CUBO A Mathematical Journal
Vol.11, $N^{\underline{o}}$03, (41–53). August 2009

Boundedness and Global Attractivity of Solutions for a System of Nonlinear Integral Equations

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ABSTRACT

It is well-known that Liapunov's direct method has been used very effectively for differential equations. The method has not, however, been used with much success on integral equations until recently. The reason for this lies in the fact that it is very difficult to relate the derivative of a scalar function to the unknown non-differentiable solution of an integral equation. In this paper, we construct a Liapunov functional for a system of nonlinear integral equations. From that Liapunov functional we are able to deduce conditions for boundedness and global attractivity of solutions. As in the case for differential equations, once the Liapunov function is constructed, we can take full advantage of its simplicity in qualitative analysis.

RESUMEN

Es conocido que el método directo de Liapunov ha sido usado de manera efectiva en ecuaciones diferenciales. Sin embargo este método no ha sido utilizado con mucho suceso en ecuaciones intergrales hasta ahora. La razón para esto reside en el hecho que es difícil relacionar la derivada de una función escalar a la solución no diferenciable desconocida. En este artículo, construimos un funcional de Liapunov para un sistema de ecuaciones integrales no lineales. Usando tal funcional de Liapunov somos capazes de deducir acotamiento y atractividad global de soluciones. Como en el caso de ecuaciones diferenciales, una vez que el funcional de Liapunov es construido, aprovechamos su simplicidad en el análisis cualitativo.

Key words and phrases: *Boundedness, global attractivity, integral equations.*

Math. Subj. Class.: *45D05, 45G15, 45G99.*

1 Introduction

This paper is concerned with a system of nonlinear integral equations

$$x(t) = h(t, x(t)) - \int_0^t D(t, s)g(s, x(s))ds \tag{1.1}$$

where $x(t) \in R^n$, $h : R^+ \times R^n \to R^n$, $D : R^+ \times R^+ \to R^{n\times n}$, $g : R^+ \times R^n \to R^n$ are continuous, and $R^+ = [0, \infty)$.

The theory of integral equations has grown tremendously in the past several decades. The growth has been strongly promoted by the advanced technology in scientific computation and the large number of applications to models in biology, economics, engineering, and other applied sciences. It is the qualitative behavior of solutions of these models that is especially important to many investigators. For the historical background, basic theory, and discussion of applications, we refer the reader to, for example, the work of Corduneanu [1], Burton [6], Gripenberg et al [7], Levin ([10]-[12]), Levin and Nohel [13], MacCamy [15], Miller [17], and references therein.

There is substantial literature on (1.1) and much of it can be traced back to the pioneering work of Levin and Nohel ([10]-[14]) in the study of asymptotic behavior of solutions of the scalar integral equation

$$x(t) = a(t) - \int_0^t D(t, s)g(x(s))ds \tag{1.2}$$

and the integro-differential equation

$$x'(t) = a(t) - \int_0^t D(t, s)g(x(s))ds. \tag{1.3}$$

These equations arise in problems related to evolutionary processes in biology, in nuclear reactors, and in control theory (see Corduneanu [1], Burton [5], Levin and Nohel [13], Kolmanovskii and

Myshkis [9]). It is often required that $a \in C(R^+, R)$ and

$$D(t,s) \geq 0, \quad D_s(t,s) \geq 0, \quad \text{and} \quad D_{st}(t,s) \leq 0. \tag{1.4}$$

with $G(x) = \int_0^x g(s)ds \to \infty$ as $|x| \to \infty$. When $D(t,s) = D(t-s)$ is of convolution type, (1.4) represents

$$D(t) \geq 0, \quad D'(t) \leq 0, \quad \text{and} \quad D''(t) \geq 0. \tag{1.5}$$

Levin's work is based on (1.5) and the construction of a Liapunov functional for (1.3). The method was further extended into a long line of investigation drawing together such different notions of positivity as Liapunov functions, completely monotonic functions, and kernels of positive type (see Corduneanu [1], Gripenberg et al [7], Levin and Nohel [14], MacCamy and Wong [16]). In a series papers ([2]-[4]), Burton obtains results on boundedness and attractivity of solutions for a scalar equation in form of (1.1) without asking the growth condition on g. Liapunov functionals play an essential role in his proofs. A good summary for recent development of the subject may be found in Burton [6].

