CUBO A Mathematical Journal
Vol.11, $N^{\underline{o}}$03, (79–100). August 2009

Positive Solutions for Systems of Three-point Nonlinear Boundary Value Problems with Deviating Arguments

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ABSTRACT

Existence of eigenvalues yielding positive solutions for a system of two second order delay differential equations along with boundary conditons is established. The results are obtained by the use of a Guo-Krasnoselskii fixed point theorem in cones.

RESUMEN

Es establecida la existencia de autovalores produciendo soluciones positivas para un sistema de dos ecuaciones diferenciales de segundo orden con retardo, con condiciones de frontera. Los resultados son obtenidos mediante el uso del Teorema de punto fijo de Guo-Krasnoselskii en conos.

Key words and phrases: *Three-point boundary value problem, system of differential equations, eigenvalue problem, positive solutions, deviating arguments.*

Math. Subj. Class.: *34B18, 34A34.*

1 Introduction

Consider the three-point boundary value problem system consisting of the second order delay differential equations,

$$\begin{aligned} u''(t) + \lambda a(t) f\left(u(\sigma_1(t)), v(\sigma_2(t))\right) = 0, \quad 0 < t < 1, \\ v''(t) + \mu b(t) g\left(u(\tau_1(t)), v(\tau_2(t))\right) = 0, \quad 0 < t < 1, \end{aligned} \tag{1}$$

along with the conditions,

$$\begin{aligned} &u(0) = 0, \qquad && u(1) = \alpha u(\eta), \\ &v(0) = 0, \qquad && v(1) = \alpha v(\eta), \\ &u(t) = \phi_1(t), && v(t) = \phi_2(t), \quad -r \le t \le 0, \end{aligned} \tag{2}$$

where $0 < \eta < 1$, $0 < \alpha < 1/\eta$, $-r = \min_{t\in[0,1]} \sigma_i(t) = \min_{t\in[0,1]} \tau_i(t), i = 1, 2$, and $\phi_1, \phi_2 : [-r, 0] \to \mathbb{R}^+$ are continuous functions, with $\phi_1(0) = \phi_2(0) = 0$. Our interest in this paper is to investigate the existence of eigenvalues λ and μ that yield positive solutions to the associated boundary value problem, (1), (2).

We assume that

(A) $f, g \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$;

(B) $a, b \in C([0, 1], \mathbb{R}^+)$, and each does not vanish identically on any subinterval;

(C) $\sigma_i, \tau_i : [0, 1] \to [-r, 1], i = 1, 2$ are continuous functions;

(D) All of

$$f_0 := \lim_{u+v\to 0^+} \frac{f(u, v)}{u + v}, \quad g_0 := \lim_{u+v\to 0^+} \frac{g(u, v)}{u + v}$$

$$f_\infty := \lim_{u+v\to \infty} \frac{f(u, v)}{u + v}, \quad g_\infty := \lim_{u+v\to \infty} \frac{g(u, v)}{u + v}$$

exist as positive real numbers;

(E) There exist an $\eta^* \in [\eta, 1]$ such that $\sigma_i(s), \tau_i(s) \in [\eta, 1]$ for all $s \in [\eta^*, 1], i = 1, 2$.

We say that a pair $(u, v) \in C([-r, 1])$ is a *solution* of the boundary value problem (BVP for short) (1), (2) if, u and v are twice continuously differentiable on $(0, 1)$, $u(t) = \phi_1(t), v(t) = \phi_2(t)$, for $-r \leq t \leq 0$, (u, v) satisfies (1) for all $t \in (0, 1)$, and $u(0) = 0, u(1) = \alpha u(\eta)$ and $v(0) = 0, v(1) = \alpha v(\eta)$.

For several years now, there has been a great deal of activity in studying positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from both a theoretical sense [4, 7, 10, 13, 20] and as applications for which only positive solutions are meaningful [1, 5, 14, 15]. These considerations are caste primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [11, 12, 17, 19, 21]. The existence of positive solutions for nonlocal three-point boundary value problems has been studied extensively in recent years. For some appropriate references we refer the reader to [17], [18].

Recently, Benchohra *et al.* [2] and Henderson and Ntouyas [8] studied the existence of positive solutions for systems of nonlinear eigenvalue problems, while Henderson and Ntouyas [9] obtained results for the case of systems with three-point nonlocal boundary conditions. The purpose of this paper is to extend the results given in [9] to the case where delays may appear in the equations of the system (1), (2).

The main tool in this paper is an application of the Guo-Krasnosel'skii fixed point theorem for operators leaving a Banach space cone invariant [7]. A Green's function plays a fundamental role in defining an appropriate operator on a suitable cone. Since, in our problem, we cannot express system (1), (2) as a single operator equation, the method used for example in [9] is not applicable here. This difficulty can be overcome by employing a method proposed by Dunninger and Wang in [3].

2 Some preliminaries

Before we state and prove our main result, we recall some useful facts that will be used in the sequel.

Concerning the boundary value problem

$$u''(t) + y(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta), \tag{4}$$

we have the following two lemmas.

Lemma 2.1. *[6] Let* (A), (B) *and* (C) *hold and assume that* $0 < \eta < 1$ *and* $0 < \alpha < 1/\eta$*. Then, for any* $y \in C[0, 1]$ *the* BVP (3), (4) *has a unique solution,*

$$u(t) = \int_0^1 k(t, s)y(s)ds, \quad t \in [0, 1],$$

where $k(t,s):[0,1]\times[0,1]\to\mathbb{R}^+$ *is the Green function defined by*

$$k(t,s)=\begin{cases}\dfrac{t(1-s)}{1-\alpha\eta}-\dfrac{\alpha t(\eta-s)}{1-\alpha\eta}-(t-s), & 0\le s\le t\le 1 \quad and \quad s\le\eta,\\ \dfrac{t(1-s)}{1-\alpha\eta}-\dfrac{\alpha t(\eta-s)}{1-\alpha\eta}, & 0\le t\le s\le\eta,\\ \dfrac{t(1-s)}{1-\alpha\eta}, & 0\le t\le s\le 1 \quad and \quad \eta\le s,\\ \dfrac{t(1-s)}{1-\alpha\eta}-(t-s), & \eta\le s\le t\le 1.\end{cases} \tag{5}$$

Lemma 2.2. *[16] Let* (A), (B) *and* (C) *hold and assume that* $0<\alpha<1/\eta$. *Then, the unique solution of the problem* (3), (4) *satisfies*

$$\inf_{t\in[\eta,1]} u(t)\ge\gamma\|u\|,$$

where $\gamma:=\min\left\{\alpha\eta,\eta,\dfrac{\alpha(1-\eta)}{1-\alpha\eta}\right\}$.

