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Fixed Point Theory and Nonlinear Periodic Systems

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ABSTRACT

This work is concerned with a nonlinear periodic system, which depends on parameters. We investigate continuity with respect to parameters of the periodic solution of the system. Applying a fixed point theorem and the results regarding parameters for C_0-semigroups, we obtained some convenient conditions for determining both existence of a unique periodic solution and continuity in parameters of the periodic solution. The results are applied to a nonlinear wave equation with forced and damped boundary conditions.

RESUMEN

Este trabajo tiene que ver con un sistema no lineal periódico que depende de parametros. Investigamos continuidad con respecto a los parametros de la solución periódica del sistema. Aplicando un teorema de punto fijo y resultados considerando parametros para C_0-semigrupos, obtenemos algunas condiciones convenientes para determinar existencia de una única solución periódica y también continuidad en terminos de los parametros para la solución periódica. Los resultados son aplicados para una ecuación de onda no lineal con condiciones de frontera de tipo forzado y damped.

Key words and phrases: C_0*-semigroup, periodic system, parameter, continuity.*

Math. Subj. Class.: *47D62, 45K05, 35L20.*

1 Introduction

Boundary value conditions in partial differential equations lead to problems involving parameters. If these equations are reformulated as abstract Cauchy problems, the equations will depend on these parameters in a way that is reflected by the domain of the operators involved. For such parameter dependent equations it is natural to want continuity and differentiability with respect to parameters of solutions of the equations. Recent studies [3, 4, 6, 7, and references therein] have established fundamental theory about continuity and differentiability with respect to parameters. In this work, we are particularly interested in a class of nonlinear periodic systems that depend on parameters and study continuity in parameters of its solutions. In our subsequent paper, we will discuss differentiability with respect to parameters of the solutions.

Consider a nonlinear wave equation with forced and damped boundary conditions

$$\begin{aligned}
&u_{tt} = u_{xx} + \eta F(u_t) && \text{for } t \geq 0,\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) && \text{for } x \in [0,1],\\
&\mu u_t(0,t) - \gamma u_x(0,t) = f_1(t), &&\\
&\delta u_t(1,t) + \gamma u_x(1,t) = f_2(t), && \mu, \gamma, \delta > 0,
\end{aligned} \tag{1.1}$$

where $f_1(t)$ and $f_2(t)$ are both ρ-periodic and continuously differentiable. F satisfies a uniform Lipschitz condition

Let $v = u_t$ and $w = u_x$, then equation (1.1) is written as

$$\begin{aligned}
&v_t = w_x + \eta F(v),\\
&w_t = v_x,\\
&\mu v(0,t) - \gamma w(0,t) = f_1(t),\\
&\delta v(1,t) + \gamma w(1,t) = f_2(t), && \mu, \gamma, \delta > 0.
\end{aligned} \tag{1.2}$$

If the change of variables

$$\begin{aligned}
\bar{v} &= v - x\delta^{-1} f_2,\\
\bar{w} &= w + (1-x)\gamma^{-1} f_1
\end{aligned}$$

is made, equation (1.2) has the form:

$$
\begin{aligned}
&\bar{v}_t = \bar{w}_x + \eta F(\bar{v} + x\delta^{-1} f_2(t)) + \gamma^{-1} f_1(t) - x\delta^{-1} f_2'(t),\\
&\bar{w}_t = \bar{v}_x + \delta^{-1} f_2(t) + (1-x)\gamma^{-1} f_1'(t),\\
&\mu\bar{v}(0,t) - \gamma\bar{w}(0,t) = 0,\\
&\delta\bar{v}(1,t) + \gamma\bar{w}(1,t) = 0, \qquad\qquad \mu,\gamma,\delta > 0.
\end{aligned}
\tag{1.3}
$$

Further, the associated abstract equation of (1.3) is given by

$$
\begin{aligned}
&\frac{dz(t)}{dt} = A(\varepsilon)z(t) + \bar{F}(t, z(t), \varepsilon),\\
&z(0) = z_0
\end{aligned}
\tag{1.4}
$$

on $X = L^2[0,1] \times L^2[0,1]$, where

$$
A(\varepsilon) = \begin{bmatrix} 0 & \partial x \\ \partial x & 0 \end{bmatrix}, \qquad \varepsilon = (\mu,\gamma,\delta,\eta) \in R^4_+, \qquad z = \begin{bmatrix} \bar{v} \\ \bar{w} \end{bmatrix},
$$

$$
D(A(\varepsilon)) = \left\{ \begin{bmatrix} \bar{v} \\ \bar{w} \end{bmatrix} \in \prod_{i=1}^{2} H^1[0,1] \;\middle|\; \begin{array}{c} \mu\bar{v}(0) = \gamma\bar{w}(0), \\ \delta\bar{v}(1) = -\gamma\bar{w}(1), \end{array} \; \mu,\gamma,\delta > 0 \right\},
$$

$$
\bar{F}(t,z,\varepsilon)x = \begin{bmatrix} \eta F(\bar{v} + x\delta^{-1} f_2(t)) + \gamma^{-1} f_1(t) - x\delta^{-1} f_2'(t) \\ \delta^{-1} f_2(t) + (1-x)\gamma^{-1} f_1'(t) \end{bmatrix}.
$$