Many investigators mentioned above frequently use the fact that (1.3) can be put into the form of (1.2) by integration. We observe that the differentiability of $x(t)$ in (1.2) is not required. For example, we may convert a system of neutral differential equations, say

$$\frac{d}{dt}\left[x(t) - g(t, x(s))\right] = Ax(t) + G(t, x(t)) \tag{1.6}$$

into a system of integral equations in the form of (1.1)

$$x(t) = \widetilde{g}(t, x(t)) + \int_0^t e^{A(t-s)}\widetilde{G}(s, x(s))ds \tag{1.7}$$

with a view of proving the existence and qualitative behavior of solutions by applying fixed point theorems. Note that the initial functions for the differential equations are absorbed into the term $\widetilde{g}(t, x(t))$. Equation (1.7) often describes actual models calling for time-dependent feedback. The integral term here may be viewed as the assumption that the future state of the process depends not only on the present, but also on the past history.

The object of this paper is to give conditions to ensure that all solutions $x = x(t)$ of (1.1) are bounded and converge to zero as $t \to \infty$ by constructing a Liapunov functional. From that Liapunov functional we are able to deduce conditions for boundedness and global attractivity of solutions. This will be done in Section 2 and 3, respectively. We generalize some classical results on boundedness and attractivity of solutions for (1.2) without asking a growth condition on g and obtain theorems for (1.1) that are parallel to those of Burton [2] for scalar equations. We notice that it is very difficult to relate the derivative of a scalar function to the unknown solution of (1.1) since the solution may not be differentiable, and this presents a significant challenge to investigators finding a suitable Liapunov function for (1.1). Even if the function was found, it might not be positive definite or decreasing along the solutions of (1.1). However, the Liapunov functional still provides us with a great deal of information on solutions of (1.1), and therefore, we can derive certain properties of solutions without actually solving the equation.

For $x \in R^n, |x|$ denotes the Euclidean norm of x. Let $C(X,Y)$ denote the space of continuous functions $\phi : X \to Y$. For an $n \times n$ matrix $B = (b_{ij})_{n\times n}$, we denote the norm of B by $\|B\| = \sup\{|Bx| : |x| \le 1\}$. If B is symmetric, we use the convention for self-adjoint positive operators to write $B \ge 0$ whenever B is positive semi-definite. Similarly, if B is negative semi-definite, then $B \le 0$. Also, if $B \ge 0$, we denote its square root by $\sqrt{B}$.

2 Boundedness

In this section, we study the boundedness of solutions of (1.1). We shall focus on *a priori* bounds of solutions. An important application of an *a priori* bound is to establish the existence of a solution of (1.1) on R^+. A well-known procedure for proving global existence of solutions calls for a local existence theorem, a continuation argument, and an *a priori* bound (see Levin [12]).

A continuous function $x : R^+ \to R^n$ is called a solution of (1.1) on R^+ if it satisfies (1.1) on R^+. It is to be understood that $x(0) = h(0, x(0))$. If $x(t)$ is specified to be a certain initial function on an initial interval, say

$$x(t) = \phi(t) \quad \text{for} \quad 0 \le t \le t_0,$$

we are then looking for a solution of

$$x(t) = h(t, x(t)) - \int_0^{t_0} D(t,s)g(s,\phi(s))ds - \int_{t_0}^{t} D(t,s)g(s,x(s))ds, \quad t \ge t_0. \tag{2.1}$$

However, a change of variable $y(t) = x(t + t_0)$ will reduce the problem back to one of form (1.1). Thus, the initial function on $[0, t_0]$ is absorbed into the forcing function, and hence, it suffices to consider (1.1) with the simple initial condition $x(0) = h(0, x(0))$.