From Lemma 2.1 and the analytical expression of k, it follows that u can be written as

$$u(t)=\frac{1}{1-\alpha\eta}\int_0^1(1-s)y(s)ds-\frac{\alpha t}{1-\alpha\eta}\int_0^\eta(\eta-s)y(s)ds-\int_0^t(1-s)y(s)ds$$

from which it follows that

$$u(t)\le\frac{1}{1-\alpha\eta}\int_0^1(1-s)y(s)ds,\quad \text{for all } t\in[0,1], \tag{6}$$

and

$$u(\eta)\ge\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)y(s)ds. \tag{7}$$

We note that a pair $(u(t),v(t))$ is a solution of the eigenvalue problem (1), (2) if, and only if, $u(t)=\phi_1(t), v(t)=\phi_2(t)$ for $-r\le t\le 0$, and

$$u(t)=\lambda\int_0^1 k(t,s)a(s)f(u(\sigma_1(s)),v(\sigma_2(s)))ds,\quad 0\le t\le 1,$$

$$v(t)=\mu\int_0^1 k(t,s)b(s)g(u(\tau_1(s)),v(\tau_2(s)))ds,\quad 0\le t\le 1.$$

The main tool in determining values of the parameters λ and μ, for which positive (with respect to a cone) solutions of the BVP (1), (2) exist, is the following fixed point theorem.

Theorem 2.1. *[7] Let* $\mathcal{B}$ *be a Banach space, and let* $\mathcal{P}\subset\mathcal{B}$ *be a cone in* $\mathcal{B}$. *Assume* Ω_1 *and* Ω_2 *are open subsets of* $\mathcal{B}$ *with* $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, *and let*

$$T:\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)\to\mathcal{P}$$

be a completely continuous operator such that, either

(i) $||Tu|| \leq ||u||, u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $||Tu|| \geq ||u||, u \in \mathcal{P} \cap \partial\Omega_2,$ *or*

(ii) $||Tu|| \geq ||u||, u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $||Tu|| \leq ||u||, u \in \mathcal{P} \cap \partial\Omega_2.$

Then T *has a fixed point in* $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Positive solutions in a cone

In this section, we apply Theorem 2.1 to obtain solutions in a cone (that is, positive solutions) of (1), (2). For our construction, we let

$$X = C([-r, 1], \mathbb{R}^+) \times C([-r, 1], \mathbb{R}^+)$$

with norm

$$\|(u, v)\| = \|u\| + \|v\|$$

where $\|u\| = \sup_{t\in[-r,1]} |u(t)|$. Then $(X, \|\cdot\|)$ is a Banach space. We will make use of the cone $\mathcal{P} \subset X$ defined by

$$\mathcal{P} = \left\{(u, v) : (u, v) \in X : u, v \geq 0 \text{ on } [-r, 1], \ \min_{t\in[\eta,1]} [u(t) + v(t)] \geq \gamma[\|u\| + \|v\|]\right\},$$

where $\gamma > 0$ is the positive constant defined in Lemma 2.2.

For our first result, define positive numbers L_1 and L_2 by

$$L_1 := \max\left\{\frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^{1}(1-r)a(r)f_\infty dr\right]^{-1}, \ \frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^{1}(1-r)b(r)g_\infty dr\right]^{-1}\right\},$$

and

$$L_2 := \min\left\{\frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_{0}^{1}(1-r)a(r)f_0 dr\right]^{-1}, \ \frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_{0}^{1}(1-r)b(r)g_0 dr\right]^{-1}\right\}.$$

Theorem 3.1. *Assume that conditions* (A), (B), (C), (D) *and* (E) *hold. Then, for each* λ, μ *satisfying*

$$L_1 < \lambda, \mu < L_2, \tag{8}$$

there exists a pair (u, v) *satisfying* (1), (2) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0, 1)$.

Proof. Let $A, B : X \to X$ and $F : X \to X$ be the integral operators defined by

$$A(u, v)(t) = \begin{cases} \phi_1(t), & -r \leq t \leq 0, \\ \lambda \displaystyle\int_0^1 k(t, s)a(s)f\left(u(\sigma_1(s)), v(\sigma_2(s))\right) ds, & 0 \leq t \leq 1, \end{cases}$$

$$B(u,v)(t) = \begin{cases} \phi_2(t), & -r \le t \le 0, \\ \mu \displaystyle\int_0^1 k(t,s)b(s)g\left(u(\tau_1(s)), v(\tau_2(s))\right) ds, & 0 \le t \le 1, \end{cases}$$

$$F(u,v)(t) = \left(A(u,v)(t), B(u,v)(t)\right), \quad t \in [-r,1].$$

Then seeking solutions to our BVP (1), (2) is equivalent to looking for fixed points of the equation

$$F(u,v) = (u,v)$$

in the Banach space X.

Choose some $(u,v) \in \mathcal{P}$. Then by Lemma 2.2 we have

$$\inf_{t\in[\eta,1]} A(u,v)(t) \ge \gamma \|A(u,v)\|, \quad \inf_{t\in[\eta,1]} B(u,v)(t) \ge \gamma \|B(u,v)\|$$

and thus

$$\begin{aligned} \inf_{t\in[\eta,1]}[A(u,v)(t) + B(u,v)(t)] &\ge \inf_{t\in[\eta,1]} A(u,v)(t) + \inf_{t\in[\eta,1]} B(u,v)(t) \\ &\ge \gamma\left[\|A(u,v)\| + \|B(u,v)\|\right] \\ &= \gamma\|(A(u,v), B(u,v)\| \end{aligned}$$

which implies that $F(\mathcal{P}) \subset \mathcal{P}$ for every $(u,v) \in \mathcal{P}$.

As A and B are integral operators, it is not difficult to see that using standard arguments we may conclude that both A and B are completely continuous; hence F is a completely continuous operator.

Let λ and μ be as in (8), and choose an $\epsilon > 0$ such that

$$\max\left\{ \frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_\eta^1 (1-r)a(r)(f_\infty - \epsilon)dr\right]^{-1}, \right.$$
$$\left. \frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_\eta^1 (1-r)b(r)(g_\infty - \epsilon)dr\right]^{-1} \right\} \le \lambda, \mu$$

and

$$\begin{aligned} \lambda, \mu \quad \le \quad \min\left\{ \frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_0^1 (1-r)a(r)(f_0 + \epsilon)dr\right]^{-1}, \right. \\ \left. \frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_0^1 (1-r)b(r)(g_0 + \epsilon)dr\right]^{-1} \right\}. \end{aligned}$$

From the definition of f_0 and g_0, there exists an $H_1 > 0$ such that

$$f(u,v) \le (f_0 + \epsilon)(u+v) \quad \text{for} \quad u, v \in \mathcal{P} \text{ with } 0 < u, v < H_1,$$

and

$$g(u,v) \le (g_0+\epsilon)(u+v) \quad \text{for} \quad u,v \in \mathcal{P} \text{ with } 0 < u,v < H_1.$$

Set

$$\Omega_1 = \{(u,v) \in X : \|(u,v)\| < H_1\}.$$

Now let $(u,v) \in \mathcal{P} \cap \partial\Omega_1$, i.e., let $(u,v) \in \mathcal{P}$ with $\| (u,v) \| = H_1$.

