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Periodic Solutions of Periodic Difference Equations by Schauder's Theorem

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ABSTRACT

In this paper, we discuss the existence problem of periodic solutions of the periodic difference equation

$$x(n+1) = f(n, x(n)), \quad n \in \mathbf{Z}$$

and the periodic difference equation with infinite delay

$$x(n+1) = f(n, x_n), \quad n \in \mathbf{Z},$$

where x and f are d-vectors, and $\mathbf{Z}$ denotes the set of integers. We show the existence of periodic solutions by using Schauder's fixed point theorem, and illustrate an example.

RESUMEN

En este artículo estudiamos el problema de existencia de soluciones periódicas para la ecuación en diferencia periódica

$$x(n+1) = f(n, x(n)), \quad n \in \mathbf{Z}$$

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y la ecuación en diferencia periódica con retardo infinito

$$x(n+1) = f(n, x_n), \;\; n \in \mathbf{Z},$$

donde x y f son d-vectores, y $\mathbf{Z}$ denota el conjunto de los números enteros. Mostramos la existencia de soluciones periódicas mediante el uso del teorema de punto fijo de Schauder, exhibimos un ejemplo.

Key words and phrases: *Periodic solutions, difference equations, fixed point theorem.*

Math. Subj. Class.: *39A10, 39A11, 39A20.*

1 Introduction

The existence problem of periodic solutions of functional equations has been discussed in many books and papers. For example, see the books [1-3, 12, 15, 28, 30, 32] and papers [4-11, 13, 14, 16-27, 29, 33], and their references. In these books and papers, many kinds of functional equations have been studied. For example, Volterra equations [2, 4-6, 13, 22, 23], ordinary and functional differential equations [1, 3, 10, 15-20, 27-29, 32, 33], integro-differential equations [7, 22], integral equations [8, 9, 21], and difference equations [11-14, 23-26, 30]. In this paper, we give some new existence results of periodic solutions for periodic difference equations by using Schauder's fixed point theorem and a convex Liapunov function, and show that the existence problem of periodic solutions of a periodic difference equation with infinite delay can be reduced to the existence problem of periodic solutions of an auxiliary difference equation with finite delay.

Fixed point theorems are very useful tools in obtaining existence theorems for periodic solutions. Since we use Schauder's second fixed point theorem later, first we state it for the sake of completeness.

Theorem 1 (Schauder's second theorem [31]). *Let $(B, \|\cdot\|)$ be a normed space, and let S be a nonempty convex subset of B. Then every continuous mapping of S into a compact set C of S has a fixed point in C.*

2 Periodic difference equations

Let $\mathbf{R}^+ = [0, \infty)$, $\mathbf{R} = (-\infty, \infty)$, and let $\mathbf{R}^d$ be the d-dimensional Euclidean space. Let $f(n, x) : \mathbf{Z} \times \mathbf{R}^d \to \mathbf{R}^d$ be continuous in x for each fixed $n \in \mathbf{Z}$, and N-periodic in n for some $N \in \mathbf{N}$ with $N > 1$, where $\mathbf{N}$ denotes the set of positive integers.

Consider the periodic difference equation

$$x(n+1) = f(n, x(n)), \quad n \in \mathbf{Z}. \tag{1}$$

For any $n_0 \in \mathbf{Z}$ and $\xi \in \mathbf{R}^d$, $x(n) = x(n, n_0, \xi)$ denotes the solution of Eq.(1) with $x(n_0) = \xi$.

When we employ Theorem 1 in order to prove the existence of a fixed point of a mapping, we need to define a suitable convex set in a Banach space. In [27], Grimmer introduced the concept of a convex Liapunov function, and proved the existence of periodic solutions of functional differential equations by employing a fixed point theorem. Moreover, in [20], the existence of periodic solutions of functional differential equations is proved by using a convex Liapunov function and Schauder's fixed point theorem. Here, first we state the definition of a convex Liapunov function for the sake of completeness.

Definition. *A function $V(n,x) : \mathbf{Z} \times \mathbf{R}^d \to \mathbf{R}^+$ is said to be a convex Liapunov function if $V(n,x)$ is continuous in x for each fixed n, and satifies the following conditions.*

(i) *$V(n,x) \geq a(|x|)$ for a continuous function $a(r)$ such that $a(r) \to \infty$ as $r \to \infty$, where $|\cdot|$ denotes the Euclidean norm of $\mathbf{R}^d$.*

(ii) *The set $X_\rho := \{x \in \mathbf{R}^d : V(n,x) \leq \rho\}$ is a convex set in $\mathbf{R}^d$ for any $n \in \mathbf{Z}$ and $\rho > 0$, provided that X_ρ is nonempty.*

Now we have the following theorem.