We shall prove the boundedness of solutions of system (1.1) by constructing a Liapunov functional which has its roots in Burton [2] and Kemp [8]. The results here generalize the theorems in Burton [2] for scalar equations. We require that

(H_1) $D(t,s)$ is a symmetric matrix with $D(t,0) \ge 0$, $D_s(t,s) \ge 0$, $D_t(t,0) \le 0$, and $D_{st}(t,s) \le 0$ with $D_s(t,s)$ and $D_{st}(t,s)$ continuous for all $t \ge s \ge 0$.

(H_2) $g(t,x)$ is bounded for x bounded and there exists a constant $K > 0$ such that

$$g^T(t,x)[x - h(t,x)] \ge |g(t,x)| \quad \text{for all} \quad |x| \ge K \quad \text{and} \quad t \ge 0 \tag{2.2}$$

where g^T is the transpose of g.

(H_3) There exists a continuous function $\psi : R^+ \to R^+$ such that $|h(t,x)| \leq \psi(|x|)$ for all $t \geq 0$ and $x \in R^n$ with $\lim_{r\to\infty}(r - \psi(r)) = \infty$.

(H_4) $\sup_{t\geq 0} \left[\|D(t,0)\|t^2 + \int_0^t \|D_s(t,s)\| \left(1+(t-s)^2\right) ds\right] < \infty.$

We observe that some of these conditions are interconnected. For example, (H_3) nearly implies (H_2) if $x^T g(t,x) \geq 0$ for $|x| \geq K$. In many applications, $h(t,x)$ is a nonlinear contraction in which ψ in (H_3) is a nondecreasing function with $\psi(r) < r$ for all $r > 0$.

Theorem 2.1. Suppose that (H_1) - (H_4) hold. Then every solution of (1.1) on R^+ is bounded.

Proof. We first define some constants to simplify notations. Let

$$\sup_{t\geq 0}\int_0^t \|D_s(t,s)\|(t-s)^2 ds = J_2 \tag{2.3}$$

and define

$$\sup_{t\geq 0}\int_0^t \|D_s(t,s)\| ds = J_1 \quad \text{and} \quad \sup_{t\geq 0}\|D(t,0)\|t^2 = D_2. \tag{2.4}$$

We now let $x = x(t)$ be a solution of (1.1) on R^+ and define a Liapunov functional

$$\begin{aligned} V(t,x(\cdot)) \quad &= \quad \int_0^t \left(\int_s^t g(v,x(v))dv\right)^T D_s(t,s)\left(\int_s^t g(v,x(v))dv\right) ds \\ &+ \left(\int_0^t g(v,x(v))dv\right)^T D(t,0)\int_0^t g(v,x(v))dv. \end{aligned} \tag{2.5}$$

Differentiate $V(t,x(\cdot))$ with respect to t to obtain

$$\begin{aligned} V'(t,x(\cdot)) \; &= \; \int_0^t \left(\int_s^t g(v,x(v))dv\right)^T D_{st}(t,s)\left(\int_s^t g(v,x(v))dv\right) ds \\ &+ 2\, g^T(t,x(t)) \int_0^t D_s(t,s)\left(\int_s^t g(v,x(v))dv\right) ds \\ &+ \left(\int_0^t g(v,x(v))dv\right)^T D_t(t,0)\int_0^t g(v,x(v))dv \\ &+ 2\, g^T(t,x(t))\, D(t,0)\int_0^t g(v,x(v))dv. \end{aligned} \tag{2.6}$$

We integrate the third to last term by parts to obtain

$$\begin{aligned}
& 2\, g^T(t,x(t)) \left[D(t,s) \int_s^t g(v,x(v))dv \Big|_{s=0}^{s=t} + \int_0^t D(t,s) g(s,x(s))ds \right] \\
= \;& 2\, g^T(t,x(t)) \left[-D(t,0) \int_0^t g(s,x(s))ds + \int_0^t D(t,s) g(s,x(s))ds \right].
\end{aligned}$$