Then, in view of the inequality (6) we have

$$\begin{aligned} A(u,v)(t) &\le \lambda \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s) f\left(u(\sigma_1(s), v(\sigma_2(s))\right) ds \\ &\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s)(f_0+\epsilon)\left[u(\sigma_1(s)+v(\sigma_2(s))\right] ds \\ &\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s)(f_0+\epsilon)[\|u\|+\|v\|] ds \\ &\le \frac{1}{2}[\|u\|+\|v\|] \\ &= \frac{1}{2}\|(u,v)\|, \end{aligned}$$

and so,

$$\|A(u,v)\| \le \frac{1}{2}\|(u,v)\|.$$

Similarily, we may take

$$\|B(u,v)\| \le \frac{1}{2}\|(u,v)\|.$$

Thus, for $(u,v) \in \mathcal{P} \cap \partial\Omega_1$ it follows that

$$\begin{aligned} \|F(u,v)\| &= \| \left(A(u,v), B(u,v)\right) \| = \|A(u,v)\| + \|B(u,v)\| \\ &\le \frac{1}{2}\|(u,v)\| + \frac{1}{2}\|(u,v)\| = \|(u,v)\|, \end{aligned}$$

that is,

$$\|F(u,v)\| \le \|(u,v)\| \quad \text{for all} \quad (u,v) \in \mathcal{P} \cap \partial\Omega_1.$$

Due to the definition of f_∞ and g_∞, there exists an $\overline{H}_2 > 0$ such that

$$f(u,v) \ge (f_\infty - \epsilon)(u+v) \quad \text{for all} \quad u,v \ge \overline{H}_2,$$

and

$$g(u,v) \ge (g_\infty - \epsilon)(u+v) \quad \text{for all} \quad u,v \ge \overline{H}_2.$$

Set

$$H_2 = \max\left\{2H_1, \frac{\overline{H}_2}{\gamma}\right\}$$

and define

$$\Omega_2 = \{(u,v) \in X : \|(u,v)\| < H_2\}.$$

As from our hypothesis on η^* it follows that

$$\inf_{t\in[\eta^*,1]}[u(\sigma_1(t))+v(\sigma_2(t))]\geq\gamma[\|u\|+\|v\|]. \tag{9}$$

By the use of (7), we have for $(u,v)\in\mathcal{P}\cap\partial\Omega_2$,

$$\begin{aligned}
A(u,v)(\eta) &\geq \lambda\frac{\eta}{1-\alpha\eta}\int_{\eta}^{1}(1-s)a(s)f\left(u(\sigma_1(s)),v(\sigma_2(s))\right)ds\\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_{\eta^*}^{1}(1-s)a(s)(f_\infty-\epsilon)\left(u(\sigma_1(s))+v(\sigma_2(s))\right)ds\\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_{\eta^*}^{1}(1-s)a(s)(f_\infty-\epsilon)\gamma[\|u\|+\|v\|]ds\\
&\geq \frac{1}{2}\|(u,v)\|,
\end{aligned}$$

that is,

$$A(u,v)(t)\geq\frac{1}{2}\|(u,v)\|\quad\text{for all } t\geq\eta$$

and so,

$$A(u,v)(t)\geq\frac{1}{2}\|(u,v)\|.$$

Similarily, we may take

$$B(u,v)(t)\geq\frac{1}{2}\|(u,v)\|.$$

Thus, for $(u,v)\in\mathcal{P}\cap\partial\Omega_2$ it follows that

$$\begin{aligned}
\|F(u,v)\| &= \|\left(A(u,v),B(u,v)\right)\|=\|A(u,v)\|+\|B(u,v)\|\\
&\geq \frac{1}{2}\|(u,v)\|+\frac{1}{2}\|(u,v)\|=\|(u,v)\|,
\end{aligned}$$

that is

$$\|F(u,v)\|\geq\|(u,v)\|\ \text{ for all }\ (u,v)\in\mathcal{P}\cap\partial\Omega_2.$$

Applying Theorem 2.1, we obtain that F has a fixed point $(u,v)\in\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$ such that $H_1\leq\|(u,v)\|\leq H_2$, and so (1), (2) has a positive solution. The proof is complete. $\square$

For our next result we define the positive numbers

$$L_3=\max\left\{\frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^{1}(1-s)a(s)f_0ds\right]^{-1},\frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^{1}(1-s)b(s)g_0ds\right]^{-1}\right\}$$

and

$$L_4=\min\left\{\frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_{0}^{1}(1-s)a(s)f_\infty dr\right]^{-1},\frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_{0}^{1}(1-s)b(s)g_\infty dr\right]^{-1}\right\}.$$

We are now ready to state and prove our main result.

Theorem 3.2. *Assume that conditions* (A), (B), (C), (D) *and* (E) *hold. Then for each* λ, μ *satisfying*

$$L_3 < \lambda, \mu < L_4, \tag{10}$$

there exists a pair (u, v) *satisfying* (1), (2) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0, 1)$.

Proof. Let λ and μ be as in (10) and choose a sufficiently small $\epsilon > 0$ so that

$$\begin{aligned} \lambda, \mu \quad \le \quad \min \Bigg\{ & \frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)(f_\infty + \epsilon)ds\right]^{-1}, \\ & \frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_0^1 (1-s)b(s)(g_\infty + \epsilon)ds\right]^{-1} \Bigg\} \end{aligned}$$

and

$$\begin{aligned} \max \Bigg\{ & \frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^1 (1-s)a(s)(f_0 - \epsilon)dr\right]^{-1}, \\ & \frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^1 (1-s)b(s)(g_0 - \epsilon)dr\right]^{-1} \Bigg\} \le \lambda, \mu. \end{aligned}$$

By the definition of f_0 and g_0, there exists an $H_1 > 0$ such that

$$f(u, v) \ge (f_0 - \epsilon)(u + v) \quad \text{for all } u, v \text{ with } 0 < u, v \le H_3,$$

and

$$g(u, v) \ge (g_0 - \epsilon)(u + v) \quad \text{for all } u, v \text{ with } 0 < u, v \le H_3.$$

Set

$$\Omega_1 = \{(x, y) \in X : \|(x, y)\| < H_3\}$$

and let $(u, v) \in \mathcal{P} \cap \partial\Omega_3$.