Theorem 2. *Let $V : \mathbf{Z} \times \mathbf{R}^d \to \mathbf{R}^+$ be an N-periodic convex Liapunov function. Suppose that there exist an $n_0 \in \mathbf{Z} \cap [0, N)$ and a constant $\rho > \max\{V(n,0) : 0 \leq n \leq N\}$ such that for any $\xi \in S := \{x \in \mathbf{R}^d : V(n_0, x) \leq \rho\}$, we have*

$$V\big(n_0, x(n_0 + N, n_0, \xi)\big) \leq \rho. \tag{2}$$

Then, Eq.(1) *has an N-periodic solution.*

Proof. Since $V(n,x)$ is a convex Liapunov function and $\rho > \max\{V(n,0) : 0 \leq n \leq N\}$, S is a nonempty compact convex subset of $\mathbf{R}^d$. Let P be a mapping on S defined by

$$P(\xi) := x(n_0 + N, n_0, \xi), \quad \xi \in S.$$

Then (2) implies that $P(S)$ is contained in S. Moreover, the continuity of $f(n,x)$ in x for each fixed $n \in \mathbf{Z}$ implies that $P : S \to S$ is a continuous mapping. Thus, by Theorem 1, P has a fixed point $\xi \in S$, and $x(n) = x(n, n_0, \xi)$ is an N-periodic solution of Eq.(1).

Now we show an example for Theorem 2.

Example. Consider the 4-periodic difference equation

$$\begin{cases} x^{(1)}(n+1) = \alpha x^{(2)}(n) + \beta \cos \frac{n\pi}{2}, \\ \\ x^{(2)}(n+1) = \gamma x^{(1)}(n) + \delta \sin \frac{n\pi}{2}, \end{cases}$$

where $n \in \mathbf{Z}$, and where α, β, γ, and δ are constants with $\sqrt{2}\max(|\alpha|, |\gamma|) < 1$. Let r be a positive constant with

$$r \geq \frac{\sqrt{2}\max(|\beta|, |\delta|)}{1 - \sqrt{2}\max(|\alpha|, |\gamma|)}, \tag{3}$$

and let $V(n, x) := (x^{(1)})^2 + (x^{(2)})^2$, where $x := (x^{(1)}, x^{(2)})$. Clearly, V is a 4-periodic convex Liapunov function with $a(r) = r^2$. The set S defined by

$$S := \{x \in \mathbf{R}^2 : |x| \leq r\}$$

is a nonempty compact convex subset of $\mathbf{R}^2$ for the constant $r > 0$. For any $\xi := (\xi^{(1)}, \xi^{(2)}) \in S$, let $x^{(1)}(n) = x^{(1)}(n, 0, \xi)$, $x^{(2)}(n) = x^{(2)}(n, 0, \xi)$, and let $x(n) = \big(x^{(1)}(n), x^{(2)}(n)\big)$. Then, (3) implies

$$-\frac{r}{\sqrt{2}} \leq -|\alpha|r - |\beta| \leq \alpha\xi^{(2)} - |\beta| \leq x^{(1)}(1) \leq \alpha\xi^{(2)} + |\beta| \leq |\alpha|r + |\beta| \leq \frac{r}{\sqrt{2}},$$
$$-\frac{r}{\sqrt{2}} \leq -|\gamma|r - |\delta| \leq \gamma\xi^{(1)} - |\delta| \leq x^{(2)}(1) \leq \gamma\xi^{(1)} + |\delta| \leq |\gamma|r + |\delta| \leq \frac{r}{\sqrt{2}},$$

which yields that $|x(1)| \leq r$. Thus we obtain $x(1) \in S$. By similar arguments, we have $V\big(4, x(4, 0, \xi)\big) \leq r^2$, and consequently $x(4) \in S$. Thus, by Theorem 2, this 4-periodic difference equation has a 4-periodic solution $x(n)$ with $|x(n)| \leq r$ for $n \in \mathbf{Z}$.

3 Periodic difference equations with finite delay

In this section, concerning the existence of periodic solutions of periodic difference equations with finite delay, we state some known results.