Cancel terms, use the sign conditions, and use (1.1) in the last step of the process to unite the Liapunov functional and the equation obtaining

$$\begin{aligned}
V'(t,x(\cdot)) \;=\; & \int_0^t \left(\int_s^t g(v,x(v))dv \right)^T D_{st}(t,s) \left(\int_s^t g(v,x(v))dv \right) ds \\
& + \left(\int_0^t g(v,x(v))dv \right)^T D_t(t,0) \int_0^t g(v,x(v))dv + 2g^T(t,x(t)) \int_0^t D(t,s) g(s,x(s))ds \\
\le \; & 2\, g^T(t,x(t)) \int_0^t D(t,s) g(s,x(s))ds \\
= \; & 2\, g^T(t,x(t))[-x(t) + h(t,x(t))]. \qquad (2.7)
\end{aligned}$$

By (H_2), we see that if $|x(t)| \ge K$, then

$$V'(t,x(\cdot)) \le -|g(t,x(t))| \qquad (2.8)$$

It is clear that $V'(t,x(\cdot))$ is bounded above for $0 \le |x(t)| \le K$ since $g(t,x)$ is bounded for x bounded and $|h(t,x)| \le \psi(|x|)$, and hence, there exists a constant $L > 0$ depending on K such that

$$V'(t,x(\cdot)) \le -|g(t,x(t))| + L \qquad (2.9)$$

By the Schwarz inequality, we have

$$\begin{aligned}
& \left| \int_0^t D_s(t,s) \int_s^t g(v,x(v))dvds \right|^2 \\
= \;& \left| \int_0^t \sqrt{D_s(t,s)} \left[\sqrt{D_s(t,s)} \int_s^t g(v,x(v))dv \right] ds \right|^2 \\
\le \;& \int_0^t \|\sqrt{D_s(t,s)}\|^2 ds \int_0^t \left| \sqrt{D_s(t,s)} \int_s^t g(v,x(v))dv \right|^2 ds \\
= \;& \int_0^t \|\sqrt{D_s(t,s)}\|^2 ds \int_0^t \left(\int_s^t g(v,x(v))dv \right)^T D_s(t,s) \left(\int_s^t g(v,x(v))dv \right) ds \\
\le \;& \int_0^t \|D_s(t,s)\| ds\, V(t,x(\cdot)) \le J_1 V(t,x(\cdot)) \qquad (2.10)
\end{aligned}$$

where J_1 is defined in (2.4). We have just integrated the left-hand side by parts, obtaining

$$\left|\int_0^t D_s(t,s)\int_s^t g(v,x(v))dvds\right|^2 = \left|-D(t,0)\int_0^t g(s,x(s))ds + \int_0^t D(t,s)g(s,x(s))ds\right|^2$$

so that by (2.10) and (1.1) we now have

$$\begin{aligned} J_1V(t,x(\cdot)) &\geq \left|\int_0^t D_s(t,s)\int_s^t g(v,x(v))dvds\right|^2 \\ &= \left|-D(t,0)\int_0^t g(s,x(s))ds + \int_0^t D(t,s)g(s,x(s))ds\right|^2 \\ &= \left|x(t)-h(t,x(t)) + D(t,0)\int_0^t g(s,x(s))ds\right|^2 \\ &\geq \frac{1}{2}|x(t)-h(t,x(t))|^2 - \left|D(t,0)\int_0^t g(s,x(s))ds\right|^2 \\ &\geq \frac{1}{2}|x(t)-h(t,x(t))|^2 - \|D(t,0)\|\left|\sqrt{D(t,0)}\int_0^t g(s,x(s))ds\right|^2 \\ &\geq \frac{1}{2}|x(t)-h(t,x(t))|^2 - D_1V(t,x(\cdot)) \end{aligned}$$

where $D_1 = \sup_{t\geq 0}\|D(t,0)\|$. Here we have used the inequality $2(a^2+b^2)\geq (a+b)^2$. It is now clear that

$$|x(t)-h(t,x(t))|^2 \leq 2(J_1+D_1)V(t,x(\cdot)). \tag{2.11}$$

We now show that $V(t,x(\cdot))$ is bounded. If $V(t,x(\cdot))$ is not bounded, then there exists a sequence $\{t_n\}\uparrow\infty$ with

$$V(t_n,x(\cdot))\geq V(s,x(\cdot)) \quad \text{for} \quad 0\leq s\leq t_n.$$

It then follows from (2.9) that

$$0\leq V(t_n,x(\cdot))-V(s,x(\cdot))\leq -\int_s^{t_n}|g(v,x(v))|dv + L(t_n-s).$$