In view of (9) and by the use of (7), we find

$$\begin{aligned} A(u, v)(\eta) \quad & \ge \quad \lambda \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)f\left(u(\sigma_1(s)), v(\sigma_2(s))\right)ds \\ & \ge \quad \lambda \frac{\eta}{1-\alpha\eta}\int_{\eta^*}^1 (1-s)a(s)f\left(u(\sigma_1(s)), v(\sigma_2(s))\right)ds \\ & \ge \quad \lambda \frac{\eta}{1-\alpha\eta}\int_{\eta^*}^1 (1-s)a(s)(f_0 - \epsilon)\left(u(\sigma_1(s)) + v(\sigma_2(s))\right)ds \\ & \ge \quad \lambda \frac{\eta}{1-\alpha\eta}\int_{\eta^*}^1 (1-s)a(s)(f_0 - \epsilon)\gamma[\|u\| + \|v\|]ds \\ & \ge \quad \frac{1}{2}\|(u, v)\|, \end{aligned}$$

that is,

$$\|A(u,v)\| \geq \frac{1}{2}\|(u,v)\|.$$

In a similar manner

$$\|B(u,v)\| \geq \frac{1}{2}\|(u,v)\|.$$

Thus, for an arbitrary $(u,v) \in \mathcal{P} \cap \partial\Omega_3$ it follows that

$$\begin{aligned}
\|F(u,v)\| &= \|(A(u,v), B(u,v))\| = \|A(u,v)\| + \|B(u,v)\| \\
&\geq \frac{1}{2}\|(u,v)\| + \frac{1}{2}\|(u,v)\| = \|(u,v)\|,
\end{aligned}$$

and so

$$\|F(u,v)\| \geq \|(u,v)\| \quad \text{for all } (u,v) \in \mathcal{P} \cap \partial\Omega_3.$$

Now let us define two functions $f^*, g^* : [0,\infty) \to [0,\infty)$ by

$$f^*(t) = \max_{0 \leq u+v \leq t} f(u,v) \quad \text{and} \quad g^*(t) = \max_{0 \leq u+v \leq t} g(u,v).$$

It follows that

$$f(u,v) \leq f^*(t) \text{ and } g(u,v) \leq g^*(t) \text{ for all } (u,v) \text{ with } 0 \leq u+v \leq t.$$

It is clear that the functions f^* and g^* are nondecreasing. Also, there is no difficulty to see that

$$\lim_{t\to\infty} \frac{f^*(t)}{t} = f_\infty \quad \text{and} \quad \lim_{t\to\infty} \frac{g^*(t)}{t} = g_\infty.$$

In view of the definitions of f_∞ and g_∞, there exists an $\overline{H}_4$ such that

$$f^*(t) < (f_\infty + \varepsilon)\, t \quad \text{for all } t \geq \overline{H}_4,$$

and

$$g^*(t) < (g_\infty + \varepsilon)\, t \quad \text{for all } t \geq \overline{H}_4.$$

Set

$$H_4 = \max\left\{2H_3, \frac{\overline{H}_4}{\gamma}\right\},$$

and

$$\Omega_4 = \{(u,v) : (u,v) \in \mathcal{P} \text{ and } \|(u,v)\| < H_4\}.$$

Let $(u,v) \in \mathcal{P} \cap \partial H_4$ and observe that, by the definition of f^*, it follows that for any $s \in [0,1]$, we have

$$f\left(u(\sigma_1(s)), v(\sigma_2(s))\right) \leq f^*\left(\|u\| + \|v\|\right) = f^*\left(\|(u,v)\|\right).$$

In view of the above observation and by the use of inequality (6) we obtain for $t \in [0,1]$

$$\begin{aligned} A(u,v)(t) &\leq \lambda \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s) f\left(u(\sigma_1(s)), v(\sigma_2(s))\right) ds \\ &\leq \lambda \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s) f^*\left(\|u\| + \|v\|\right) ds \\ &\leq \lambda \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s)(f_\infty + \varepsilon)\left(\|u\| + \|v\|\right) dr \\ &\leq \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s)(f_\infty + \varepsilon) dr \, \|(u,v)\| \\ &\leq \frac{1}{2} \|(u,v)\|, \end{aligned}$$

which implies

$$\|A(u,v)\| \leq \frac{1}{2} \|(u,v)\|.$$

In a similar manner, we take

$$\|B(u,v)\| \leq \frac{1}{2} \|(u,v)\|.$$

Thus, for $(u,v) \in \mathcal{P} \cap \partial\Omega_4$ it follows that

$$\begin{aligned} \|F(u,v)\| &= \|(A(u,v), B(u,v))\| = \|A(u,v)\| + \|B(u,v)\| \\ &\leq \frac{1}{2}\|(u,v)\| + \frac{1}{2}\|(u,v)\| = \|(u,v)\|, \end{aligned}$$

and so

$$\|F(u,v)\| \leq \|(u,v)\| \quad \text{for all} \quad (u,v) \in \mathcal{P} \cap \partial\Omega_4$$

Applying Theorem 2.1, we obtain that F has a fixed point $(u,v) \in \mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$ such that $H_3 \leq \|(u,v)\| \leq H_4$, and so (1), (2) has a positive solution. The proof is complete. □

4 A General Application

In this section we apply Theorems 3.1 and 3.2 to the case where each one of the functions f and g is the sum of two (nonlinear) functions of a single argument, i.e., we consider the three-point boundary value system

$$\begin{aligned} u''(t) + \lambda a(t)[\widetilde{f}_1(u(\sigma_1(t))) + \widetilde{f}_2(v(\sigma_2(t)))] = 0, \quad 0 < t < 1, \\ v''(t) + \mu b(t)[\widetilde{g}_1(u(\tau_1(t))) + \widetilde{g}_2(v(\tau_2(t)))] = 0, \quad 0 < t < 1, \end{aligned} \tag{11}$$

$$\begin{aligned} u(0) = 0, \quad u(1) = \alpha u(\eta), \\ v(0) = 0, \quad v(1) = \alpha v(\eta), \\ u(t) = \phi_1(t), v(t) = \phi_2(t), \quad -r \leq t \leq 0, \end{aligned} \tag{12}$$

where $0 < \eta < 1$, $0 < \alpha < 1/\eta$, r is a positive number, $\phi_1, \phi_2 : [-r, 0] \to \mathbb{R}^+$, with $\phi_1(0) = \phi_2(0) = 0$ and $\sigma_i, \tau_i : [0,1] \to [-r, 1], i = 1,2$ are continuous functions.

