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Bounded Solutions and Periodic Solutions of Almost Linear Volterra Equations

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ABSTRACT

This article addresses boundedness and periodicity of solutions of certain Volterra type equations. These equations are studied under a set of assumptions on the functions involved in the equations. The equations will be called almost linear when these assumptions hold.

RESUMEN

Este artículo es concerniente a acotomiento y periocidad de ciertas ecuaciones de tipo Volterra. Estas ecuaciones son estudiadas bajo un conjunto de condiciones sobre las funciones envolvidas en las ecuaciones. Las ecuaciones serán llamadas casi lineales cuando estas condiciones sean válidas.

Key words and phrases: *Volterra integral equation, integrodifferential equation, resolvent, Krasnoselsii's fixed point theorem, bounded solution, periodic solution.*

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1 Introduction.

Consider the following scalar equations:

$$x(t) = a(t) + \int_0^t C(t,s)g(x(s))ds, \; t \geq 0, \tag{1.1}$$

and

$$x'(t) = a(t)h(x(t)) + \int_{-\infty}^t C(t,s)g(x(s))ds + p(t), \; t \in (-\infty, \infty). \tag{1.2}$$

We assume that the functions h and g are continuous and that there exist positive constants H, H^*, G, G^* such that

$$|h(x) - Hx| \leq H^*, \tag{1.3}$$

and

$$|g(x) - Gx| \leq G^*. \tag{1.4}$$

Equations (1.1) and (1.2) will be called almost linear if (1.3) and (1.4) hold. In [2] Burton introduced this concept of almost linear equations and studied certain important properties of the resolvent kernel of a linear Volterra equation. Throughout this paper we assume $a(t)$ in (1.1) is continuous for $t \geq 0$, and $a(t), p(t)$ in (1.2) are continuous for $-\infty < t < \infty$. Also, we assume that $C(t,s)$ in (1.1) is continuous for $0 \leq s \leq t < \infty$, and $C(t,s)$ in (1.2) is continuous for $-\infty < s \leq t < \infty$.

In Section 2 we obtain the boundedness of solutions of (1.1) using the respective resolvent kernels. In Section 3 we study (1.2) and show the existence of a periodic solution by employing Krasnoselskii's fixed point theorem.

The literature on the resolvent is massive. However, for many interesting results on resolvents of Volterra integral and integrodifferential equations we refer to [1], [3], [4], [6–8], [10–16], [18] and [19]. Burton [8] contains a large number of existing studies on the resolvents of Volterra integral equations which also includes many recent works related to the resolvent. On Krasnoselskii's fixed point theorem and it's application in integral equations we refer the reader to [5], [9] and [17].

2 On Solutions of (1.1).

We rewrite (1.1),

$$x(t) = a(t) + \int_0^t C(t,s)[g(x(s)) - Gx(s)]ds + \int_0^t C(t,s)Gx(s)ds. \tag{2.1}$$

Let

$$A(t) = a(t) + \int_0^t C(t,s)[g(x(s)) - Gx(s)]ds \tag{2.2}$$

and

$$B(t,s) = GC(t,s).$$

Then (2.1) becomes

$$x(t) = A(t) + \int_0^t B(t,s)x(s)ds. \tag{2.3}$$

Let $R(t,s)$ be the resolvent kernel associated with (2.3). Then $R(t,s)$ exists and satisfies

$$R(t,s) = -B(t,s) + \int_s^t R(t,u)B(u,s)du. \tag{2.4}$$

Then any solution $x(t)$ of (2.3) satisfies

$$x(t) = A(t) - \int_0^t R(t,s)A(s)ds. \tag{2.5}$$

Theorem 2.1 Assume $a(t)$ is bounded for $t \geq 0$. Also assume

$$\sup_{t\geq 0} \int_0^t |C(t,s)|ds < \infty, \tag{2.6}$$

and

$$\sup_{t\geq 0} \int_0^t |R(t,s)|ds < \infty. \tag{2.7}$$

Then any solution $x(t)$ of (1.1) is bounded.

Proof. From (2.2),using (1.4) and (2.6) we obtain

$$|A(t)| \leq |a(t)| + G^* \int_0^t |C(t,s)|ds < \infty.$$

Therefore from (2.5) and (2.7), we get

$$|x(t)| \leq |A(t)| + \int_0^t |R(t,s)||A(s)|ds < \infty.$$

This concludes the proof of Theorem 2.1.