This implies

$$\int_s^{t_n}|g(v,x(v))|dv\leq L(t_n-s). \tag{2.12}$$

Substitute (2.12) into $V(t_n, x(\cdot))$ to obtain

$$\begin{aligned} V(t_n, x(\cdot)) &\le \int_0^{t_n} \|D_s(t_n,s)\| \left| \int_s^{t_n} g(v,x(v))dv \right|^2 ds + \|D(t_n,0)\| \left| \int_0^{t_n} g(v,x(v))dv \right|^2 \\ &\le \int_0^{t_n} \|D_s(t_n,s)\| \left[L^2(t_n-s)^2 \right] ds + \|D(t_n,0)\| L^2(t_n)^2 \\ &\le (J_2+D_2)L^2. \end{aligned}$$

This implies that $V(t,x(\cdot)) \le (J_2+D_2)L^2$ for all $t \ge 0$, and therefore, by (2.11) we have

$$|x(t) - h(t,x(t))|^2 \le 2(J_1+D_1)(J_2+D_2)L^2. \tag{2.13}$$

Observe that

$$|x(t) - h(t,x(t))| \ge |x(t)| - \psi(|x(t)|). \tag{2.14}$$

Since $r - \psi(r) \to \infty$ as $r \to \infty$ by (H_3), there exists a constant $B > 0$ such that $r \ge B$ implies $r - \psi(r) > 0$ and

$$[r - \psi(r)]^2 > 2(J_1+D_1)(J_2+D_2)L^2. \tag{2.15}$$

We now claim that $|x(t)| < B$ for all $t \ge 0$. Suppose there exists $t^* \ge 0$ with $|x(t^*)| \ge B$. Then by (2.13)-(2.15), we have

$$\begin{aligned} 2(J_1+D_1)(J_2+D_2)L^2 &< [|x(t^*)| - \psi(|x(t^*)|)]^2 \\ &\le |x(t^*) - h(t^*,x(t^*))|^2 \le 2(J_1+D_1)(J_2+D_2)L^2 \end{aligned}$$

a contradiction, and thus, $|x(t)| < B$ for all $t \ge 0$, whenever x is a solution of (1.1) on R^+. This completes the proof.

3 Attractivity

In this section, we study the global attractivity of solutions of (1.1). We shall show that every solution of (1.1) on R^+ tends to zero as $t \to \infty$ regardless of its initial condition. We may view $h(t,x) = u(t,x), u \in \mathcal{G}$, as a perturbation term (or control) of the system where $\mathcal{G}$ is a pre-described class of functions. The project is to characterize $\mathcal{G}$ so that the stability property ($x(t) \to 0$ as $t \to \infty$) is independent of the special choice of $u \in \mathcal{G}$. To arrive at these conclusions, we assume that

(P_1) There exists a function $\tilde{\psi} \in C(R^+, R^+)$ with $\tilde{\psi}(0) = 0$ and $\tilde{\psi}(r) > 0$ for $r > 0$ such that

$$g^T(t,x)[\, x - h(t,x) + h(t,0)\,] \;\ge\; |g(t,x)|\,\tilde{\psi}(|x|) \quad \text{for all} \;\; t \ge 0,\; x \in R,$$

(P$_2$) For each $\mu > 0$ and $\alpha > 0$, there exists $\beta > 0$ such that $|x| \le \mu$ implies

$$|g(t,x)| \le \alpha + \beta\,\tilde{\psi}(|x|)) \quad \text{for all} \;\; t \ge 0,$$

(P$_3$) $h(t,0) \to 0$ as $t \to \infty$,

(P$_4$) $\|D(t,0)\|\, t^2 \to 0$ as $t \to \infty$,

(P$_5$) $\displaystyle\int_0^p \|D_s(t,s)\|(t-s)^2 ds \to 0$ as $t \to \infty$ for each fixed $p > 0$,

One may notice from (P$_1$) and (P$_2$) that $g(t,x)$ is almost independent of $h(t,x)$ and it can be highly nonlinear. We will discuss these conditions in details later after presenting the main theorem of this section.

