume first that $m = 0$ and let P be any function of time. Introduce the new function

$$P_0(t) = \int_0^t \frac{1}{T} e^{-\frac{t-s}{T}} P(s) ds.$$

By simple differentiation,

$$\dot{P}_0(t) = \frac{1}{T} \left(P(t) - P_0(t) \right).$$

Assume next that $m \geq 1$, and for all $l = 0, 1, \ldots, m$ introduce the functions

$$P_l(t) = \int_0^t \frac{1}{l!} \left(\frac{m}{T} \right)^{l+1} (t-s)^l e^{-\frac{m(t-s)}{T}} P(s) ds.$$

Then by differentiation,

$$\dot{P}_l(t) = \frac{m}{T} \left(P_{l-1}(t) - P_l(t) \right)$$

and

$$\dot{P}_0(t) = \frac{m}{T} \left(P(t) - P_0(t) \right).$$

By selecting $P_k(s) = R_k(Q_k(s))$, the integro-differential equation system is equivalent to the following system of ordinary differential equations:

$$\begin{aligned}
\dot{x}_k(t) &= K_k \left(P_{km}(t) - x_k(t) \right) \quad (1 \leq k \leq N) \\
\dot{P}_{kl}(t) &= \frac{q_k}{T_k} \left(P_{k,l-1}(t) - P_{kl}(t) \right) \quad (1 \leq k \leq N, 1 \leq l \leq m_k) \\
\dot{P}_{k0}(t) &= \frac{q_k}{T_k} \left(P_k(t) - P_{k0}(t) \right) \quad (1 \leq k \leq N)
\end{aligned}$$

with

$$q_k = \begin{cases} 1, & \text{if } m_k = 0; \\ m_k, & \text{if } m_k \geq 1. \end{cases} \tag{6.4}$$

□

Linearizing equation (6.2) we have for all k,

$$\dot{x}_{k\delta}(t) = K_k \left(r_k \int_0^t w(t-s, T_k, m_k) \cdot \sum_{l \neq k} x_{l\delta}(s) ds - x_{k\delta}(t) \right) \tag{6.5}$$

where $r_k = R_k'(Q_k^*)$, and $x_{k\delta}(t)$ is the deviation of $x_k(t)$ from its equilibrium level. As it is usual in the theory of ordinary differential equations we look for the solution as $x_{k\delta}(t) = v_k e^{\lambda t}$. By substituting this form into (6.5) and letting $t \to \infty$ we have

$$(\lambda + K_k) v_k - K_k r_k \int_0^t w(s, T_k, m_k) e^{-\lambda s} ds \sum_{l \neq k} v_l = 0$$

and by using simple integration and the definition of the gamma function this equation simplifies as

$$(\lambda + K_k)v_k - K_k r_k \left(1 + \frac{\lambda T_k}{q_k}\right)^{-(m_k+1)} \sum_{l \neq k} v_l = 0. \tag{6.6}$$

Introduce function

$$A_k(\lambda) = (\lambda + K_k)\left(1 + \frac{\lambda T_k}{q_k}\right)^{m_k+1}$$

to see that equations (6.6) are equivalent to a determinantal equation

$$\det \begin{pmatrix} A_1(\lambda) & -K_1 r_1 & \cdots & -K_1 r_1 \\ -K_2 r_2 & A_2(\lambda) & \cdots & -K_2 r_2 \\ \vdots & \vdots & \ddots & \vdots \\ -K_N r_N & -K_N r_N & \cdots & A_N(\lambda) \end{pmatrix} = 0. \tag{6.7}$$

Notice that by introducing

$$\underline{D} = \operatorname{diag}\left(A_1(\lambda) + K_1 r_1, \ldots, A_N(\lambda) + K_N r_N\right), \quad \underline{a} = (-K_1 r_1, \ldots, -K_N r_N)^T$$

and $\underline{b}^T = (1, \ldots, 1)$ this equation can be rewritten as

$$\begin{aligned} \det(\underline{D} + \underline{a}\underline{b}^T) &= \det(\underline{D}) \cdot \det(\underline{I} + \underline{D}^{-1}\underline{a}\underline{b}^T) \\ &= \prod_{k=1}^{N} (A_k(\lambda) + K_k r_k) \cdot \left[1 - \sum_{k=1}^{N} \frac{K_k r_k}{A_k(\lambda) + K_k r_k}\right] = 0. \end{aligned}$$

First we prove that all roots of equation

$$A_k(\lambda) + K_k r_k = 0 \tag{6.8}$$

have negative real parts. Clearly $\lambda \neq 0$, and assume that with some root λ, $R_e \lambda \geq 0$. Then

$$|\lambda + K_k| > K_k \quad \text{and} \quad \left|1 + \frac{\lambda T_k}{q_k}\right| > 1$$

implying that

$$K_k > |K_k r_k| = |A_k(\lambda)| > K_k$$

which is an obvious contradiction. As the asymptotic behavior of the equilibrium is concerned we have to examine the locations of the solutions of equation

$$1-\sum_{k=1}^{N}\frac{K_k r_k}{A_k(\lambda)+K_k r_k}=0. \tag{6.9}$$

Notice first that it is equivalent to a polynomial equation, so there are finitely many roots. In the general case computer methods are needed to locate the roots, however in special cases analytic results can be obtained.