For convenience, write $\bar{v}$ as v,$\bar{w}$ as w, $\bar{F}$ as F_1, and $\alpha = \frac{\mu}{\gamma}, \beta = \frac{\gamma}{\delta}$, we then have

$$
\begin{aligned}
&\frac{dz(t)}{dt} = A(\varepsilon)z(t) + F_1(t, z(t), \varepsilon),\\
&z(0) = z_0
\end{aligned}
\tag{1.5}
$$

on $X = L^2[0,1] \times L^2[0,1]$, where

$$
A(\varepsilon) = \begin{bmatrix} 0 & \partial x \\ \partial x & 0 \end{bmatrix}, \qquad \varepsilon = (\alpha,\beta,\delta,\eta) \in R^4_+, \qquad z = \begin{bmatrix} v \\ w \end{bmatrix},
$$

$$
D(A(\varepsilon)) = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \prod_{i=1}^{2} H^1[0,1] \;\middle|\; \begin{array}{c} \alpha v(0) = w(0), \\ v(1) = -\beta w(1), \end{array} \; \alpha,\beta > 0 \right\},
$$

$$
F_1(t,z,\varepsilon)x = \begin{bmatrix} \eta F(v + x\delta^{-1} f_2(t)) + \gamma^{-1} f_1(t) - x\delta^{-1} f_2'(t) \\ \delta^{-1} f_2(t) + (1-x)\gamma^{-1} f_1'(t) \end{bmatrix}.
$$

By examining equation (1.5), we see that a) F_1 is a nonlinear and ρ-periodic function, b) ε is a multi-parameter, c) $A(\varepsilon)$ is linear and densely defined, and d) $D(A(\varepsilon))$ is dependent on the parameter ε.

Our work is motivated by this example to consider the abstract nonlinear periodic equation

$$\begin{aligned}&\frac{dz(t)}{dt} = A(\varepsilon)z(t) + F(t, z(t), \varepsilon),\\ &z(0) = z_0\end{aligned} \tag{1.6}$$

on a Banach space X with norm $\|\cdot\|$, where $A(\varepsilon)$ is linear and densely defined, $\varepsilon \in P$ is a parameter (where P is an open subset of a finite-dimensional normed linear space $\mathcal{P}$ with norm $|\cdot|$), $F(t+\rho, z, \varepsilon) = F(t, z, \varepsilon)$ for some $\rho > 0$, and F is continuous in $(t, z, \varepsilon) \in R \times X \times P$. Note that the above example illustrates a fact that the occurrence of parameters in boundary conditions causes the domain of the operator $A(\varepsilon)$ to depend on the parameters. This phenomenon is not considered in any papers known to the authors. The goal of this work is to determine conditions concerning a) existence and uniqueness of the periodic (weak) solution of equation (1.6) and b) continuity with respect to parameter ε of the solution of (1.6).

Based on semigroup theory, when $A(\varepsilon)$ generates a C_0-semigroup $T(t, \varepsilon)$, the weak solution of (1.6) will have the form

$$z(t, \varepsilon) = T(t, \varepsilon)z_0 + \int_0^t T(t-s, \varepsilon)F(s, z(s, \varepsilon), \varepsilon)\, ds.$$

It is clear that the continuity in parameter ε of semigroup $T(t, \varepsilon)$ will play a key role in attaining our goals. To this end we first discuss the parameter properties of C_0-semigroup $T(t, \varepsilon)$ and present a method which can be especially useful for dealing with operator $A(\varepsilon)$ that has a domain dependent on ε in Section 2. We then apply a fixed point theorem to show both existence of a unique periodic solution and continuity in ε of the periodic solution of (1.6) in Section 3. In the last section, we shall illustrate the results and technique in an application to a nonlinear wave equation with variable boundary conditions.

2 Continuity in Parameters of C_0-Semigroups

As noted in Section 1, the semigroup theory indicates that the continuity in parameters of C_0-semigroup $T(t, \varepsilon)$ generated by $A(\varepsilon)$ in (1.6) can easily derive the analogous result for the solution of equation (1.6). Hence, we will focus on discussing continuity property of C_0-semigroup $T(t, \varepsilon)$. In this section, we will first state a continuity result for the case that operators have constant domains (the domain is independent of parameters). Then we will present a method that is especially useful in handling the case that operators have variable domains (the domain is dependent of parameters).

In the sequel we use "A is a Hille-Yosida operator" to mean that there are constants $M \geq 1$ and $\omega \in R$ such that $\lambda > \omega$ implies $\lambda \in \rho(A)$ (the resolvent set of A) and

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for } \lambda > \omega,\ n \in N.$$

In the special case when an operator has a constant domain, one has the result which states that the strong continuity in parameters of the operator implies the continuity in parameters of its semigroup.