For a fixed $\kappa \in \mathbf{N}$, let $\mathbf{B}$ be the set of sequences $\phi : \mathbf{Z} \cap [-\kappa, 0] \to \mathbf{R}^d$. For any $\phi \in \mathbf{B}$, define $\|\phi\|$ by

$$\|\phi\| := \sup\{|\phi(k)| : k \in \mathbf{Z} \cap [-\kappa, 0]\}.$$

For any $\alpha > 0$, the set $\mathbf{B}_\alpha$ defined by

$$\mathbf{B}_\alpha := \{\phi \in \mathbf{B} : \|\phi\| \leq \alpha\}$$

is compact. For any sequence $x(k) : \mathbf{Z} \to \mathbf{R}^d$ and any fixed $n \in \mathbf{Z}$, the symbol x_n denotes the restriction of $x(k)$ on $\mathbf{Z} \cap [n - \kappa, n]$, that is, x_n is an element of $\mathbf{B}$ defined by

$$x_n(k) := x(n + k), \ \ k \in \mathbf{Z} \cap [-\kappa, 0].$$

Consider the difference equation with finite delay

$$x(n+1) = f(n, x_n), \ \ n \in \mathbf{Z}, \tag{4}$$

where $f : \mathbf{Z} \times \mathbf{B} \to \mathbf{R}^d$ is continuous in ϕ for each fixed $n \in \mathbf{Z}$, and N-periodic in n for some $N \in \mathbf{N}$ with $N > 1$. For any $n_0 \in \mathbf{Z}$ and any initial sequence $\phi \in \mathbf{B}$, there is a unique solution of Eq.(4), denoted by $x(n, n_0, \phi)$, such that it satisfies Eq.(4) for $n \in \mathbf{Z} \cap [n_0, \infty)$ and

$$x(n_0 + k, n_0, \phi) = \phi(k) \ \text{ for } \ k \in \mathbf{Z} \cap [-\kappa, 0].$$

In [26], concerning the existence of periodic solutions of Eq.(4), the following theorem is proved by employing Browder's fixed point theorem.

Theorem 3 ([26]). *If $f(n, \phi)$ in* Eq.(4) *is N-periodic in n for some $N \in \mathbf{N}$ with $N > 1$, and if the solutions of* Eq.(4) *are uniformly ultimately bounded for bound X, then* Eq.(4) *has an N-periodic solution $x(n)$ such that $|x(n)| < X$ for $n \in \mathbf{Z}$.*

Here the solutions of Eq.(4) are said to be uniformly ultimately bounded for bound X, if there exists an X and if corresponding to any $n_0 \in \mathbf{Z}$ and $\alpha > 0$, there exists a $\nu = \nu(\alpha) \in \mathbf{N}$ such that $\phi \in \mathbf{B}_\alpha$ implies that $|x(n, n_0, \phi)| < X$ for $n \in \mathbf{Z} \cap [n_0 + \nu, \infty)$.

In Theorem 3, uniform ultimate boundedness of solutions of Eq.(4) is an important assumption. Here we state a boundedness theorem due to Shunian Zhang without a proof.

Theorem 4 ([34]). *Suppose that there exists a Liapunov function $V : \mathbf{Z} \times \mathbf{R}^d \to \mathbf{R}^+$, which satifies the following conditions;*

(i) *$a(|x|) \leq V(n, x) \leq b(|x|)$, where $a, \ b : \mathbf{R}^+ \to \mathbf{R}^+$, $a(r)$ and $b(r)$ are continuous, increasing and $a(r) \to \infty$ as $r \to \infty$,*

(ii) $\Delta V_{(4)}(n, x(n)) := V(n+1, x(n+1)) - V(n, x(n)) \leq M - c(|x(n)|)$
whenever

$$P\big(V(n+1, x(n+1))\big) > V(k, x(k)) \ \text{ for } \ k \in \mathbf{Z} \cap [n - \kappa, n],$$

where $x(n)$ is a solution of Eq.(4)*, M is a positive constant, $c : \mathbf{R}^+ \to \mathbf{R}^+$ is continuous, increasing and $c(r) \to \infty$ as $r \to \infty$, and $P : \mathbf{R}^+ \to \mathbf{R}^+$ is continuous, $P(u) > u$ for $u > 0$, and $\kappa \in \mathbf{N}$.*

Then the solutions of Eq.(4) *are uniformly ultimately bounded for a bound X.*

4 Periodic difference equations with infinite delay

By combining Liapunov's method and Theorems 3 and 4 in Section 3, we can obtain a theorem which assures the existence of periodic solutions of periodic difference equations. But Theorem 4 is applicable to difference equations with finite delay, and it seems to be open whether we can prove a theorem similar to Theorem 4 for difference equations with infinite delay or not. In this section, we show that the existence problem of periodic solutions of periodic difference equations with infinite delay can be reduced to the existence problem of periodic solutions of auxiliary difference equations whose delay is equal to its period.