Assume now that the firms are identical and the equilibrium is symmetric. Then $K_1=\ldots=K_N=K$, $T_1=\ldots=T_N=T$, $m_1=\ldots=m_N=m$, $q_1=\ldots=q_N=q$, and $r_1=\ldots=r_N=r$ showing that equation (6.9) reduces to the following:

$$(\lambda+K)\left(1+\frac{\lambda T}{q}\right)^{m+1}+(1-N)Kr=0. \tag{6.10}$$

Consider first the case of $T=0$, which corresponds to models without information delay. In this case (6.10) has a unique root $\lambda<0$, so the equilibrium is locally asymptotically stable. Note that this symmetric case is a special case of Theorem 1.

Assume next that $T>0$ and $m=0$. Then (6.10) becomes quadratic:

$$\lambda^2 T+\lambda(1+KT)+K\left(1+(1-N)r\right)=0.$$

Since all coefficients are positive, all roots have negative real parts implying the local asymptotic stability of the equilibrium (see for example, [8]).

Consider next the case of $T>0$ and $m=1$. In this case (6.10) is a cubic equation:

$$\lambda^3T^2+\lambda^2(2T+T^2K)+\lambda(1+2KT)+K\left(1+(1-N)r\right)=0. \tag{6.11}$$

All coefficients are positive and the Routh-Hurwitz criterion implies that all roots have negative real parts if and only if

$$(2T+T^2K)(1+2KT)>T^2K\left(1+(1-N)r\right), \tag{6.12}$$

which can be rewritten as

$$2T^2K^2+TK\left(4+r(N-1)\right)+2>0. \tag{6.13}$$

The discriminant of the left hand side is

$$r(N-1)\left[8+r(N-1)\right],$$

where $r(N-1)<0$. Therefore we have the following cases.

<u>Case 1.</u> If $8+r(N-1)>0$, then the left hand side of (6.13) has no real root, so it always holds. Therefore the equilibrium is locally asymptotically stable.

Case 2. If $8 + r(N-1) = 0$, then there is a unique root

$$TK = \frac{-4 - r(N-1)}{4} = \frac{-(8 + r(N-1)) + 4}{4} = 1$$

and if $TK \neq 1$, then the equilibrium is locally asymptotically stable.

Case 3. If $8 + r(N-1) < 0$, then there are two real roots

$$(TK)^*_{1,2} = \frac{-4 - r(N-1) \pm \sqrt{(4 + r(N-1))^2 - 16}}{4T^2}.$$

Notice that $-4 - r(N-1) = -(8 + r(N-1)) + 4 > 0$, so both roots are positive. Hence the equilibrium is locally asymptotically stable if

$$TK < (TK)^*_1 \text{ or } TK > (TK)^*_2,$$

where $(TK)^*_1 < (TK)^*_2$. The equilibrium is unstable if

$$(TK)^*_1 < TK < (TK)^*_2.$$

Figure 2 shows the stability region of the equilibrium.

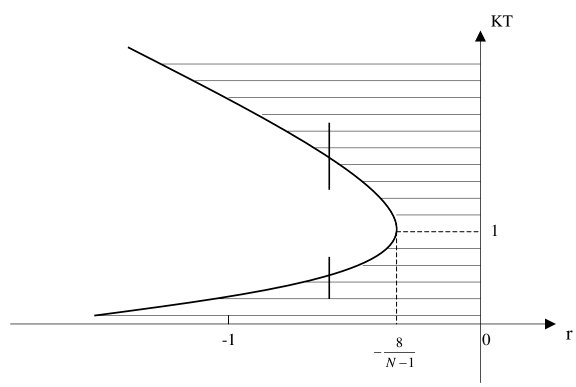

Figure 2: Stability region in the (r, KT) space

From the above analysis we can draw the following interesting conclusions. If $N < 9$, then $-8/(N-1) < -1$, so Case 1. occurs regardless of the value of r, so the equilibrium is always locally asymptotically stable. That is, we need at least 9 firms to have instability. If $N = 9$, then Case 1. occurs for $r > -1$ and Case 2. occurs for $r = -1$. Assume next that $N \geq 10$. If $r > -\frac{8}{N-1}$, then the equilibrium is always locally asymptotically stable, if $r = -\frac{8}{N-1}$, then it occurs if $KT \neq 1$ (that is, always except a particular value of KT), and if $r < -\frac{8}{N-1}$, then KT has to be sufficiently small or sufficiently large to guarantee local asymptotical stability. It is also interesting to note that the stability conditions depend on only the product KT and not on the individual values of these parameters. It shows a certain compensation between the speed of adjustment and average information delay.