Theorem 2.1. Assume that

(2.1) $D(A(\varepsilon)) = D$ for all $\varepsilon \in P$.

(2.2) There are constants $M \geq 1$ and $\omega \in R$ such that

$$||(\lambda I - A(\varepsilon))^{-n}|| \leq \frac{M}{(\lambda - \omega)^n} \text{ for } \lambda > \omega,\ n \in N, \text{ and all } \varepsilon \in P.$$

(2.3) For each $x \in D$, $A(\varepsilon)x$ is continuous in $\varepsilon \in P$.

Then $(\lambda I - A(\varepsilon))^{-1}x$ is continuous in ε for each $x \in X$. Further, the C_0-semigroup $T(t, \varepsilon)$ generated by $A(\varepsilon)$ is strongly continuous in $\varepsilon \in P$, and the continuity is uniform on bounded t-intervals. In particular, for any $\varepsilon \in P, h \in \mathcal{P}$ with $\varepsilon + h \in P$, and any $t_0 \in [0, \infty)$,

$$\sup_{0 \leq t \leq t_0} \|T(t, \varepsilon + h)x - T(t, \varepsilon)x\| = \circ(1) \quad \text{as } |h| \to 0, \text{ for each } x \in X.$$

Proof. See [8] for the proof of continuity of $(\lambda I - A(\varepsilon))^{-1}x$. The proof of continuity of C_0-semigroup $T(t, \varepsilon)$ is similar to the proof of the C_0-semigroup approximation theorem in [1, Theorem 3.17], and it is omitted. □

The next theorem gives a convenient condition to determine the continuity in parameters of C_0-semigroup when the domain of the operator is dependent on parameters.

We remark that the following assumption is naturally possessed by many hyperbolic and parabolic types of boundary value problems (see an example in Section 4).

ASSUMPTION Q. Let $\varepsilon_0 \in P$ be given. Then for each $\varepsilon \in P$ there exists bounded operators $Q_1(\varepsilon), Q_2(\varepsilon) : X \to X$ with bounded inverses $Q_1^{-1}(\varepsilon)$ and $Q_2^{-1}(\varepsilon)$, such that $A(\varepsilon) = Q_1(\varepsilon)A(\varepsilon_0)Q_2(\varepsilon)$.

Note that if $A(\varepsilon_1) = Q_1(\varepsilon_1)A(\varepsilon_0)Q_2(\varepsilon_1)$, then

$$\begin{aligned} A(\varepsilon) &= Q_1(\varepsilon)A(\varepsilon_0)Q_2(\varepsilon) \\ &= Q_1(\varepsilon)Q_1^{-1}(\varepsilon_1)Q_1(\varepsilon_1)A(\varepsilon_0)Q_2(\varepsilon_1)Q_2^{-1}(\varepsilon_1)Q_2(\varepsilon) \\ &= \tilde{Q}_1(\varepsilon)A(\varepsilon_1)\tilde{Q}_2(\varepsilon). \end{aligned}$$

Thus, having such a relationship for some ε_0 implies a similar relationship at any other $\varepsilon_1 \in P$. Without loss of generality then, we may just consider the continuity of the semigroup $T(t, \varepsilon)$ at $\varepsilon = \varepsilon_0 \in P$.

Consider an auxiliary operator

$$\tilde{A}(\varepsilon) = Q_2(\varepsilon)A(\varepsilon)Q_2^{-1}(\varepsilon) = Q_2(\varepsilon)Q_1(\varepsilon)A(\varepsilon_0).$$

Lemma 2.2. Assume that Assumption Q and (2.2) are satisfied and suppose that

(2.4) $Q_i(\varepsilon)x$ and $Q_2^{-1}(\varepsilon)x$ are continuous in ε for $x \in X$, $i = 1, 2$.

Then the C_0-semigroup $\tilde{T}(t,\varepsilon)$ generated by $\tilde{A}(\varepsilon)$ is strongly continuous at ε_0, and the continuity is uniform on bounded t-intervals. In particular, for any $h \in \mathcal{P}$ with $\varepsilon_0 + h \in P$, and any $t_0 \in [0,\infty)$,

$$\sup_{0\le t\le t_0} \|\tilde{T}(t,\varepsilon_0+h)x - \tilde{T}(t,\varepsilon_0)x\| = \circ(1) \quad \text{as} \quad |h| \to 0, \text{ for each } x \in X.$$

Proof. Note that $\tilde{A}(\varepsilon) = Q_2(\varepsilon)Q_1(\varepsilon)A(\varepsilon_0)$. Clearly, $D(\tilde{A}(\varepsilon)) = D(A(\varepsilon_0))$ for all $\varepsilon \in P$. Thus, $\tilde{A}(\varepsilon)$ has a fixed domain. Also (2.4) implies that $\tilde{A}(\varepsilon)$ is continuous in ε, that is, $\tilde{A}(\varepsilon)$ satisfies (2.3). Since

$$(\lambda I - \tilde{A}(\varepsilon))^n = Q_2(\varepsilon)(\lambda I - A(\varepsilon))^n Q_2^{-1}(\varepsilon),$$

thus

$$(\lambda I - \tilde{A}(\varepsilon))^{-n} = Q_2(\varepsilon)(\lambda I - A(\varepsilon))^{-n} Q_2^{-1}(\varepsilon).$$