Let $\mathcal{B}$ be the set of bounded sequences $\phi : \mathbf{Z}^- \to \mathbf{R}^d$, where $\mathbf{Z}^-$ denotes the set of nonpositive integers. For any $\phi \in \mathcal{B}$, define $\|\phi\|$ by

$$\|\phi\| := \sup\{|\phi(k)| : k \in \mathbf{Z}^-\}.$$

For any sequence $x(k) : \mathbf{Z} \to \mathbf{R}^d$ and any fixed $n \in \mathbf{Z}$, the symbol x_n denotes the restriction of $x(k)$ on $\mathbf{Z} \cap (-\infty, n]$, that is, x_n is an element of $\mathcal{B}$ defined by

$$x_n(k) = x(n+k), \ \ k \in \mathbf{Z}^-.$$

Consider the periodic difference equation with infinite delay

$$x(n+1) = f(n, x_n), \ \ n \in \mathbf{Z}, \tag{5}$$

where $f(n,\phi) : \mathbf{Z} \times \mathcal{B} \to \mathbf{R}^d$ is continuous in ϕ for each fixed $n \in \mathbf{Z}$, and N-periodic in n for some $N \in \mathbf{N}$ with $N > 1$. For any $n_0 \in \mathbf{Z}$ and any initial sequence $\phi \in \mathcal{B}$, there is a unique solution of Eq.(5), denoted by $x(n, n_0, \phi)$, such that it satisfies Eq.(5) for $n \in \mathbf{Z} \cap [n_0, \infty)$ and

$$x(n_0 + k, n_0, \phi) = \phi(k) \ \text{ for } \ k \in \mathbf{Z}^-.$$

Corresponding to $\mathcal{B}$, let $\mathcal{B}_N$ be the set of sequences $\psi : \mathbf{Z} \cap [-N, 0] \to \mathbf{R}^d$. For any $\psi \in \mathcal{B}_N$, define a mapping $\rho(\psi) : \mathcal{B}_N \to \mathcal{B}_N$ by

$$\rho(\psi)(k) := \psi(k) + \frac{N+k}{N}\big(\psi(-N) - \psi(0)\big), \ \ k \in \mathbf{Z} \cap [-N, 0],$$

and a mapping $\sigma(\psi) : \mathcal{B}_N \to \mathcal{B}$ by

$$\sigma(\psi)(k) := \begin{cases} \psi(k), & -N \le k \le 0, \\ \\ \rho(\psi)(k + jN), & -(j+1)N \le k < -jN, \ \ j \in \mathbf{N}. \end{cases}$$

Then, we have the following lemma.

Lemma. *The functional $\sigma(\psi)$ is continuous. If $\psi \in \mathcal{B}_N$ satisfies $\psi(-N) = \psi(0)$, then $\sigma(\psi)(k)$ is N-periodic on $\mathbf{Z}^-$.*

Proof. From the definition of $\sigma(\psi)$, it is clear that the functional $\sigma(\psi)$ is continuous. Next, from the definition of $\rho(\psi)$, if $\psi(-N) = \psi(0)$, then we have

$$\rho(\psi)(k) \equiv \psi(k) \text{ for } k \in \mathbf{Z} \cap [-N, 0],$$

which together with the definition of $\sigma(\psi)$, implies that $\sigma(\psi)(k)$ is N-periodic on $\mathbf{Z}^-$.

For the functional $f(n, \phi)$ in Eq.(4), define the functional $g(n, \psi) : \mathbf{Z} \times \mathcal{B}_N \to \mathbf{R}^d$ by

$$g(n, \psi) := f\big(n, \sigma(\psi)\big), \quad (n, \psi) \in \mathbf{Z} \times \mathcal{B}_N.$$

Then, $g(n, \psi)$ is continuous in ψ for each fixed $n \in \mathbf{Z}$, and N-periodic in n. Corresponding to Eq.(5), consider the auxiliary difference equation

$$y(n+1) = g(n, y_n), \quad n \in \mathbf{Z}, \tag{6}$$

where $y_n \in \mathcal{B}_N$, that is,

$$y_n(k) = y(n+k), \quad k \in \mathbf{Z} \cap [-N, 0].$$

Then, we have the following theorem.

Theorem 5. *If* Eq.(5) *has an N-periodic solution, then it is an N-periodic solution of* Eq.(6)*, and vice versa.*

Proof. Let $x(n)$ be an N-periodic solution of Eq.(5), and let $y_n \in \mathcal{B}_N$ be the restriction of $x(k)$ on $\mathbf{Z} \cap [n-N, n]$. Then we have $y_n(-N) = y_n(0)$, which together with Lemma, implies $\sigma(y_n) = x_n$. Thus, we obtain

$$y(n+1) = x(n+1) = f(n, x_n) = f\big(n, \sigma(y_n)\big) = g(n, y_n),$$

which shows that $y(n)$ is an N-periodic solution of Eq.(6).

The converse part can be proved similarly.

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