We also know from the above analysis that by crossing the critical values $(TK)_1^*$ and $(TK)_2^*$ with the value of TK, stability is lost or gained. It is interesting to examine what happens at these critical values. We will show that Hopf-bifurcation occurs (see for example, [4]) implying the possibility of limit cycles around the equilibrium. At the critical values inequality (6.13) as well as (6.12) become equality implying that equation (6.11) can be rewritten as

$$\begin{aligned} 0 &= \lambda^3 T^2 + \lambda^2(2T + T^2K) + \lambda \frac{T^2K\,(1+(1-N)r)}{2T+T^2K} + K(1+(1-N)r) \\ &= \left(\lambda T^2 + (2T+T^2K)\right)\left(\lambda^2 + \frac{K(1+(1-N)r)}{2T+T^2K}\right) \end{aligned}$$

showing that one eigenvalue is negative, $\lambda_1 = -\frac{2+TK}{T}$, and the other two are pure complex. Differentiating equation (6.11) implicitly with respect to T we have

$$3\lambda^2\dot{\lambda}T^2 + \lambda^3 2T + 2\lambda\dot{\lambda}(2T+T^2K) + \lambda^2(2+2TK) + \dot{\lambda}(1+2TK) + \lambda 2K = 0$$

implying that

$$\dot{\lambda} = \frac{-2T\lambda^3 - \lambda^2(2+2TK) - 2K\lambda}{3\lambda^2T^2 + 2\lambda(2T+T^2K) + (1+2TK)}. \tag{6.14}$$

By letting

$$\alpha^2 = \frac{K(1+(1-N)r)}{2T+T^2K}$$

clearly

$$\dot{\lambda} = \frac{2T\alpha^3 i + \alpha^2(2+2TK) - 2K\alpha i}{-3\alpha^2T^2 + 2\alpha i(2T+T^2K) + (1+2TK)}$$

with real part

$$R_e\dot{\lambda} = \frac{-2\alpha^4T^2(2+2TK) + 4\alpha^2T(T\alpha^2 - K)(2+TK)}{(-2\alpha^2T^2)^2 + 4\alpha^2(2T+T^2K)^2}$$

where we used the simple fact that $1 + 2TK = T^2\alpha^2$. The numerator is

$$4\alpha^2T(\alpha^2T - TK^2 - 2K).$$

The firs factor is positive, and the second factor can be rewritten as

$$\begin{aligned}
& \frac{K(1+(1-N)r)}{2+TK} - TK^2 - 2K = \frac{K}{2+TK} \cdot \left((1+(1-N)r) - (2+TK)^2\right) \\
= \;& \frac{K}{2+TK}\left(1 + \frac{2(TK+1)^2}{TK} - (2+TK)^2\right) \\
= \;& \frac{1}{(2+TK)T}\left(-(TK)^3 - 2(TK)^2 + (TK) + 2\right) \\
= \;& \frac{1-(TK)^2}{T} \neq 0 \qquad (6.15)
\end{aligned}$$

where we used the equality form of inequality (6.13). Since $R_e\dot{\lambda} \neq 0$, there is the possibility of a limit cycle around the equilibrium.

The discrete version of this model and its asymptotical properties can be discussed similarly.

7 Learning in Oligopoly Models

For the sake of simplicity we will discuss the classical Cournot model again and assume linear cost and price functions. Therefore assume that the cost function of firm k is $C_k(x_k) = \alpha_k x_k + \beta_k$, and the true price function is $f(Q) = B - AQ$, where $Q = \sum_{k=1}^{N} x_k$ as before. Assume that the firms have only limited knowledge on the price function, and during the dynamic process they repeatedly update their beliefs of the price function giving rise of a learning process. In this session we will examine three cases.

Case 1. Assume that the firms know the value of Q, where the price becomes zero. In this case firm k believes that the price function is $f_k(Q) = \varepsilon_k\left(\frac{B}{A} - Q\right)$, but does not know that $\varepsilon_k = A$ is the true value.

Case 2. If the firms know only the slope of the price function, then firm k believes that the price function is $f_k(Q) = \varepsilon_k - AQ$, but does not know that $\varepsilon_k = B$ is the true value.

Case 3. If the firms know the price at $Q = 0$ but they are uncertain about the slope, then firm k believes that the price function is $f_k(Q) = B - \varepsilon_k Q$, but does not know the true value $\varepsilon_k = A$.

As we will see, the dynamic learning processes will be significantly different in the above cases.

Starting with Case 1. we examine the game first from the viewpoint of firm k. It believes that the profit of each firm (including itself) is

$$\varphi_l^{(k)}(x_1,\ldots,x_N) = x_l\varepsilon_k\left(\frac{B}{A} - Q\right) - (\alpha_l x_l + \beta_l) \quad (l = 1,2,\ldots N). \qquad (7.1)$$

By assuming interior optima, the believed best response of firm l is

$$x_l = \frac{B}{A} - \frac{\alpha_l}{\varepsilon_k} - Q. \qquad (7.2)$$

By adding these equations

$$Q = \frac{NB}{A} - \frac{1}{\varepsilon_k}\sum_{l=1}^{N}\alpha_l - NQ$$

implying that firm k believes that at the equilibrium the total production of the industry is

$$Q^{(k)} = \frac{1}{N+1}\left(\frac{NB}{A} - \frac{\sum_{l=1}^{N}\alpha_l}{\varepsilon_k}\right)$$

and the corresponding equilibrium price is

$$f_k(Q^{(k)}) = \varepsilon_k\left(\frac{B}{A} - Q^{(k)}\right) = \frac{1}{N+1}\left(\frac{B\varepsilon_k}{A} + \sum_{l=1}^{N}\alpha_l\right). \tag{7.3}$$