Assume $a'(t)$ and $C_t(t,s)$ both exist and are continuous. Now differentiating (1.1), one gets

$$\begin{aligned} x'(t) &= a'(t) + C(t,t)g(x(t)) + \int_0^t C_t(t,s)g(x(s))ds \\ &= a'(t) + C(t,t)[g(x(t)) - Gx(t)] + \int_0^t C_t(t,s)[g(x(s)) - Gx(s)]ds \\ &\quad +C(t,t)Gx(t) + \int_0^t C_t(t,s)Gx(s)ds. \end{aligned} \tag{2.8}$$

Let
$$F(t) = a'(t) + C(t,t)[g(x(t)) - Gx(t)] + \int_0^t C_t(t,s)[g(x(s)) - Gx(s)]ds. \tag{2.9}$$
Then (2.8) becomes
$$x'(t) = GC(t,t)x(t) + \int_0^t GC_t(t,s)x(s)ds + F(t). \tag{2.10}$$
Let
$$B(t,s) = GC_t(t,s), \quad A(t) = GC(t,t).$$
Then (2.10) becomes
$$x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s)ds + F(t), \; x(0) = a(0). \tag{2.11}$$

Let $Z(t,s)$ be the resolvent kernel associated with (2.11). Then $Z(t,s)$ exists and satisfies
$$Z_s(t,s) = -Z(t,s)A(s) - \int_s^t Z(t,u)B(u,s)du, \quad Z(t,t) = 1. \tag{2.12}$$
Then from the variation of parameters formula, any solution $x(t)$ of (2.11) has the form
$$x(t) = Z(t,0)a(0) + \int_0^t Z(t,s)F(s)ds. \tag{2.13}$$

Theorem 2.2 Assume $a'(t)$ is bounded. Also assume
$$\sup_{t\geq 0} \int_0^t |C_t(t,s)|ds < \infty, \tag{2.14}$$
and
$$\sup_{t\geq 0} \int_0^t |Z(t,s)|ds < \infty. \tag{2.15}$$
In addition, we assume that $|C(t,t)|$ and $|Z(t,0)|$ are bounded. Then any solution $x(t)$ of (2.11) is bounded.

Proof. Applying (1.4) and (2.14) in (2.9), we get
$$|F(t)| \leq |a'(t)| + |C(t,t)|G^* + \int_0^t |C_t(t,s)|G^* ds < \infty.$$
Therefore from (2.13) one obtains
$$|x(t)| \leq |Z(t,0)||a(0)| + \int_0^t |Z(t,s)||F(s)|ds < \infty.$$

This concludes the proof of Theorem 2.2.

Properties in (2.7) and (2.15) are known as integrability properties of resolvent. Conditions to ensure (2.7) can be found in [11], [14] and [18], and conditions to ensure (2.15) can be found in [10], [12], [13] and [19].

3 Periodic Solutions of (1.2)

In this section we investigate the existence of a periodic solution of (1.2) using Krasnoselskii's fixed point theorem.

We start with a statement of Krasnoselskii's fixed point theorem.

Theorem Krasnoselskii [17]. Let K be a closed convex non-empty subset of a Banach space M. Suppose that A and B map K into M such that

(i) $x, y \in K$, implies $Ax + By \in K$,

(ii) A is continuous and AK is contained in a compact subset of M,

(iii) B is a contraction mapping.

Then there exists $z \in K$ with $z = Az + Bz$.

In this section we assume that

$$\sup_{-\infty<t<\infty} \int_{-\infty}^{t} |C(t,s)|\, ds < \infty. \tag{3.1}$$

For convenience we rewrite (1.2),

$$x'(t) = a(t)h(x(t)) + \int_{-\infty}^{t} C(t,s)g(x(s))ds + p(t),\ t \in (-\infty, \infty), \tag{3.2}$$

from which we get

$$\begin{aligned} x'(t) - Ha(t)x(t) &= -Ha(t)x(t) + a(t)h(x(t)) + p(t) \\ + \int_{-\infty}^{t} C(t,s)\big[g(x(s)) - Gx(s)\big]ds &+ \int_{-\infty}^{t} C(t,s)Gx(s)ds. \end{aligned} \tag{3.3}$$

Suppose there exists a constant $T > 0$ such that

$$a(t+T) = a(t),\ p(t+T) = p(t),\ C(t+T, s+T) = C(t,s). \tag{3.4}$$

We assume that

$$\int_0^T a(t)dt \neq 0. \tag{3.5}$$

Let M be the complete metric space of continuous T-periodic functions $\phi : (-\infty, \infty) \to (-\infty, \infty)$ with the supremum metric. Then, for any positive constant m the set

$$P_T = \{f \in M : ||f|| \le m\}, \tag{3.6}$$

is a closed convex subset of M. Let

$$k(t) = p(t) + \int_{-\infty}^t C(t,s)\big[g(x(s)) - Gx(s)\big]\, ds + \int_{-\infty}^t C(t,s)Gx(s)ds.$$