Theorem 3.1. Suppose that (H$_1$) - (H$_4$) and (P$_1$) - (P$_5$) hold. Then every solution of (1.1) on R^+ tends to zero as $t \to \infty$.

Proof. Let $x = x(t)$ be a solution of (1.1) on R^+. Since (H$_1$) - (H$_4$) hold, by Theorem 2.1, all solutions of (1.1) on R^+ are bounded. There exists a constant $\mu > 0$ such that $|x(t)| \le \mu$ for all $t \ge 0$. For this fixed solution, let $V(t, x(\cdot))$ be defined in (2.5). Then by (2.7), we have

$$\begin{aligned} V'(t,x(\cdot))) &\le 2\, g^T(t,x(t))[-x(t) + h(t,x(t))] \\ &= -2\, g^T(t,x(t))[x(t) - h(t,x(t)) + h(t,0)] + 2\, g^T(t,x(t))h(t,0) \\ &\le -2\,|g(t,x(t))|\,\tilde{\psi}(|x(t)|) + M_\mu(t) \end{aligned} \tag{3.1}$$

where $M_\mu(t) = \sup\{\, 2\, g^*_\mu |h(s,0)| : s \ge t \,\}$ with $g^*_\mu = \sup\{\, |g(t,x)| : t \in R^+, |x| \le \mu \,\}$. Note that $M_\mu(t)$ is decreasing and converges to zero as $t \to \infty$ by (P$_3$). We first claim that $V(t,x(\cdot)) \to 0$ as $t \to \infty$. Observe that $V(t,x(\cdot))$ is bounded since x is bounded. Now let

$$\limsup_{t\to\infty} V(t,x(\cdot)) = P \ge 0.$$

Then for any $\varepsilon > 0$, there exists a positive constant $K > 0$ and a sequence $\{t_n\} \uparrow \infty$ with

$$V(t_n, x(\cdot)) \ge V(s, x(\cdot)) - \varepsilon \;\;\text{for}\;\; K \le s \le t_n. \tag{3.2}$$

In fact, by the definition of $\limsup_{t\to\infty} V(t,x(\cdot))$, for any $\varepsilon > 0$, there exists $K > 0$ such that $t \ge K$ implies

$$-\frac{\varepsilon}{2} < \sup_{s\ge t} V(s,x(\cdot)) - P < \frac{\varepsilon}{2}.$$

Thus, there exists a sequence $\{t_n\} \uparrow \infty$ with $t_1 \geq K$ such that

$$-\frac{\varepsilon}{2} < V(t_n, x(\cdot)) - P < \frac{\varepsilon}{2}.$$

and therefore

$$V(t_n, x(\cdot)) > P - \frac{\varepsilon}{2} = \left(P + \frac{\varepsilon}{2}\right) - \varepsilon > V(s, x(\cdot)) - \varepsilon$$

for all $K \leq s \leq t_n$ and for $n = 1, 2, \cdots$. By (3.1) and (3.2), we now see that

$$\begin{aligned} -\varepsilon &\leq V(t_n, x(\cdot)) - V(s, x(\cdot)) \\ &\leq -\int_s^{t_n} |g(s, x(v))|\, \tilde{\psi}(|x(v)|)dv + M_\mu(K)(t_n - s) \end{aligned}$$

or

$$\int_s^{t_n} |g(s, x(v))|\, \tilde{\psi}(|x(v)|)dv \leq \varepsilon + M_\mu(K)(t_n - s) \tag{3.3}$$