Firm k also produces the corresponding believed equilibrium level

$$x_k = \frac{B}{A} - \frac{\alpha_k}{\varepsilon_k} - Q^{(k)} = \frac{B}{(N+1)A} - \frac{\alpha_k}{\varepsilon_k} + \frac{\sum_{l=1}^{N}\alpha_l}{\varepsilon_k(N+1)}. \tag{7.4}$$

Therefore in reality, the total production level of the industry becomes

$$Q = \sum_{k=1}^{N} x_k = \frac{NB}{(N+1)A} - \sum_{k=1}^{N}\frac{\alpha_k}{\varepsilon_k} + \left(\sum_{l=1}^{N}\alpha_l\right)\left(\sum_{k=1}^{N}\frac{1}{\varepsilon_k}\right)\frac{1}{N+1} \tag{7.5}$$

with the corresponding equilibrium price

$$P = B - AQ = \frac{B}{N+1} + A\sum_{k=1}^{N}\frac{\alpha_k}{\varepsilon_k} - \frac{A}{N+1}\left(\sum_{l=1}^{N}\alpha_l\right)\left(\sum_{k=1}^{N}\frac{1}{\varepsilon_k}\right). \tag{7.6}$$

The actual price is usually different than the believed prices by the firms. For firm k, the discrepancy between the actual and believed price is

$$D^{(k)} = P - f_k(Q^{(k)}) = \frac{B}{N+1}\left(1 - \frac{\varepsilon_k}{A}\right) + A\sum_{k=1}^{N}\frac{\alpha_k}{\varepsilon_k} - \frac{1}{N+1}\left(\sum_{l=1}^{N}\alpha_l\right)\left[A\sum_{k=1}^{N}\frac{1}{\varepsilon_k} + 1\right]. \tag{7.7}$$

Based on this price discrepancy firm k thinks as follows. If $D^{(k)} = 0$, then the believed price equals the actual price, so the believed price is considered correct. If $D^{(k)} > 0$, then the believed price is too low, so firm k wants to increase its estimate on the price function by increasing ε_k. If $D^{(k)} < 0$, then its price estimate was too high, so the firm wants to decrease it by decreasing the value of ε_k. By assuming continuous time scales this adjustment process can be modelled as

$$\dot{\varepsilon}_k = K_k D^{(k)} \quad (k = 1, 2, \ldots, N), \tag{7.8}$$

and in the discrete case as

$$\varepsilon_k(t+1) = \varepsilon_k(t) + K_k D^{(k)} \quad (k = 1, 2, \ldots, N). \tag{7.9}$$

Here $K_k > 0$ is the speed of adjustment of firm k.

First we prove that systems (7.8) and (7.9) have only one steady state $\varepsilon_k = A$ $(k = 1, 2, \dots, N)$ which is the full information case. Notice first that $D^{(k)} = 0$ for all k may occur only if the ε_k values are identical. Let ε denote this common value, then

$$\begin{aligned} 0 &= \frac{B}{N+1}\left(1-\frac{\varepsilon}{A}\right) + \frac{A}{\varepsilon}\sum_{k=1}^{N}\alpha_k - \frac{1}{N+1}\left(\sum_{k=1}^{N}\alpha_k\right)\left[\frac{AN}{\varepsilon}+1\right] \\ &= \frac{B}{N+1}\left(1-\frac{\varepsilon}{A}\right) + \left(\sum_{k=1}^{N}\alpha_k\right)\left(\frac{A}{\varepsilon}-1\right)\frac{1}{N+1}. \end{aligned}$$

If $\varepsilon > A$, then both terms are negative; if $\varepsilon < A$, then both are positive, and if $\varepsilon = A$, then they are zero. Hence the only steady state is $\varepsilon_k = A$ for all k.

In order to analyze the asymptotical stability of systems (7.8) and (7.9) notice first that

$$\frac{\partial D^{(k)}}{\partial \varepsilon_k} = -\frac{B}{(N+1)A} + \frac{A}{\varepsilon_k^2}\left(-\alpha_k + \frac{1}{N+1}\sum_{l=1}^{N}\alpha_l\right)$$

and for $l \neq k$,

$$\frac{\partial D^{(k)}}{\partial \varepsilon_l} = \frac{A}{\varepsilon_l^2}\left(-\alpha_l + \frac{1}{N+1}\sum_{k=1}^{N}\alpha_k\right).$$