Then we may write (3.3) as

$$x'(t) - Ha(t)x(t) = -Ha(t)x(t) + a(t)h(x(t)) + k(t). \tag{3.7}$$

Assume (3.4) and (3.5) hold. Multiply both sides of (3.7) with $e^{-H\int_0^t a(s)ds}$ and then integrate both sides from $t-T$ to t, to obtain

$$\begin{aligned} & x(t)[e^{-H\int_{t-T}^t a(s)ds} - 1]e^{-H\int_0^{t-T} a(s)ds} \\ = & \int_{t-T}^t \big[-Ha(u)x(u) + a(u)h(x(u)) + k(u)\big]e^{-H\int_0^u a(s)ds}du. \end{aligned}$$

Now, multiplying both sides by $e^{H\int_0^{t-T} a(s)ds}$, we get

$$\begin{aligned} & x(t)[e^{-H\int_{t-T}^t a(s)ds} - 1] \\ = & \int_{t-T}^t \big[-Ha(u)x(u) + a(u)h(x(u)) + k(u)\big]e^{-H\int_{t-T}^u a(s)ds}du. \end{aligned}$$

Due to the periodicity of $a(t)$ we note that $e^{-H\int_{t-T}^t a(s)ds} = e^{-H\int_0^T a(s)ds}$. Substituting k by the expression given earlier and then dividing by $e^{-H\int_{t-T}^t a(s)ds} - 1$, we arrive at

$$\begin{aligned} x(t) = & \frac{1}{e^{-H\int_0^T a(s)ds} - 1}\Big\{\int_{t-T}^t a(u)[h(x(u)) - Hx(u)]e^{-H\int_{t-T}^u a(s)ds}du \\ & + \int_{t-T}^t \int_{-\infty}^u C(u,s)[g(x(s)) - Gx(s)]\, ds\; e^{-H\int_{t-T}^u a(s)ds}du \\ & + \int_{t-T}^t \int_{-\infty}^u C(u,s)Gx(s)\, ds\; e^{-H\int_{t-T}^u a(s)ds}du \\ + & \int_{t-T}^t p(u)e^{-H\int_{t-T}^u a(s)ds}du\Big\}. \end{aligned} \tag{3.8}$$

Define mappings A and B from P_T into M as follows.

For $\phi \in P_T$,

$$\begin{aligned}(A\phi)(t) &= \frac{1}{e^{-H\int_0^T a(s)ds} - 1}\Big\{\int_{t-T}^{t} a(u)[h(\phi(u)) - H\phi(u)]e^{-H\int_{t-T}^{u} a(s)ds}du \\ &\quad + \int_{t-T}^{t}\int_{-\infty}^{u} C(u,s)[g(\phi(s)) - G\phi(s)]\,ds\; e^{-H\int_{t-T}^{u} a(s)ds}du\Big\}\end{aligned}$$

and for $\psi \in P_T$,

$$\begin{aligned}(B\psi)(t) &= \frac{1}{e^{-H\int_0^T a(s)ds} - 1}\Big\{\int_{t-T}^{t}\int_{-\infty}^{u} C(u,s)G\psi(s)\,ds\; e^{-H\int_{t-T}^{u} a(s)ds}du \\ &+ \int_{t-T}^{t} p(u)e^{-H\int_{t-T}^{u} a(s)ds}du\Big\}.\end{aligned}$$

It can easily be verified that both $(A\phi)(t)$ and $(B\psi)(t)$ are T-periodic and continuous in t. Assume

$$\sup_{-\infty<t<\infty}\Big|\frac{1}{e^{-H\int_0^T a(s)ds} - 1}\Big|\int_{t-T}^{t}\int_{-\infty}^{u} |C(u,s)|G\;ds\;\; e^{-H\int_{t-T}^{u} a(s)ds}du \le \alpha < 1, \tag{3.9}$$

and

$$\begin{aligned}&\sup_{-\infty<t<\infty}\Big|\frac{1}{e^{-H\int_0^T a(s)ds} - 1}\Big|\Big\{\int_{t-T}^{t} |a(u)|H^* e^{-H\int_{t-T}^{u} a(s)ds}du \\ &+ \int_{t-T}^{t}\int_{-\infty}^{u} G^*|C(u,s)|\,ds\; e^{-H\int_{t-T}^{u} a(s)ds}du\Big\} \le \beta < \infty.\end{aligned} \tag{3.10}$$

Choose the constant m of (3.6) satisfying

$$\sup_{-\infty<t<\infty}\Big|\frac{1}{e^{-H\int_0^T a(s)ds} - 1}\Big|\int_{t-T}^{t} |p(u)|e^{-H\int_{t-T}^{u} a(s)ds}du + \alpha m + \beta \le m. \tag{3.11}$$

Lemma 3.1 Assume (3.4), (3.5), (3.9) and (3.11). Then map B is a contraction from P_T into P_T.