for all $K \leq s \leq t_n$. Apply (P_2) and (3.3) in the following argument to obtain

$$\begin{aligned}
V(t_n, x(\cdot)) \quad \leq \quad & \int_0^K \|D_s(t_n,s)\| \left| \int_s^{t_n} g(v,x(v))dv \right|^2 ds \\
& + \int_K^{t_n} \|D_s(t_n,s)\| \left| \int_s^{t_n} g(v,x(v))dv \right|^2 ds + \|D(t_n,0)\| \left| \int_0^{t_n} g(v,x(v))dv \right|^2 \\
\leq \quad & (g_\mu^*)^2 \int_0^K \|D_s(t_n,s)\|(t_n{-}s)^2 ds + (g_\mu^*)^2 \, \|D(t_n,0)\| t_n^2 \\
& + \int_K^{t_n} \|D_s(t_n,s)\| \left[(t_n{-}s) \int_s^{t_n} |g(v,x(v))|^2 dv \right] ds \\
\leq \quad & (g_\mu^*)^2 \int_0^K \|D_s(t_n,s)\|(t_n{-}s)^2 ds + (g_\mu^*)^2 \, \|D(t_n,0)\| t_n^2 \\
& + \int_K^{t_n} \|D_s(t_n,s)\| \left\{ (t_n{-}s) \int_s^{t_n} |g(v,x(v))| \left[\alpha + \beta \tilde{\psi}(|x(v)|) \right] dv \right\} ds \\
\leq \quad & (g_\mu^*)^2 \left\{ \int_0^K \|D_s(t_n,s)\|(t_n{-}s)^2 ds + \|D(t_n,0)\| t_n^2 \right\} + \alpha g_\mu^* \int_0^{t_n} \|D_s(t_n,s)\|(t_n{-}s)^2 ds \\
& + \beta \int_K^{t_n} \|D_s(t_n,s)\|(t_n{-}s) \left[\int_s^{t_n} |g(v,x(v))| \, \tilde{\psi}(|x(v)|) dv \right] ds \\
\leq \quad & (g_\mu^*)^2 \left\{ \int_0^K \|D_s(t_n,s)\|(t_n{-}s)^2 ds + \|D(t_n,0)\| t_n^2 \right\} + \alpha g_\mu^* J_2 \\
& + \beta \int_K^{t_n} \|D_s(t_n,s)\|(t_n{-}s)[\varepsilon + M_\mu(K)(t_n - s)] ds \\
\leq \quad & (g_\mu^*)^2 \left\{ \int_0^K \|D_s(t_n,s)\|(t_n{-}s)^2 ds + \|D(t_n,0)\| t_n^2 \right\} + \alpha g_\mu^* J_2 \\
& + \varepsilon\beta(J_1 + J_2) + M_\mu(K)\beta J_2 \qquad (3.4)
\end{aligned}$$

where J_1 and J_2 are defined in (2.4) and (2.3), respectively. Now, for a given $\delta > 0$, choose $K > 0$ so large, $\varepsilon > 0$ and $\alpha > 0$ so small that $\alpha J_2 \, g_\mu^* + \varepsilon\beta(J_1 + J_2) < \delta$, and $M_\mu(K)\beta J_2 < \delta$. Since $\int_0^K \|D_s(t_n,s)\|(t_n{-}s)^2 ds + \|D(t_n,0)\| t_n^2 \to 0$ as $n \to \infty$ by (P_4) and (P_5), so that as $\delta \to 0$, we see that $V(t_n, x(\cdot)) \to 0$ as $n \to \infty$. This implies that $P = 0$, and therefore, $V(t, x(\cdot)) \to 0$ as $t \to \infty$.