For the sake of simple notation introduce the variable

$$\gamma_k = \frac{1}{N+1}\sum_{l=1}^{N}\alpha_l - \alpha_k. \tag{7.10}$$

The Jacobian of the continuous system (7.8) has the form

$$\begin{aligned} \underline{J}^C &= \begin{pmatrix} K_1\left(-\frac{B}{(N+1)A}+\frac{A\gamma_1}{\varepsilon_1^2}\right) & \frac{K_1A\gamma_1}{\varepsilon_1^2} & \cdots & \frac{K_1A\gamma_1}{\varepsilon_1^2} \\ \frac{K_2A\gamma_2}{\varepsilon_2^2} & K_2\left(-\frac{B}{(N+1)A}+\frac{A\gamma_2}{\varepsilon_2^2}\right) & \cdots & \frac{K_2A\gamma_2}{\varepsilon_2^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{K_NA\gamma_N}{\varepsilon_N^2} & \frac{K_NA\gamma_N}{\varepsilon_N^2} & \cdots & K_N\left(-\frac{B}{(N+1)A}+\frac{A\gamma_N}{\varepsilon_N^2}\right) \end{pmatrix} \\ &= \underline{D} + \underline{a}\underline{b}^T \end{aligned} \tag{7.11}$$

with

$$D = \operatorname{diag}\left(\frac{-K_1B}{(N+1)A}, \dots, \frac{-K_NB}{(N+1)A}\right), \quad \underline{a} = \left(\frac{K_1A\gamma_1}{\varepsilon_1^2}, \dots, \frac{K_NA\gamma_N}{\varepsilon_N^2}\right)^T$$

and $\underline{b}^T = (1, \dots, 1)$. Therefore the characteristic polynomial of this matrix is the following:

$$\varphi(\lambda)=\prod_{k=1}^{N}\left(\frac{-K_kB}{(N+1)A}-\lambda\right)\left[1+\sum_{k=1}^{N}\frac{\frac{K_kA\gamma_k}{\varepsilon_k^2}}{\frac{-K_kB}{(N+1)A}-\lambda}\right]. \tag{7.12}$$

The eigenvalues are $\lambda=\frac{-K_kB}{(N+1)A}<0$ and the solutions of equation

$$\sum_{k=1}^{N}\frac{\frac{K_kA\gamma_k}{\varepsilon_k^2}}{-\frac{K_kB}{(N+1)A}-\lambda}+1=0$$

which has the same form as equation (3.5) by assuming that $\gamma_k\le 0$ for all k. Notice that this condition holds if the marginal costs α_k are close to each other. Then by repeating the proof of Theorem 1 we can show that all eigenvalues have negative real parts implying the local asymptotical stability of the equilibrium.

The Jacobian of the discrete system (7.9) is similarly $\underline{J}^D=\underline{I}+\underline{J}^C$, which has the same structure as $\underline{J}^C$, but the identity matrix has to be added to the diagonal matrix $\underline{D}$. Therefore its characteristic polynomial has the form

$$\prod_{k=1}^{N}\left(1-\frac{K_kB}{(N+1)A}-\lambda\right)\left[1+\sum_{k=1}^{N}\frac{\frac{K_kA\gamma_k}{\varepsilon_k^2}}{1-\frac{K_kB}{(N+1)A}-\lambda}\right]. \tag{7.13}$$

By repeating the proof of Theorem 2 we can show that all eigenvalue of $\underline{J}^D$ are inside the unit circle at the equilibrium $\varepsilon_k=A$ if $\gamma_k\le 0$ and $\frac{K_kB}{(N+1)A}<2$ for all k, furthermore

$$\sum_{k=1}^{N}\frac{K_k\gamma_k(N+1)}{2(N+1)A-K_kB}>-1. \tag{7.14}$$

In Case 2. we assume that firm k believes that the price function is $f_k(Q)=\varepsilon_k-AQ$, and the firms learn about the value of ε_k. Then firm k believes that the profit of any firm l (including itself) is

$$\varphi_l^{(k)}(x_1,\dots x_N)=x_l(\varepsilon_k-AQ)-(\alpha_lx_l+\beta_l) \tag{7.15}$$

so the believed best response of firm l is

$$x_l=\frac{\varepsilon_k-\alpha_l}{A}-Q, \tag{7.16}$$

the total output of the industry is believed to be

$$Q^{(k)}=\frac{N\varepsilon_k-\sum_{l=1}^{N}\alpha_l}{(N+1)A} \tag{7.17}$$

with price

$$f_k(Q^{(k)})=\frac{\varepsilon_k+\sum_{l=1}^{N}\alpha_l}{N+1}. \tag{7.18}$$

Based on this belief firm k produces the amount

$$x_k = \frac{\varepsilon_k - \alpha_k}{A} - Q^{(k)} = \frac{\varepsilon_k - (N+1)\alpha_k + \sum_{l=1}^N \alpha_l}{(N+1)A}. \tag{7.19}$$

In reality however each firm thinks in the same way but believes in its own ε_l value in the price function, so the actual total production level of the industry becomes

$$Q = \sum_{k=1}^N x_k = \frac{1}{(N+1)A}\left(\sum_{k=1}^N \varepsilon_k - \sum_{l=1}^N \alpha_l\right) \tag{7.20}$$

with actual equilibrium price

$$P = B - AQ = B - \frac{1}{N+1}\left(\sum_{k=1}^N \varepsilon_k - \sum_{l=1}^N \alpha_l\right). \tag{7.21}$$

Based on the discrepancy

$$D^{(k)} = P - f_k(Q^{(k)}) = \frac{1}{N+1}\left((N+1)B - \sum_{l=1}^N \varepsilon_l - \varepsilon_k\right) \tag{7.22}$$

the dynamic process become similar to (7.8) and (7.9) with the only difference that in this case $D^{(k)}$ is given in equation (7.22).