Proof. For $\phi \in P_T$,

$$\begin{aligned}|(B\phi)(t)| &\le m\Big|\frac{1}{e^{-H\int_0^T a(s)ds} - 1}\Big|\int_{t-T}^{t}\int_{-\infty}^{u} |C(u,s)|G\;ds\;\; e^{-H\int_{t-T}^{u} a(s)ds}du \\ &+ \Big|\frac{1}{e^{-H\int_0^T a(s)ds} - 1}\Big|\int_{t-T}^{t} |p(u)|e^{-H\int_{t-T}^{u} a(s)ds}du \\ &\le \sup_{-\infty<t<\infty}\Big|\frac{1}{e^{-H\int_0^T a(s)ds} - 1}\Big|\int_{t-T}^{t} |p(u)|e^{-H\int_{t-T}^{u} a(s)ds}du + \alpha m \\ &< m.\end{aligned}$$

For $\phi, \psi \in P_T$, we obtain, using (3.9),

$$|(B\phi)(t) - (B\psi)(t)| \le \alpha||\phi - \psi||.$$

This proves that B is a contraction mapping from P_T into P_T.

Lemma 3.2 Assume (1.2), (1.3), (3.1), (3.4), (3.5), (3.10) and (3.11). Then map A from P_T into P_T is continuous, and AP_T is contained in a compact subset of M.

Proof. For any $\phi \in P_T$, it follows from (3.10) and (3.11) that

$$|(A\phi)(t)| \le \beta \le m. \tag{3.12}$$

So, A maps from P_T into P_T, and the set $\{A\phi\}$ for $\phi \in P_T$ is uniformly bounded. To show that A is a continuous map, let $\{\phi_n\}$ be any sequence of functions in P_T with $||\phi_n - \phi|| \to 0$ as $n \to \infty$. Then one can easily verify that

$$||A\phi_n - A\phi|| \to 0 \text{ as } n \to \infty.$$

This proves that A is a continuous mapping.

Now, we will show that the set $\{A\phi\}$ for $\phi \in P_T$ is equicontinuous by showing that $|(A\phi)'(t)|$ is bounded. Taking the derivative of $(A\phi)(t)$ and then using (1.3), (1.4) and (3.12) we get

$$\begin{aligned}
|(A\phi)'(t)| &\le |a(t)||h(\phi(t)) - H\phi(t)| + |Ha(t)(A\phi)(t)| \\
&+ \Big|\frac{e^{-H\int_0^T a(s)ds}}{e^{-H\int_0^T a(s)ds} - 1}\int_{-\infty}^{t} C(t,s)(g(\phi(s)) - G\phi(s))ds \\
&- \frac{1}{e^{-H\int_0^T a(s)ds} - 1}\int_{-\infty}^{t-T} C(t,s)(g(\phi(s)) - G\phi(s))ds\Big| \\
&\le ||a||(H^* + mH) \\
&+ \frac{G^*}{\left|e^{-H\int_0^T a(s)ds} - 1\right|}\left(1 + e^{-H\int_0^T a(s)ds}\right)\int_{-\infty}^{t} |C(t,s)|\, ds \\
&\le ||a||(H^* + mH) + l.
\end{aligned}$$

Here we assumed

$$\frac{G^*}{\left|e^{-H\int_0^T a(s)ds} - 1\right|}\left(1 + e^{-H\int_0^T a(s)ds}\right)\int_{-\infty}^{t} |C(t,s)|\, ds < l < \infty$$

for all $t \in (-\infty, \infty)$. Since $a(t)$ is a bounded function, this shows that the set $\{A\phi\}$ for $\phi \in P_T$ is equicontinuous. Therefore, by the Arzela-Ascoli Theorem, AP_T is contained in a compact subset of M.

We are now ready to use the fixed point theorem of Krasnoselskii to show the existence of a continuous T-periodic solution of (3.2).

Theorem 3.1 Suppose assumptions of Lemmas 3.1 and 3.2 hold. Then (3.2) has a continuous T-periodic solution.

Proof. For $\phi, \psi \in P_T$, we get

$$\begin{aligned} |(A\phi)(t) + (B\psi)(t)| &\leq \sup_{t\geq 0} \Big| \frac{1}{e^{-H\int_0^T a(s)ds} - 1} \Big| \int_{t-T}^{t} |p(u)| e^{-H\int_{t-T}^{u} a(s)ds} du \\ &+ \alpha m + \beta \\ &\leq m. \end{aligned}$$

which proves that $A\phi + B\psi \in P_T$.

Therefore, by Krasnoselskii's theorem there exists a function $x(t)$ in P_T such that

$$x(t) = Ax(t) + Bx(t).$$

This proves that (3.2) has a continuous T-periodic solution $x(t)$.

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