We now show that $x(t) \to 0$ as $t \to \infty$. Observe (P_1) implies that

$$|x - h(t,x) + h(t,0)| \geq \tilde{\psi}(|x|).$$

It then follows from (2.11) that

$$\begin{aligned} 2(J_1 + D_1)V(t, x(\cdot)) &\geq |x(t) - h(t, x(t))|^2 \\ &= |x(t) - h(t, x(t)) + h(t, 0) - h(t, 0)|^2 \\ &\geq \frac{1}{2}|x(t) - h(t, x(t)) + h(t, 0)]^2 - |h(t, 0)|^2 \\ &\geq \frac{1}{2}\,\tilde{\psi}^2(|x(t)|) - |h(t, 0)|^2. \end{aligned}$$

This yields that

$$\tilde{\psi}^2(|x(t)|) \leq 4(J_1 + D_1)V(t, x(\cdot)) + 2\,|h(t, 0)|^2 \to 0 \quad \text{as} \quad t \to \infty. \tag{3.5}$$

By (P_1), we see that $x(t) \to 0$ as $t \to \infty$. This completes the proof.

Remark 3.1. We point out that (P_1) and (P_2) are quite mild conditions which allow $g(t, x)$ to be highly nonlinear and nearly independent of $h(t, x)$. Note also that (P_5) is a fading memory condition of the integral $\int_0^t \|D_s(t, s)\|(t - s)^2 ds$. If $D(t, s) = D(t - s)$ is of convolution type, then (H_4) implies (P_5).

Example 3.1. Let $g(t, x) = (x_1^3, x_2^3)^T$ for $x = (x_1, x_2)^T \in R^2$, and let $h(t, x)$ be continuous satisfying

$$|h(t, x) - h(t, 0)| \leq \phi(|x|)|x| \tag{3.6}$$

where $\phi \in C(R^+, R^+)$ with $\phi(r) < 1/2$. Then (P_1) and (P_2) hold.

Proof. We will use the following inequalities for $x = (x_1, x_2)^T$

$$|g(t, x)| = \sqrt{|x_1|^6 + |x_2|^6} \leq |x_1|^3 + |x_2|^3 \tag{3.7}$$

and

$$\left(|x_1|^3 + |x_2|^3\right)|x| \leq \left(|x_1|^3 + |x_2|^3\right)(|x_1| + |x_2|) \leq 2\left(|x_1|^4 + |x_2|^4\right). \tag{3.8}$$

Note that we are not seeking the best estimate here. Use these inequalities to obtain

$$\begin{aligned} & g^T(t,x)[x-h(t,x)+h(t,0)] \\ = \; & x^T g(t,x) - g^T(t,x)[h(t,x)-h(t,0)] \\ \geq \; & \left(|x_1|^4+|x_2|^4\right) - |g(t,x)||h(t,x)-h(t,0)| \\ \geq \; & \left(|x_1|^4+|x_2|^4\right) - \left(|x_1|^3+|x_2|^3\right)\phi(|x|)|x| \\ \geq \; & \left(|x_1|^3+|x_2|^3\right)\left[\frac{1}{2}|x| - \phi(|x|)|x|\right] \\ \geq \; & |g(t,x)|\tilde{\psi}(|x|) \end{aligned}$$

where $\tilde{\psi}(r) = \frac{1}{2}r[1-2\phi(r)]$. Thus, ($\mathrm{P}_1$) holds. To prove ($\mathrm{P}_2$), let $\mu > 0$ and $\alpha > 0$ be given. If $|x| \leq \mu$, then

$$\begin{aligned} |g(t,x)| \; & \leq \; |x_1|^3+|x_2|^3 \leq |x|^2\mu \leq \alpha + \left(\frac{\mu^2}{\alpha}\right)|x|^4 \\ & = \; \alpha + \left(\frac{\mu^2}{\alpha}\right)\frac{2\,|x|^3}{1-2\,\phi(|x|)}\,\tilde{\psi}(|x|)) \leq \alpha + \beta\,\tilde{\psi}(|x|)) \end{aligned}$$

where $\beta = 2(\mu^2/\alpha)\mu^3/\phi_\mu^*$ with $\phi_\mu^* = \inf\{\,1-2\phi(r) : 0 \leq r \leq \mu\,\}$. Thus, ($\mathrm{P}_2$) holds and the proof is complete.

Received: October 12, 2008. Revised: October 31, 2008.

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