Because $Q_2(\varepsilon)$ and $Q_2^{-1}(\varepsilon)$ are bounded, (2.2) implies that $\tilde{A}(\varepsilon)$ is a Hille-Yosida operator for all $\varepsilon \in P$. Now applying Theorem 2.1, we have that $\tilde{T}(t,\varepsilon)$ is continuous in ε. □

Theorem 2.3. Assume that Assumption Q, (2.2) and (2.4) are satisfied, then the C_0-semigroup $T(t,\varepsilon)$ generated by $A(\varepsilon)$ is strongly continuous at ε_0, and the continuity is uniform on bounded t-intervals. In particular, for any $h \in \mathcal{P}$ with $\varepsilon_0 + h \in P$, and any $t_0 \in [0,\infty)$,

$$\sup_{0\le t\le t_0} \|T(t,\varepsilon_0+h)x - T(t,\varepsilon_0)x\| = \circ(1) \quad \text{as} \quad |h| \to 0, \text{ for each } x \in X.$$

Proof. In fact, $\tilde{A}(\varepsilon) = Q_2(\varepsilon)A(\varepsilon)Q_2^{-1}(\varepsilon)$. By the definition of C_0-semigroup [2], it is clear that $T(t,\varepsilon) = Q_2^{-1}(\varepsilon)\tilde{T}(t,\varepsilon)Q_2(\varepsilon)$.

Further, for $x \in X$ and $\varepsilon_0 + h \in B_\delta(\varepsilon_0) = \{\varepsilon \,|\, |\varepsilon - \varepsilon_0| \le \delta\}$,

$$\begin{aligned}
& \|T(t,\varepsilon_0+h)x - T(t,\varepsilon_0)x\| \\
= \quad & \|Q_2^{-1}(\varepsilon_0+h)\tilde{T}(t,\varepsilon_0+h)Q_2(\varepsilon_0+h)x - Q_2^{-1}(\varepsilon_0)\tilde{T}(t,\varepsilon_0)Q_2(\varepsilon_0)x\| \\
\le \quad & \|Q_2^{-1}(\varepsilon_0+h)\tilde{T}(t,\varepsilon_0+h)\| \, \|Q_2(\varepsilon_0+h)x - Q_2(\varepsilon_0)x\| \\
& + \|Q_2^{-1}(\varepsilon_0+h)\| \, \|(\tilde{T}(t,\varepsilon_0+h) - \tilde{T}(t,\varepsilon_0))Q_2(\varepsilon_0)x\| \\
& + \|(Q_2^{-1}(\varepsilon_0+h) - Q_2^{-1}(\varepsilon_0))\tilde{T}(t,\varepsilon_0)Q_2(\varepsilon_0)x\| \,.
\end{aligned}$$

Note that $\|Q_2^{-1}(\varepsilon_0+h)\tilde{T}(t,\varepsilon_0+h)\|$, $\|Q_2^{-1}(\varepsilon_0+h)\|$ are uniformly bounded. Moreover, from Lemma 2.2 and (2.4), $\tilde{T}(t,\varepsilon), Q_2(\varepsilon)$, and $Q_2^{-1}(\varepsilon)$ are continuous in ε. Thus, we have $T(t,\varepsilon)$ is continuous in ε. □

3 Continuity in Parameters of Periodic Solutions of (1.6)

The goal of this section is to obtain existence and continuity in parameters of a unique periodic solution $z(t,\varepsilon)$ of (1.6). We first study a special case of (1.6)

$$\begin{aligned}\frac{dz(t)}{dt} &= A(\varepsilon)z(t) + F(t,\varepsilon),\\ z(0) &= z_0\end{aligned} \tag{3.1}$$

on a Banach space X, where $A(\varepsilon)$ is linear and densely defined, $\varepsilon \in P$ is a parameter, $F(t+\rho,\varepsilon) = F(t,\varepsilon)$ for some $\rho > 0$, and $F(t,\varepsilon)$ is continuous in $(t,\varepsilon) \in R \times P$.

We aim for obtaining the existence and continuity in ε of the periodic solution of (3.1) and then apply this result together with a fixed point theorem to show that (1.6) has a unique periodic solution, which is continuous in parameter ε.

For convenient reference, we state the fixed point theorem that will be needed in the later proofs.

Theorem 3.1.[5, p7] If $\mathcal{F}$ is a closed subset of a Banach space $\mathcal{X}$, $\mathcal{G}$ is a subset of a Banach space $\mathcal{Y}$, $T_y : \mathcal{F} \to \mathcal{F}$, $y \in \mathcal{G}$ is a uniform contraction on $\mathcal{F}$ and $T_y x$ is continuous in y for each fixed x in $\mathcal{F}$, then the unique fixed point $g(y)$ of T_y, y in $\mathcal{G}$, is continuous in y.