We assume that

(A1) $\widetilde{f}_i, \widetilde{g}_i \in C([0,\infty),[0,\infty)), i = 1,2;$

(B1) $a, b \in C([0,1],[0,\infty))$ and each function does not vanish on any subinterval of $[0,1]$;

(C1) All of

$$\widetilde{f}_0 := \lim_{t\to 0+} \frac{\widetilde{f}_i(t)}{t}, \quad \widetilde{f}_\infty := \lim_{t\to\infty} \frac{\widetilde{f}(t)}{t},\ i = 1,2,$$

$$\widetilde{g}_0 := \lim_{t\to 0+} \frac{\widetilde{g}_i(t)}{t}, \quad \widetilde{g}_\infty := \lim_{t\to\infty} \frac{\widetilde{g}_i(t)}{t},\ i = 1,2$$

exist as positive real numbers.

We say that a pair $(u,v) \in C([-r,1])$ is a solution of the BVP (11), (12) if

(i) $u(t) = \phi_1(t),\ v(t) = \phi_2(t), \quad \text{for } -r \le t \le 0,$

(ii) (u,v) satisfies (11) for all $t \in (0,1)$, and

(iii) $u(0) = v(0) = 0,\ u(1) = \alpha u(\eta)$ and $v(1) = \alpha v(\eta)$.

Before we state our existence results for the BVP (11), (12), we prove an elementary lemma.

Lemma 4.1. *Let $h_i : [0,\infty) \to [0,\infty), i = 1,2$ be continuous functions for which*

$$\lim_{t\to 0+} \frac{h_i(t)}{t} = k \in (0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} \frac{h_i(t)}{t} = m \in (0,\infty),\ i = 1,2.$$

Then for the function $\widehat{h} : [0,\infty)\times[0,\infty) \to [0,\infty)$ with $\widehat{h}(u,v) = h_1(u) + h_2(v)$, it holds that

$$\lim_{u+v\to 0^+} \frac{\widehat{h}(u,v)}{u+v} = k \quad \textit{and} \quad \lim_{u+v\to\infty} \frac{\widehat{h}(u,v)}{u+v} = m.$$

Proof. By $\lim_{t\to 0+} \frac{h_i(t)}{t} = k, i = 1,2$ for an arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$(k-\varepsilon)\,u \le h_1(u) \le (k+\varepsilon)\,u \quad \text{for all} \quad u \in (0,\delta),$$

$$(k-\varepsilon)\,v \le h_2(v) \le (k+\varepsilon)\,v \quad \text{for all} \quad v \in (0,\delta),$$

and so, for any (u,v) with $u, v \in \left(0, \frac{\delta}{2}\right)$, we have

$$\begin{aligned} k - \varepsilon = \frac{(k-\varepsilon)\,u + (k-\varepsilon)\,v}{u+v} \quad &\le \quad \frac{\widehat{h}(u,v)}{u+v} = \frac{h_1(u) + h_2(v)}{u+v} \\ &\le \quad \frac{(k+\varepsilon)\,u + (k+\varepsilon)\,v}{u+v} = k + \varepsilon, \end{aligned}$$

i.e., it holds that

$$\left|\frac{\widehat{h}(u,v)}{u+v}-k\right| \le \varepsilon \quad \text{for any} \quad u,v>0 \quad \text{with} \quad u+v<\delta$$

which implies that

$$\lim_{u+v\to 0+}\frac{\widehat{h}(u,v)}{u+v}=k.$$

Now let us assume that $\lim\limits_{t\to\infty}\dfrac{h_i(t)}{t}=m\in(0,\infty)\,, i=1,2$. It follows that, for an arbitrarily small $\varepsilon>0$, there exists an $M_0>0$ such that

$$(m-\varepsilon)\,u\le h_1(u)\le(m+\varepsilon)\,u, \quad \text{for all} \quad u>M_0,$$

$$(m-\varepsilon)\,v\le h_2(v)\le(m+\varepsilon)\,v, \quad \text{for all} \quad v>M_o.$$

Let $u,v\ge 0$ with $u+v>2M_0$. Then either $u>M_0$ and $v>M_0$ or one of u,v is greater than M_0 while the other is less than M_0.

If $u>M_0$ and $v>M_0$, then by the last two inequalities we have

$$\begin{aligned} m-\varepsilon=\frac{(m-\varepsilon)\,u+(m-\varepsilon)\,v}{u+v} &\le \frac{\widehat{h}(u,v)}{u+v}=\frac{h_1(u)+h_2(v)}{u+v} \\ &\le \frac{(m+\varepsilon)\,u+(m+\varepsilon)\,v}{u+v}=m+\varepsilon, \end{aligned}$$

which implies that

$$\left|\frac{\widehat{h}(u,v)}{u+v}-m\right| \le \varepsilon \quad \text{for any} \quad u,v\ge 0 \quad \text{with} \quad u>M_0 \text{ and } v>M_0. \tag{13}$$

Now let us deal with the case that one of the arguments u and v is less than M_0 and the other one is (necessarily) greater that M_0. We consider only the case $u\le M_0$ and $v>M_0$, as the conclusion for the dual case $u>M_0$ and $v\le M_0$ follows by similar arguments.

Set $M^*=\sup_{u\in[0,M]}h_1(u)$. Then, as $\lim\limits_{v\to\infty}\dfrac{M^*}{v}=0$ and $\lim\limits_{v\to\infty}\dfrac{mv}{M_0+v}=m$, and $\lim_{v\to\infty}\frac{h_2(v)}{v}=m$ we may consider an $M>2M_0$ such that

$$\frac{M^*}{v}<\frac{\varepsilon}{2} \quad \text{and} \quad m-\varepsilon<\frac{mv}{M_0+v} \quad \text{and} \quad \frac{h_2(v)}{v}<\frac{\varepsilon}{2}+m.$$

Then for any $u,v\ge 0$ with $u\le M_0$ and $v>M$, we find

$$m-\varepsilon\le\frac{mv}{M_0+v}\le\frac{\widehat{h}(u,v)}{u+v}=\frac{h_1(u)+h_2(v)}{u+v}\le\frac{M^*+h_2(v)}{v}\le\varepsilon+m,$$

which implies that

$$\left|\frac{\widehat{h}(u,v)}{u+v}-m\right| \le \varepsilon \quad \text{for any} \quad u,v\ge 0 \quad \text{with} \quad u\le M_0,\, v>M. \tag{14}$$

In view of (13) and (14) we see that for any arbitrarily small positive real number ε, we can always find an $M > 0$ such that

$$\left|\frac{\widehat{h}(u,v)}{u+v} - m\right| \leq \varepsilon \quad \text{for any} \quad u, v > 0 \quad \text{with} \quad u + v > 2M.$$

Consequently, it holds

$$\lim_{u+v\to\infty} \frac{\widehat{h}(u,v)}{u+v} = m,$$

which completes the proof of the lemma. □

Applying our main results to the case of the BVP (11), (12), we obtain the following two theorems.