Similarly to the previous case we can prove that there is a unique steady state $\varepsilon_k = B$ for all k, which corresponds to the full information case. Clearly $D^{(k)} = 0$ for all k, if the ε_k values are identical. Let ε denote this common value, then $(N+1)B - N\varepsilon - \varepsilon = 0$ implying that $\varepsilon = B$.

Notice that systems (7.8) and (7.9) are both linear in this case, so local asymptotical stability implies global asymptotical stability. The coefficient matrix in the continuous case is

$$\underline{J}^C = \frac{1}{N+1}\begin{pmatrix} -2K_1 & -K_1 & \cdots & -K_1 \\ -K_2 & -2K_2 & \cdots & -K_2 \\ \vdots & \vdots & \ddots & \vdots \\ -K_N & -K_N & \cdots & -2K_N \end{pmatrix} \tag{7.23}$$

and in the discrete case

$$\underline{J}^D = \underline{I} + \underline{J}^C.$$

Similarly to Theorems 1 and 2 we can easily prove that the continuous system is always asymptotically stable and the discrete system is asymptotically stable if and only if for all k, $K_k < 2(N+1)$, and

$$\sum_{k=1}^N \frac{K_k}{2(N+1) - K_k} < 1.$$

We turn our attention next to Case 3, when the firms learn about the slope of the price function. In this case we assume the firm k believes that the price function is $f_k(Q) = B - \varepsilon_k Q$, the profit of firm l $(l = 1, 2, \ldots, N)$ is believed by firm k to be

$$\varphi_l^{(k)}(x_1, \ldots, x_N) = x_l(B - \varepsilon_k Q) - (\alpha_l x_l + \beta_l), \tag{7.24}$$

so the best believed output choice is

$$x_l = \frac{B - \alpha_l}{\varepsilon_k} - Q,$$

and the believed total production of the industry is

$$Q^{(k)} = \frac{NB - \sum_{l=1}^N \alpha_l}{(N+1)\varepsilon_k}. \tag{7.25}$$

The believed equilibrium price,

$$f_k(Q^{(k)}) = \frac{B + \sum_{l=1}^N \alpha_l}{N+1}, \tag{7.26}$$

is the same for all firms. Firm k also believes that its equilibrium output is

$$x_k = \frac{B - (N+1)\alpha_k + \sum_{l=1}^N \alpha_l}{(N+1)\varepsilon_k}. \tag{7.27}$$

Therefore in reality the total production of the industry becomes

$$Q = \sum_{k=1}^N x_k = \frac{1}{N+1}\left(\left(B + \sum_{l=1}^N \alpha_l\right)\sum_{k=1}^N \frac{1}{\varepsilon_k} - (N+1)\sum_{k=1}^N \frac{\alpha_k}{\varepsilon_k}\right) \tag{7.28}$$

with actual equilibrium price

$$P = B - AQ = B - \frac{A}{N+1}\left(\left(B + \sum_{l=1}^N \alpha_l\right)\sum_{k=1}^N \frac{1}{\varepsilon_k} - (N+1)\sum_{k=1}^N \frac{\alpha_k}{\varepsilon_k}\right). \tag{7.29}$$

Based on the discrepancy between the actual and believed price

$$D^{(k)} = \frac{1}{N+1}\left(NB - A\left(B + \sum_{l=1}^N \alpha_l\right)\sum_{k=1}^N \frac{1}{\varepsilon_k} + A(N+1)\sum_{k=1}^N \frac{\alpha_k}{\varepsilon_k} - \sum_{l=1}^N \alpha_l\right) \tag{7.30}$$

the dynamic process is similar to the previous cases (7.8) and (7.9). Note that $D^{(k)}$ is the same for all firms, and therefore a set of ε_k $(k = 1, 2, \ldots, N)$ values is a steady state of the system if and only if

$$B\left(N - A\sum_{k=1}^N \frac{l}{\varepsilon_k}\right) + (N+1)A\sum_{k=1}^N \frac{\alpha_k}{\varepsilon_k} - \left(\sum_{l=1}^N \alpha_l\right)\left(A\sum_{k=1}^N \frac{1}{\varepsilon_k} + 1\right) = 0.$$

Since this is a single equation for N unknowns with a feasible solution $\varepsilon_k = A$, there are infinitely many steady states. That is, there is the possibility that all firms believe in wrong price functions but none of them notices it since believed and actual prices still coincide. In this case no learning is possible in this way.