As discussed in Section 2, several convenient conditions have been obtained for determining continuity in parameter ε of C_0-semigroup $T(t,\varepsilon)$. Thereby, in this section, we will just assume that $T(t,\varepsilon)$ is continuous in parameter ε.

Theorem 3.2. Assume that

(3.2) $T(t,\varepsilon)z$ is continuous in ε for each $z \in X$, and

$$\|T(t,\varepsilon)\| \le M(t_0)$$

for some $M(t_0) > 0$ and all $\varepsilon \in P, t \in [0,t_0]$, and

(3.3) $\|T(N\rho,\varepsilon)\| \le k < 1$ for some integer N with $N\rho < t_0$ and all $\varepsilon \in P$.

Then there exists a unique ρ-periodic solution of (3.1), say $z(t,\varepsilon)$, which is continuous in ε for $\varepsilon \in P$.

Proof. First, we know that the weak solution of (3.1) can be expressed as

$$z(t,\varepsilon) = T(t,\varepsilon)z_0 + \int_0^t T(t-s,\varepsilon)F(s,\varepsilon)\,ds.$$

To show that there is a unique ρ-periodic solution which is continuous in ε, it suffices to show that the operator $K(\varepsilon)$ has a unique fixed point where

$$K(\varepsilon)z = T(\rho,\varepsilon)z + \int_0^\rho T(\rho-s,\varepsilon)F(s,\varepsilon)\,ds.$$

Consider the Nth-iterate, $K^N(\varepsilon)$. We first note that $K^N(\varepsilon)$ is a uniform contraction on X. In fact, note that

$$K^N(\varepsilon)z = T(N\rho,\varepsilon)z + \int_0^{N\rho} T(N\rho - s,\varepsilon)F(s,\varepsilon)\,ds$$

and then, for all $\varepsilon \in P$ and $z_1, z_2 \in X$,

$$\begin{aligned} & \|K^N(\varepsilon)z_1 - K^N(\varepsilon)z_2\| = \|T(N\rho,\varepsilon)(z_1 - z_2)\| \\ \leq \quad & \|T(N\rho,\varepsilon)\| \cdot \|z_1 - z_2\| \leq k\|z_1 - z_2\| \quad (\text{since } \|T(N\rho,\varepsilon)\| \leq k < 1). \end{aligned}$$

In addition, $K^N(\varepsilon)$ is continuous in ε for each fixed $z \in X$. To see this, let ε be an arbitrary point in P. It is clear, from (3.2) and continuity of $F(t,\varepsilon)$, that $T(N\rho - s,\varepsilon)F(s,\varepsilon)$ is continuous at $\varepsilon = \varepsilon_0$.

Furthermore,

$$\int_0^{N\rho} T(N\rho - s,\varepsilon)F(s,\varepsilon)\,ds \,\text{is continuous at}\, \varepsilon = \varepsilon_0 \tag{3.4}$$

In fact, for each fixed $s \in [0, N\rho]$ and $h \in \mathcal{P}$ with $\varepsilon_0 + h \in P$,

$$\begin{aligned} & \|T(N\rho - s,\varepsilon_0 + h)F(s,\varepsilon_0 + h) - T(N\rho - s,\varepsilon_0)F(s,\varepsilon_0)\| \\ \leq \quad & \|T(N\rho - s,\varepsilon_0 + h)\| \cdot \|F(s,\varepsilon_0 + h) - F(s,\varepsilon_0)\| \\ & +\|(T(N\rho - s,\varepsilon_0 + h) - T(N\rho - s,\varepsilon_0))F(s,\varepsilon_0)\| \\ \rightarrow \quad & 0 \quad \text{as} \quad |h| \rightarrow 0, \quad (\text{by (3.2) and continuity of } F(t,\varepsilon)). \end{aligned}$$

Also, by continuity of $F(t,\varepsilon)$, there is $B(\varepsilon_0, \delta(\varepsilon_0))$ $(\delta(\varepsilon_0) > 0)$ such that

$$\|F(s,\varepsilon)\| \leq N(\varepsilon_0) \qquad \text{for some } N(\varepsilon_0) > 0 \text{ and } (s,\varepsilon) \in [0,\rho] \times \overline{B(\varepsilon_0, \delta(\varepsilon_0))}.$$

Thus

$$\|T(\rho - s,\varepsilon_0 + h)f(s,\varepsilon_0 + h) - T(\rho - s,\varepsilon_0)f(s,\varepsilon_0)\| \leq 4M(t_0) \cdot N(\varepsilon_0)$$

and (3.4) follows from the Dominated Convergence Theorem. Above all, $K^N(\varepsilon)z$ is continuous at $\varepsilon = \varepsilon_0$. Since ε_0 is arbitrarily chosen, $K^N(\varepsilon)z$ is continuous in $\varepsilon \in P$.