Theorem 4.1. *Assume that conditions* (A1), (B1), (C1), (C) *and* (E) *hold. Then, for any* λ, μ *satisfiyng*

$$L_1 < \lambda, \mu < L_2, \tag{15}$$

the BVP (11), (12) *has at least one solution* (u, v) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0, 1)$*, where we have set*

$$L_1 = \max\left\{\frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^1 (1-r)\, a(r)\widetilde{f}_\infty dr\right]^{-1}, \frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^1 (1-r)\, b(r)\widetilde{g}_\infty dr\right]^{-1}\right\}$$

and

$$L_2 = \min\left\{\frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_0^1 (1-r)\, a(r)\widetilde{f}_0 dr\right]^{-1}, \frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_0^1 (1-r)\, b(r)\widetilde{g}_0 dr\right]^{-1}\right\}.$$

Theorem 4.2. *Assume that conditions* (A1), (B1), (C1), (C) *and* (E) *hold. Then, for any* λ, μ *satisfiyng*

$$L_3 < \lambda, \mu < L_4, \tag{16}$$

there exists a pair (u, v) *satisfying the BVP* (11), (12) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0, 1)$*, where*

$$\begin{aligned} L_3 \quad &= \quad \max\left\{\frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^1 (1-s)a(s)\widetilde{f}_0 ds\right]^{-1},\right. \\ &\qquad \left.\frac{1}{2}\left[\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^1 (1-s)b(s)\widetilde{g}_0 ds\right]^{-1}\right\} \end{aligned}$$

and

$$\begin{aligned} L_4 \quad &= \quad \min\left\{\frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)\widetilde{f}_\infty dr\right]^{-1},\right. \\ &\qquad \left.\frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_0^1 (1-s)b(s)\widetilde{g}_\infty dr\right]^{-1}\right\}. \end{aligned}$$

5 Examples

In this section, we present some examples that illustrate the breadth of our results. In particular, we give two examples, from which the first one concerns our general application while the second one concerns Theorems 3.1 and 3.2.

Example 5.1. *For the sake of simplicity, we assume that* $a = b, f_1 = f_2$ *and* $g_1 = g_2$, $\sigma_1 = \sigma_2$, $\tau_1 = \tau_2$, *i.e., we consider the BVP*

$$\begin{aligned} &u''(t) + \lambda a(t)[\widetilde{f}\,(u(\sigma(t))) + \widetilde{f}\,(v(\sigma(t)))] = 0, \quad 0 < t < 1,\\ &v''(t) + \mu a(t)\,[\widetilde{g}\,(u(\tau(t))) + \widetilde{g}\,(v(\tau(t)))] = 0, \quad 0 < t < 1, \end{aligned} \tag{17}$$

$$\begin{aligned} &u(0) = 0, \qquad u(1) = 2u\left(\tfrac{1}{3}\right),\\ &v(0) = 0, \qquad v(1) = 2v\left(\tfrac{1}{3}\right),\\ &u(t) = \phi_1(t), \quad v(t) = \phi_2(t), \quad -r \le t \le 0, \end{aligned} \tag{18}$$

where

$$\widetilde{f}(t) = p_1(t) + q_1 \sin(t),\ \ t \in \mathbb{R},$$

$$\widetilde{g}(t) = p_2(t) + q_2 \sin(t),\ \ t \in \mathbb{R},$$

with $p_i, p_i + q_i > 0, i = 1, 2,\ \phi_1, \phi_2 : [-r, 0] \to \mathbb{R}^+$, *and* $\sigma, \tau : [0, 1] \to [-\frac{1}{4}, 1]$ *are given by*

$$\sigma(t) = \begin{cases} \sqrt{t}, & t \in [0, 1/4],\\ \frac{1}{2}, & t \in [1/4, 1/2],\\ t, & t \in [1/2, 3/4],\\ \frac{1}{2}\left(t + \frac{3}{4}\right), & t \in [3/4, 1], \end{cases}$$

and

$$\tau(t) = t - \frac{1}{4}, t \in [0, 1].$$

It is not difficult to see that the argument σ is advanced on the interval $[0, 1/4]$ (nonconstant on $[0, 1/4]$ and constant on $[1/4, 1/2]$), retarded on the interval $[3/4, 1]$ while neither retarded nor advanced on the interval $[1/4, 1/2]$.

By the definition of $\widetilde{f}$ and $\widetilde{g}$ we may verify that

$$\widetilde{f}_\infty = p_1, \quad \widetilde{g}_\infty = p_2, \quad \widetilde{f}_0 = p_1 + q_1 \quad \widetilde{g}_0 = p_2 + q_2.$$

As $\alpha = 2$ and $\eta = \dfrac{1}{3}$ we find

$$\gamma := \min\left\{\alpha\eta, \eta, \frac{\alpha(1-\eta)}{1-\alpha\eta}\right\} = \min\left\{\frac{2}{3}, \frac{1}{3}, \frac{2\left(1-\frac{1}{3}\right)}{1-\frac{2}{3}}\right\} = \frac{1}{3}.$$

Note that $\sigma(t) \geq \eta = \dfrac{1}{3}$, for all $t \in \left[\dfrac{1}{9}, 1\right]$, while $\tau(t) = t - \dfrac{1}{4} \geq \eta = \dfrac{1}{3}$, for all $t \in \left[\dfrac{7}{12}, 1\right]$, and so

$$\eta^* = \frac{7}{12}.$$

Thus

$$\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^{1}(1-r)\,a(r)\widetilde{f}_\infty dr = \frac{\frac{1}{3}\cdot\frac{1}{3}}{1-\frac{2}{3}}\int_{\frac{7}{12}}^{1}(1-r)\,a(r)p_1 dr = \frac{1}{3}p_1\int_{\frac{7}{12}}^{1}(1-r)\,a(r)dr$$

and

$$\frac{\gamma\eta}{1-\alpha\eta}\int_{\eta^*}^{1}(1-r)\,b(r)\widetilde{g}_\infty dr = \frac{1}{3}p_2\int_{\frac{7}{12}}^{1}(1-r)\,a(r)dr.$$