In the above discussed cases we always assumed that the firms use instantaneous information about prices, however in reality price informations are always delayed. The effect of information lag in the learning process can be similarly examined to the cases being demonstrated in Session 2.6.

8 Laboratory Experiments

In November 2002 the Nobel Committee awarded Vernon Smith the prize in Economics for a body of work spanning a half-century that demonstrated that controlled laboratory experiments could be used to study economic behavior, exactly as experiments in the hard sciences study physical phenomena. Smith summarized his methodological breakthrough in a highly referenced 1982 paper, "Microeconomic Systems as an Experimental Science." Today, economic experiments are widely undertaken to explore three main avenues of research: to inform economic theory, to test-bed newly designed institutions under stressful environmental conditions, and to understand how brain activity leads to economic behavior. As economic theorists we should be very concerned with rescuing our theories from the doldrums of mathematical curiosity by subjecting them to the rigors of laboratory scrutiny. This chapter has so far presented a review and some new developments in oligopoly theory that are enmeshed with an abundant theoretical literature in this area, but have we not discussed how to assess whether those results relate to what people really do. We can greatly increase the value of our theories to the society that invests in them by becoming concerned with how people really organize themselves to form, sustain, and adapt rules of order in order to generate beneficial outcomes for themselves.

Each laboratory that conducts experiments with human subjects approaches its research with different auxiliary hypotheses but uses the common analytic framework that has three main components: the environment, the institution, and the behavior of the human subjects. The environment includes the subjects' preferences or incentives for achieving various allocations in the exchange system, subjects' productive capabilities and system technical constraints on achieving those allocations, and knowledge about the initial conditions and allocation in the exchange system in which subjects will participate. The experimenter controls the environment using induced values, a mapping of outcomes to different monetary earnings, and carefully worded instructions. For example, in a simple oligopoly environment the experimenter may privately inform each experimental subject i through computerized instructions that he will be playing the role of a producer of a fictitious good, and that he will be able to, in any given period during the upcoming experiment, produce up to u_i units of the good at a cost of $\$c_i$ per unit, and that if he can sell those units he will earn a cash profit, paid to him by the experimenter, which will be the difference between the market price, p, he sells each unit for, and his cost of production.

The institution consists of a set of rules that completely specify (1) what messages subjects are allowed to send and when they can send them, (2) how these messages are translated into reallocations of environmental conditions, and (3) what feedback the subject receives about the messages that were sent and the reallocations they produced. The experimenter typically controls for the institution with a computer network that instantiates these rules. For example, in one simple oligopoly environment a 'producer' may only be able to make only one offer of a given total quantity that he is willing to sell at a particular per unit price in a given period, and the institution may gather those offers, order them from lowest to highest price, and sell as many of them as possible at a uniform price to buyers whose bids to buy have been ordered from highest to lowest. The institution in this simple example may decide not to reveal publicly what offers and bids are submitted or what was the volume traded in the period.

The behavior is what the experimenter can then observe in the form of messages that are actually sent by subjects as they participate in the experiment. By controlling the environment, E, and the institution, I, and creating experimental 'treatments' which can selectively alter either of them, the experimenter can estimate the behavioral response function $b_i(E, I)$ for each subject, i. Institutions where there is repeated interaction among subjects precipitate learning and the behavioral response functions themselves become functions of time, $b_i^t(E, I)$, that depend upon the sequence of information delivered to the subject by the institution and the perceived reallocations in the environment.

The experimental economics laboratory typically uses economic theory to predict the form of the behavioral functions $b_i(\cdot, \cdot)$, and thus predicts the environmental reallocations that will be produced as the subjects interact. It becomes possible to compare predicted outcomes to actual outcomes and predicted behavioral functions to estimated behavioral functions to ask how well does the theory perform in the laboratory? Consider the following example of a two-player game that has been extensively studied in the laboratory. In this case the messages are very simple. Subject One must move first and decide whether to stop the game immediately, in which case he receives a payment of $10 and Subject Two also receives a payment of $10, or pass the decision on to Subject Two. If Subject One chooses to pass, Subject Two now must choose between an allocation which pays Subject One $0 and himself $40, or an allocation which pays Subject One $ 15 and himself $25. Using game theory, Nash equilibrium predicts that when this game is played once by anonymous traders who have complete information, Subject 2 would always choose the $(0, 40)$, and that Subject 1, realizing what Subject 2 would do, will always to stop the game immediately for a $(\$10, \$10)$ allocation. In fact, when this experiment is run typically only 50% of the Subject 1's elect to stop, and, conditional on passing to Subject 2, 75% of the Subject 2's choose the allocation $(\$15, \$25)$. Thus in the typical population the expected payoff of Subject 1 if he passes is $.75 \times 15 = 11.25$, greater than 10. This simple experiment demonstrates the failure of a theory that does not fully account for the evolved tendency of human subjects to divine opportunity through understanding the history and intentions of those with whom they interact. A greater irony reveals itself in this scenario when we relax the Nash assumptions by repeating the game and giving subjects incomplete information in the form of their only their own payoffs: then the Nash prediction is robust!