Taking $\mathcal{F} = X, y = \varepsilon$, and $T_y = K^N(\varepsilon)$, we have that Theorem 3.1 implies that there exists a unique fixed point $z_0(\varepsilon)$ of $K^N(\varepsilon)$ which is continuous in ε. Thus, $z_0(\varepsilon)$ is the unique fixed point of $K(\varepsilon)$ and it is continuous in ε.

Finally, using the same argument as above, we have that the unique ρ-periodic weak solution of (3.10)

$$z(t,\varepsilon) = T(t,\varepsilon)z_0(\varepsilon) + \int_0^t T(t - s,\varepsilon)F(s,\varepsilon)\,ds.$$

is continuous in ε. $\square$

Now we discuss equation (1.6).

Lemma 3.3. Assume that (3.2) and (3.3) are satisfied. Then $(I - T(N\rho,\varepsilon))^{-1}z$ is continuous in ε for each $z \in X$.

Proof. First note that from (3.3), it is easy to show that $(I - T(\rho,\varepsilon))^{-1}$ exists. Also,

$$\|(I - T(N\rho,\varepsilon))^{-1}\| \leq \frac{1}{1-k} \doteq H.$$

Next consider the operator defined on X:

$$J(\varepsilon)z = T(N\rho,\varepsilon)z + y \quad \text{where } y \text{ is a given point in } X.$$

Then we have

$$\|J(\varepsilon)z_1 - J(\varepsilon)z_2\| \leq \|T(N\rho,\varepsilon)\| \cdot \|z_1 - z_2\| \leq k\|z_1 - z_2\|.$$

Thus, $J(\varepsilon)$ is a uniform contraction. Also it is obvious that $J(\varepsilon)z$ is continuous in ε by (3.2). From Theorem 3.1 it follows that there is a unique fixed point of $J(\varepsilon)$, say $z(\varepsilon)$, such that $z(\varepsilon) = T(N\rho,\varepsilon)z(\varepsilon) + y$ and $z(\varepsilon)$ is continuous in ε.

Furthermore, $z(\varepsilon) = (I - T(N\rho,\varepsilon))^{-1}y$ and $(I - T(N\rho,\varepsilon))^{-1}y$ is continuous in ε. □

Let $PC[R,\rho] = \{g \in C(R,X) \,|\, g(t+\rho) = g(t), t \in R\}$ together with the sup norm, $\|\cdot\|_\infty$. Consider the equation

$$\begin{aligned} z'(t) &= A(\varepsilon)z(t) + f(t,g(t),\varepsilon) \\ z(0) &= z_0 \end{aligned} \tag{3.5}$$

on a Banach space $(X, \|\cdot\|)$, where $f(t+\rho,g,\varepsilon) = F(t,g,\varepsilon)$ for some $\rho > 0$ and $g \in PC[R,\rho]$, and $f(t,g,\varepsilon)$ is continuous in $(t,g,\varepsilon) \in R \times PC[R,\rho] \times P$.

Lemma 3.4. Assume that (3.2) and (3.3) are satisfied.

Then there exists a unique ρ-periodic solution of (3.5), say $z(t,g,\varepsilon)$, which is continuous with respect to ε for $\varepsilon \in P$. Also

$$z(0,g,\varepsilon) = (I - T(N\rho,\varepsilon))^{-1} \int_0^{N\rho} T(\rho - s,\varepsilon)F(s,g(s),\varepsilon)\,ds. \tag{3.6}$$

Proof. Let $F_2(t,\varepsilon) = F(t,g(t),\varepsilon)$. Then $F_2(t+\rho,\varepsilon) = F_2(t,\varepsilon)$. Also it is obvious that $F_2(t,\varepsilon)$ is continuous in (t,ε) because $F(t,z,\varepsilon)$ is continuous in (t,z,ε). Therefore, by Theorem 3.2, there is a unique ρ-periodic weak solution $z(t,\varepsilon,g)$ of (3.5) which is continuous in ε. In particular, $z(0,\varepsilon,g)$ is continuous in ε. Moreover, using the same argument as that in the proof of Theorem 3.2 we see that

$$z(0,\varepsilon,g) = T(N\rho,\varepsilon)z(0,\varepsilon,g) + \int_0^{N\rho} T(N\rho - s,\varepsilon)F(s,g(s),\varepsilon)\,ds.$$

Thus, (3.6) holds. □

Define $J_1(\varepsilon) : PC[R,\rho] \to PC[R,\rho]$ by

$$J_1(\varepsilon)g(t) = T(t,\varepsilon)z(0,\varepsilon,g) + \int_0^t T(t-s,\varepsilon)F(s,g(s),\varepsilon)\,ds.$$

Lemma 3.5. Assume that (3.2)and (3.3) are satisfied . In addition, assume that

$$\|F(t,z_1,\varepsilon) - F(t,z_2,\varepsilon)\| \le L(\varepsilon^1)\|z_1 - z_2\|, \tag{3.7}$$

where $L(\varepsilon^1)$ is continuous in $\varepsilon^1 \in P$ in a neighborhood of $\varepsilon = 0$ and $L(0) = 0$.