Hence

$$\begin{aligned} L_1 &= \max\left\{\frac{1}{2}\left[\frac{1}{3}p_1\int_{\frac{7}{12}}^{1}(1-r)\,a(r)dr\right]^{-1}, \frac{1}{2}\left[\frac{1}{3}p_2\int_{\frac{7}{12}}^{1}(1-r)\,a(r)dr\right]^{-1}\right\} \\ &= \frac{3}{2\min\{p_1,p_2\}\displaystyle\int_{\frac{7}{12}}^{1}(1-r)\,a(r)dr} \end{aligned}$$

and

$$\begin{aligned} L_2 &= \min\left\{\frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_{0}^{1}(1-r)\,a(r)\widetilde{f}_0 dr\right]^{-1}, \frac{1}{2}\left[\frac{1}{1-\alpha\eta}\int_{0}^{1}(1-r)\,a(r)\widetilde{g}_0 dr\right]^{-1}\right\} \\ &= \frac{1}{6}\min\left\{\left[\int_{0}^{1}(1-r)\,a(r)\,(p_1+q_1)\,dr\right]^{-1}, \left[\int_{0}^{1}(1-r)\,a(r)\,(p_2+q_2)\,dr\right]^{-1}\right\} \\ &= \frac{1}{6\max\{p_1+q_1,p_2+q_2\}\displaystyle\int_{0}^{1}(1-r)\,a(r)dr}. \end{aligned}$$

Therefore, assuming that p_1, q_1, p_2, q_2 have been chosen so that

$$\frac{3}{2} < \min\{p_1,p_2\}\int_{\frac{7}{12}}^{1}(1-r)\,a(r)dr$$

and

$$9\max\{p_1+q_1,p_2+q_2\}\int_{0}^{1}(1-r)\,a(r)dr < \min\{p_1,p_2\}\int_{\frac{7}{12}}^{1}(1-r)\,a(r)dr,$$

it follows that $L_1 < L_2$, and from Theorem 4.1, we derive that, for any λ, μ satisfiyng

$$L_1 < \lambda, \mu < L_2,$$

the BVP (17), (18) has at least one solution (u,v) such that $u(t) > 0$ and $v(t) > 0$ on $(0,1)$.

Example 5.2. *Consider the BVP*

$$
\begin{aligned}
&u''(t) + \lambda a(t) f\left(u(\sigma(t)), v(\sigma(t))\right) = 0, \quad 0 < t < 1,\\
&v''(t) + \mu b(t) g\left(u(\tau(t)), v(\tau(t))\right) = 0, \quad 0 < t < 1,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&u(0) = 0, \qquad u(1) = 2u\left(\frac{1}{3}\right),\\
&v(0) = 0, \qquad v(1) = 2v\left(\frac{1}{3}\right),\\
&u(t) = \phi_1(t), \quad v(t) = \phi_2(t), \quad -1 \leq t \leq 0,
\end{aligned}
\tag{20}
$$

where $\phi_1, \phi_2 : [-1, 0] \to \mathbb{R}^+$ *with* $\phi_1(0) = 0 = \phi_2(0)$*, and* $\sigma, \tau : [0, 1] \to [-1, 1]$ *are given by*

$$
\sigma(t) = \begin{cases} -\sin\left(3\pi t\right), & t \in [0, 1/3],\\ \dfrac{9}{4}\left(-t^2 + 2t - \dfrac{5}{9}\right), & t \in [1/3, 1], \end{cases}
$$

and

$$
\tau(t) = \sqrt{t}, \quad t \in [0, 1].
$$

As $\alpha = 2$ and $\eta = \dfrac{1}{3}$, we find

$$
\gamma := \min\left\{\alpha\eta, \eta, \frac{\alpha(1-\eta)}{1-\alpha\eta}\right\} = \min\left\{\frac{2}{3}, \frac{1}{3}, \frac{2\left(1-\frac{1}{3}\right)}{1-\frac{2}{3}}\right\} = \frac{1}{3}.
$$

Since $\sigma(t) \geq \eta = \dfrac{1}{3}$ is equivalent to

$$
\frac{9}{4}\left(-t^2 + 2t - \frac{5}{9}\right) \geq \frac{1}{3},
$$

from which

$$
\sigma(t) \geq \frac{1}{3} \quad \text{for all} \quad t \in \left[1 - \frac{2}{3}\sqrt{\frac{2}{3}}, 1\right],
$$

while $\tau(t) = \sqrt{t} \geq \eta = \dfrac{1}{3}$, for all $t \in \left[\dfrac{1}{9}, 1\right]$, we conclude that

$$
\eta^* = 1 - \frac{2}{3}\sqrt{\frac{2}{3}}.
$$

We mention that the argument τ is advanced while the argument σ is retarded on $\left[0, \dfrac{5}{9}\right]$ and delayed on $\left[\dfrac{5}{9}, 1\right]$.

Now we calculate the positive numbers L_3 and L_4. As in Example 5.1, we have $\alpha = 2$, $\eta = \dfrac{1}{3}$ and $\gamma = \dfrac{1}{3}$ as well as $\dfrac{\gamma\eta}{1-\alpha\eta} = \dfrac{1}{3}$ and $\dfrac{1}{1-\alpha\eta} = 3$. We find

$$L_3 = \frac{3}{2\min\left\{\displaystyle\int_{1-\frac{2}{3}\sqrt{\frac{2}{3}}}^{1}(1-s)a(s)\widetilde{f}_0 ds, \int_{1-\frac{2}{3}\sqrt{\frac{2}{3}}}^{1}(1-s)b(s)\widetilde{g}_0 ds\right\}}$$

and

$$L_4 = \min\left\{1, \frac{1}{6\max\left\{\displaystyle\int_0^1 (1-s)a(s)\widetilde{f}_\infty dr, \int_0^1 (1-s)b(s)\widetilde{g}_\infty dr\right\}}\right\}.$$

Applying Theorem 4.1, we find that if

$$\frac{3}{2} < \min\left\{\int_{1-\frac{2}{3}\sqrt{\frac{2}{3}}}^{1}(1-s)a(s)\widetilde{f}_0 ds, \int_{1-\frac{2}{3}\sqrt{\frac{2}{3}}}^{1}(1-s)b(s)\widetilde{g}_0 ds\right\}$$

and

$$\begin{aligned}9\max&\left\{\int_0^1 (1-s)a(s)\widetilde{f}_\infty dr, \int_0^1 (1-s)b(s)\widetilde{g}_\infty dr\right\}\\ &< \min\left\{\int_{1-\frac{2}{3}\sqrt{\frac{2}{3}}}^{1}(1-s)a(s)\widetilde{f}_0 ds, \int_{1-\frac{2}{3}\sqrt{\frac{2}{3}}}^{1}(1-s)b(s)\widetilde{g}_0 ds\right\},\end{aligned}$$

then for any λ, μ satisfiyng

$$L_3 < \lambda, \mu < L_4,$$

there exists a pair (u, v) satisfying the BVP (19), (20) such that $u(t) > 0$ and $v(t) > 0$ on $(0, 1)$.