Fouraker and Siegel [3] conducted the first extensive experimental study of basic oligopoly theory more than 40 years ago. They hypothesized that although the original quantity adjusting Cournot model does not directly discuss the information conditions of the agents, those conditions as well as the number of agents might have important consequences during repeated economic interactions. They were right: In Cournot quantity adjusting experiments where subjects either knew (complete info.) or didn't know (incomplete info.) their rivals payoff function, the distribution of outcomes suffered more variability under complete information, while the Cournot prediction was more robust with less information. There are many experiments in various other environments that verify that when provided information subjects attempt to use it in not always an entirely predictable manner. Further analysis provided by the Fouraker/Siegel data showed that both rivalistic (the tendency to increase your output when you observe others are producing more than you) and cooperative (the tendency to decrease your output when you observe others are producing more than you) signaling behaviors are more prevalent when information is complete. This result was most prevalent in duopolies and transferred strongly to a Bertrand price-setting environment.

Since the original oligopoly experiments, there has been an exponential growth of published oligopoly theory that deals with a multitude of mathematical nuances and assumptions, but comparatively few experimental tests that reign in the relevance of all those theories to human behavior. An interesting experimental study executed by [2] points out an environmental condition, low profitability under competition, that may confound the predictions of extant theory in a price setting scenario. They create a simple single product multi-period environment where 5 independent producers have identical cost structures such that they can each produce 100 units at cost c per unit, 100 units at cost $c+d$ per unit, and 100 additional units at cost $c+d+e$ per unit. The demand function is linear and represented by more than a hundreds robot buyers who simply reveal their value for consuming the product during the experiment. The demand function intersects the aggregate supply function at 1200 units, where the marginal cost of production is $c+d+e$. At the competitive price the sellers' share of surplus is an order of magnitude less than the buyers'. Each period during the experiment the oligopolists must post a single price at which they are willing to sell their product and the infinitesimal buyers queue up electronically and buy in order of lowest price available. Buyers are rationed uniformly amongst producers tied in price. A competitive Nash equilibrium exists that has each producer offering his 300 units at price $c+d+e$ and earning $100d+200e$, since a higher price would exclude the producer completely and a lower price would produce a loss on his higher cost units. But this is not what the average group of oligopolists does in this environment. They engage in an offering strategy that creates price cycles that tend to rise rapidly and fall slowly. The highest price producer (perhaps 2 or 3) is always excluded in a given period, but the remaining active sellers are rewarded with supra-competitive profits. These subjects, under much more complicated environmental and institutional conditions than in the simple two person game discussed above, and without direct communication, create a continuous tension between competitive (gradual undercutting) and cooperative (raise prices) behavior that serves them well in the long run.

Our experimental philosophy has been shaped largely by our interpretation of Friedrich Von Hayek's view that economics is the study of the co-emergent order of the brain and our social institutions. This philosophical framework leads us to look for ecologically rational explanations of both individual and institutional behavior. In particular we assume that individuals are adapted to specific functional needs that arose over evolutionary time, but that are now expressed by our modern brains through interactions with our current institutions and environments. However, as our brains learn to cope with their modern scenario they encounter cognitive opportunity costs that we attempt to overcome by the development of institutions that can perform the requisite computations and reallocations cheaply and efficiently. Economic experiments are crucial in testing the theory used to prescribe institutional innovation and in test-bedding the prescription before its implementation.

9 Conclusions

In this paper dynamic oligopoly models were examined. After introducing the classical Cournot model and its extensions, the stability of single product oligopoly was discussed with the assumption that full and instantaneous information was available to all firms concerning the market price. The equilibrium is always locally asymptotically stable with continuous time scales, however in order to preserve stability in the discrete case we have to assume that either the derivatives of the best response functions of all firms are sufficiently small, or the firms do not change much their output levels. Then we assumed that there was partial cooperation among the firms when each firm took a certain portion of the profits of the rivals into account in its payoff function. Under realistic conditions we could obtain the same stability results as in the previous case. The firms are usually uncertain in the market as they use only an estimate of the price function in their decisions on the best choices on production levels. We could show that similar stability conditions hold for this case as well, however the production level might converge to a steady state which differs from the Nash equilibrium. We also investigated the effect of information delay in market price, and showed that stability can be lost. We have derived conditions for stability and instability of the equilibrium and showed that at the critical value of the bifurcation parameter Hopf bifurcation occurs giving the possibility of the birth of limit cycles. Three particular learning processes were then introduced and examined when the firms had only limited information on the linear price function. It was interesting to see that the number of steady states (that is, the possibility of learning) and the asymptotic properties of the learning process were different for different types of uncertainty.

All theoretical results discussed in these sessions were based on certain specific assumptions on cost and market demand structure as well as on particular assumptions on the behavior of the decision makers. Such special conditions are not always satisfied in real economies, and in addition, the decision making managers do not think and decide always as we expect them to do. The most appropriate methodology in examining human decision making and economic processes without special conditions is based on laboratory experiments with realistic environment. Experimental economy is this very important procedure in which the actual decisions of the participants are repeated, observed and analyzed and hence we are able to gain the right insight into the minds of the decision making humans.

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