Then for small $|\varepsilon^1|$, the operator $J_1(\varepsilon)$ has a unique fixed point $g(\cdot,\varepsilon) \in PC[R,\rho]$ which is continuous in ε.

Proof. First note that it is clear that $(PC[R,\rho], \|\cdot\|_\infty)$ is a Banach space. Next since $L(0) = 0$, then, by continuity of $L(\varepsilon^1)$, there is δ_0 such that $|\varepsilon^1| < \delta_0$ implies

$$L(\varepsilon^1) \le \frac{1}{4}\min\{\frac{1}{HM^2(t_0)t_0}, \frac{1}{M(t_0)t_0}\}.$$

Now for $\varepsilon \in P$ with $|\varepsilon^1| < \delta_0$,

$$\begin{aligned}
& \|T(t,\varepsilon)\| \cdot \|z(0,g_1,\varepsilon) - z(0,g_2,\varepsilon)\| \\
= \quad & \|T(t,\varepsilon)\| \cdot \|(I - T(N\rho,\varepsilon))^{-1} \int_0^{N\rho} T(N\rho - s,\varepsilon)[F(s,g_1,\varepsilon) - F(s,g_2,\varepsilon)]\,ds\| \\
\le \quad & M(t_0)\|(I - T(N\rho,\varepsilon))^{-1}\| \int_0^{N\rho} \|T(N\rho - s,\varepsilon)\| \cdot \|F(s,g_1,\varepsilon) - F(s,g_2,\varepsilon)\|\,ds \\
\le \quad & M(t_0) \cdot H \cdot M(t_0) \int_0^{N\rho} \|F(s,g_1,\varepsilon) - F(s,g_2,\varepsilon)\|\,ds \\
\le \quad & H \cdot M^2(t_0) \cdot t_0 \cdot L(\varepsilon^1)\|g_1 - g_2\| \le \frac{1}{4}\|g_1 - g_2\|,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^t \|T(t-s,\varepsilon)\| \cdot \|F(s,g_1(s),\varepsilon) - F(s,g_2(s),\varepsilon)\|\,ds \\
\le \quad & M(t_0)L(\varepsilon^1) \int_0^t \|g_1(s) - g_2(s)\|\,ds \\
\le \quad & M(t_0) \cdot t_0 L(\varepsilon^1)\|g_1 - g_2\| \le \frac{1}{4}\|g_1 - g_2\|.
\end{aligned}$$

Hence

$$\begin{aligned}
& \|J_1(\varepsilon)g_1 - J_1(\varepsilon)g_2\| \\
\le \quad & \|T(t,\varepsilon)\| \cdot \|z(0,g_1,\varepsilon) - z(0,g_2,\varepsilon)\| + \int_0^t \|T(t-s,\varepsilon)\| \cdot \|F(s,g_1(s),\varepsilon) - F(s,g_2(s),\varepsilon)\|\,ds \\
\le \quad & \frac{1}{4}\|g_1 - g_2\| + \frac{1}{4}\|g_1 - g_2\| \\
\le \quad & \frac{1}{2}\|g_1 - g_2\|.
\end{aligned}$$

Therefore $J_1(\varepsilon)$ is a uniform contraction.

Furthermore, $J_1(\varepsilon)g$ is continuous in ε for fixed g. Therefore from Theorem 3.1, it follows that $J_1(\varepsilon)$ has a unique fixed point, say $g(\cdot, \varepsilon) \in PC[R, \rho]$, which is continuous in ε. $\square$

Theorem 3.6 Assume that (3.2), (3.3) and (3.7) are satisfied.

Then for small $|\varepsilon^1|$, there exists a unique ρ-periodic weak solution of (1.6), say $z(t, \varepsilon)$, which is continuous in ε for $\varepsilon \in P$.

Proof. This is an immediate result from Lemma 3.4 and Lemma 3.5.

4 Application to a Nonlinear Wave Equation

Consider the nonlinear wave equation mentioned in Section 1

$$\begin{aligned}
&u_{tt} = u_{xx} + \eta F(u_t), && \text{for } t \geq 0,\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) && \text{for } x \in [0,1],\\
&\mu u_t(0,t) - \gamma u_x(0,t) = f_1(t), &&\\
&\delta u_t(1,t) + \gamma u_x(1,t) = f_2(t), && \mu, \gamma, \delta > 0,
\end{aligned} \tag{4.1}$$

where $f_1(t)$ and $f_2(t)$ are both ρ-periodic and continuously differentiable. F satisfies a uniform Lipschitz condition.