6 Remarks

(1) Similar results to those of Theorems 3.1 and 3.2 can be proved for the following system of two point boundary value problems with deviating arguments

$$\begin{aligned} u''(t) + \lambda a(t) f\left(u(\sigma_1(t)), v(\sigma_2(t))\right) = 0, \quad 0 < t < 1,\\ v''(t) + \mu b(t) g\left(u(\tau_1(t)), v(\tau_2(t))\right) = 0, \quad 0 < t < 1, \end{aligned} \tag{21}$$

$$\begin{array}{lll} \alpha u(0) - \beta u'(0) = 0, & \gamma u(1) + \delta u'(1) = 0, & \\ \alpha v(0) - \beta v'(0) = 0, & \gamma v(1) + \delta v'(1) = 0, & \\ u(t) = \phi_1(t), & v(t) = \phi_2(t), & -r \le t \le 0, \end{array} \tag{22}$$

where $\alpha, \beta, \gamma, \delta \ge 0$ with $\alpha + \beta + \gamma + \delta > 0, \rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

(2) Nondecreasingness may be used to give a sufficient condition that yields the existence of a positive number η^* such as the one described in Theorem 3.1. It is not difficult to see that, if $\sigma_i(t) \leq t, \tau_i(t) \leq t$, for all $t \in [0,1]$, σ_i and τ_i are nondecreasing and $\tau_i(1) > \eta,\ \sigma_i(1) > \eta$, then there always exists an $\eta^* \in [\eta, 1]$ such that $\min_{s\in[\eta^*,1]} \{\sigma_i(s)\} \in [\eta,1], \min_{s\in[\eta^*,1]} \{\tau_i(s)\} \in [\eta,1], i = 1,2$.

(3) A requirement equivalent to the one in Theorems 3.1 and 3.2 is the following: *there exists an* $\eta^* \in [\eta,1]$ *such that* $\min_{s\in[\eta^*,1]} \{\sigma_i(s)\} \in [\eta,1], \min_{s\in[\eta^*,1]} \{\tau_i(s)\} \in [\eta,1], i = 1,2$.

(4) In the case of advanced arguments $\sigma_i(t) > t, \tau_i(t) > t$ for all $t \in [0,1]$, $i = 1,2$, inequality (9) also holds, since

$$\inf_{t\in[\eta,1]} u(\sigma(t)) \geq \inf_{t\in[\eta,1]} u(t) \geq \gamma\|u\|.$$

Consequently we can deduce similar results to those of Theorems 3.1 and 3.2 for the case of advanced arguments.

(5) We can easily find necessary conditions in order to have $L_1 < L_2$ and $L_3 < L_4$. For example, $L_1 < L_2$ gives

$$\max\left\{\left[\int_{\eta^*}^1 (1-r)\,a(r)f_\infty dr\right]^{-1}, \left[\int_{\eta^*}^1 (1-r)\,b(r)g_\infty dr\right]^{-1}\right\}$$
$$< \gamma\eta\min\left\{\left[\int_0^1 (1-r)\,a(r)f_0 dr\right]^{-1}, \left[\int_0^1 (1-r)\,b(r)g_0 dr\right]^{-1}\right\}$$

and

$$\frac{1}{\min\left\{\int_{\eta^*}^1 (1-r)\,a(r)f_\infty dr, \int_{\eta^*}^1 (1-r)\,b(r)g_\infty dr\right\}}$$
$$< \frac{\gamma\eta}{\max\left\{\int_0^1 (1-r)\,a(r)f_0 dr, \int_0^1 (1-r)\,b(r)g_0 dr\right\}}$$

or

$$\text{(i)} \qquad \max\left\{\int_0^1 (1-r)\,a(r)f_0 dr, \int_0^1 (1-r)\,b(r)g_0 dr\right\}$$
$$< \gamma\eta\min\left\{\int_{\eta^*}^1 (1-r)\,a(r)f_\infty dr, \int_{\eta^*}^1 (1-r)\,b(r)g_\infty dr\right\}.$$

In a similar manner from $L_3 < L_4$ it follows that

$$\text{(ii)} \qquad \max\left\{\int_0^1 (1-s)a(s)f_\infty dr, \int_0^1 (1-s)b(s)g_\infty dr\right\}$$
$$< \gamma\eta\min\left\{\int_{\eta^*}^1 (1-s)a(s)f_0 ds, \int_{\eta^*}^1 (1-s)b(s)g_0 ds\right\}.$$

In order that (i) holds, it is necessary that

$$\begin{aligned}\int_0^1 (1-r)\, b(r) g_0 dr &< \gamma\eta \int_{\eta^*}^1 (1-r)\, b(r) g_\infty dr \\ 1 < \frac{\int_0^1 (1-r)\, b(r) dr}{\displaystyle\int_{\eta^*}^1 (1-r)\, b(r) dr} &< \gamma\eta \frac{g_\infty}{g_0},\end{aligned}$$

and similarly, we take

$$1 < \frac{\int_0^1 (1-r)\, a(r) dr}{\int_{\eta^*}^1 (1-r)\, a(r) dr} < \gamma\eta \frac{f_\infty}{f_0}.$$

From these relations it follows that

$$\frac{1}{\eta^2} \leq \frac{1}{\gamma\eta} < \frac{g_\infty}{g_0}, \frac{f_\infty}{f_0}$$

which gives a (first) estimation of the bound for η, i.e.,

$$\sqrt{\frac{g_0}{g_\infty}}, \sqrt{\frac{f_0}{f_\infty}} < \eta \leq 1.$$

Clearly, from the above necessary inequalities it follows that:

I) At most one of Theorems 3.1 and 3.2 may be applicable.

II) It is possible that both Theorems 3.1 and 3.2 may fail as both $L_1 < L_2$ and $L_3 < L_4$ may not be satisfied: if $f_\infty = g_\infty = f_0 = f_0$, then by the last inequality above, neither (i) nor (ii) holds.

Sufficient conditions so that (i) or (ii) hold may be easily obtained in terms of g_∞, g_0 f_∞, f_0, a, b, η.

Received: November 8, 2007. Revised: April 29, 2008.

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