As shown in Section 1, the associated abstract equation of (4.1) is given by:

$$\begin{aligned}
&\frac{dz(t)}{dt} = A(\varepsilon)z(t) + F_1(t, z(t), \varepsilon),\\
&z(0) = z_0
\end{aligned} \tag{4.2}$$

on $X = L^2[0,1] \times L^2[0,1]$, where

$$A(\varepsilon) = \begin{bmatrix} 0 & \partial x \\ \partial x & 0 \end{bmatrix}, \qquad \varepsilon = (\alpha, \beta, \delta, \eta) \in R_+^4, \qquad z = \begin{bmatrix} v \\ w \end{bmatrix},$$

$$D(A(\varepsilon)) = \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \prod_{i=1}^{2} H^1[0,1] \;\middle|\; \begin{matrix} \alpha v(0) = w(0), \\ v(1) = -\beta w(1), \end{matrix} \quad \alpha, \beta > 0 \right\}.$$

$$F_1(t, z, \varepsilon)x = \begin{bmatrix} \eta F(v + x\delta^{-1} f_2(t)) + \gamma^{-1} f_1(t) - x\delta^{-1} f_2'(t) \\ \delta^{-1} f_2(t) + (1-x)\gamma^{-1} f_1'(t) \end{bmatrix}$$

Now make the change of variables

$$\tilde{z} = \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} = U \begin{bmatrix} v \\ w \end{bmatrix} \qquad \text{where } U = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Then,

$$A_1(\varepsilon) \equiv UA(\varepsilon)U^{-1} = \begin{bmatrix} \partial x & 0 \\ 0 & -\partial x \end{bmatrix}.$$

where

$$D(A_1(\varepsilon)) = \left\{ \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} \in \prod_{i=1}^{2} H^1[0,1] \;\middle|\; \begin{array}{c} (1-\alpha)\tilde{v}(0) = (1+\alpha)\tilde{w}(0), \\ (1+\beta)\tilde{v}(1) = -(1-\beta)\tilde{w}(1). \end{array} \right\}$$

Define

$$\tilde{\alpha} = \frac{1-\alpha}{1+\alpha}, \qquad \tilde{\beta} = \frac{1-\beta}{1+\beta} \qquad \tilde{\alpha}, \tilde{\beta} \in (-1,1].$$

$A_1(\varepsilon)$ has the domain

$$D(A_1(\varepsilon)) = \left\{ \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} \in \prod_{i=1}^{2} H^1[0,1] \;\middle|\; \begin{array}{c} \tilde{\alpha}\tilde{v}(0) = \tilde{w}(0), \\ \tilde{v}(1) = -\tilde{\beta}\tilde{w}(1), \end{array} \; \tilde{\alpha},\, \tilde{\beta} \in (-1,1] \right\}.$$

If $|\tilde{\alpha}\tilde{\beta}| < 1$, then $A_1(\varepsilon)$ generates a C_0-semigroup, $T_1(t,\varepsilon)$, on $X = L^2[0,1] \times L^2[0,1]$, endowed with the norm $\|(f,g)\| \equiv \|f\|_2 + \|g\|_2$. It can easily be shown using the method of characteristics that this semigroup satisfies $\|T_1(t,\varepsilon)\| = 1$ for $0 \le t < 1$, $\|T_1(t,\varepsilon)\| = \max\{|\tilde{\alpha}|, |\tilde{\beta}|\} \le 1$ for $1 \le t < 2$, and $\|T_1(t,\varepsilon)\| \le |\tilde{\alpha}\tilde{\beta}| < 1$ for $t \ge 2$. Thus the semigroup is eventually contracting.

It follows from $T(t,\varepsilon) = UT_1(t,\varepsilon)U^{-1}$ that $T(t,\varepsilon)$ must have the same properties with respect to the norm

$$|||z||| \equiv \|Uz\|$$

on $X = L^2[0,1] \times L^2[0,1]$ when $\tilde{\alpha}, \tilde{\beta} > 0$.

Now consider the operator $A_1(\varepsilon)$. Take $\varepsilon_0 = (0,0)$. For $\varepsilon = (\tilde{\alpha},\tilde{\beta})$ with $\tilde{\alpha},\tilde{\beta} \in (-1,1]$

$$A_1(\varepsilon) = \frac{1}{1+\tilde{\alpha}\tilde{\beta}} Q(\varepsilon) A_1(0) Q(\varepsilon)$$

where

$$Q(\varepsilon) = \begin{bmatrix} 1 & \tilde{\beta} \\ -\tilde{\alpha} & 1 \end{bmatrix}.$$

obviously, $Q(\varepsilon)$ is continuous in ε and is bounded. So is $Q^{-1}(\varepsilon)$. Let $Q_2(\varepsilon) = \frac{1}{1+\tilde{\alpha}\tilde{\beta}} Q(\varepsilon)$ and $Q_1(\varepsilon) = Q(\varepsilon)$. It is clear that the hypotheses of Theorem 2.3 are satisfied for any $\varepsilon_0 = (\tilde{\alpha}_0, \tilde{\beta}_0)$ in $R^+ \times R^+$. That is, we have strong continuity in parameter ε of semigroup $T_1(t,\varepsilon)$ and thus strong continuity of semigroup $T(t,\varepsilon)$.

Above all, the associated semigroup is eventually contracting and is continuous in ε. Thus, it follows from Theorem 3.6 that when $|\eta|$ is small there is a unique ρ-periodic weak solution of (4.2) and thus also a unique ρ-periodic weak solution of (4.1) and it is L^2 continuous in $\varepsilon = (\alpha, \delta, \gamma, \eta)$.

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