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On an inequality related to the radial growth of subharmonic functions

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ABSTRACT

It is a classical result that every subharmonic function, defined and $\mathcal{L}^p$-integrable for some p, $0 < p < +\infty$, on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ is for almost all θ of the form $o((1-|z|)^{-1/p})$, uniformly as $z \to e^{i\theta}$ in any Stolz domain. Recently Pavlović gave a related integral inequality for absolute values of harmonic functions, also defined on the unit disk in the complex plane. We generalize Pavlović's result to so called quasi-nearly subharmonic functions defined on rather general domains in $\mathbb{R}^n$, $n \geq 2$.

RESUMEN

Es un resultado clásico que toda función subarmónica definida y $\mathcal{L}^p$-integrable para algún p, $0 < p < +\infty$, sobre el disco unitario $\mathbb{D}$ del plano complejo $\mathbb{C}$ es para casi todo θ de la forma $o((1-|z|)^{-1/p})$, uniformemente cuando $z \to e^{i\theta}$ en cualquier dominio de Stolz. Recientemente, Pavlović encontró una desigualdad integral relacionada para valores absolutas de funciones armónicas, también definidas en el disco unitario del plano complejo. Generalizamos el resultado de Pavlović a las así llamada funciones subarmónicas casi-cercanas definidas en dominios bastante generales en $\mathbb{R}^n$, $n \geq 2$.

Key words and phrases: *Subharmonic function, quasi-nearly subharmonic function, accessible boundary point, approach region, integrability condition, radial order.*

Math. Subj. Class.: *31B25, 31B05.*

1 Introduction

1.1 Previous results. The following theorem is a special case of the original result of Gehring [4, Theorem 1, p. 77], and of Hallenbeck [5, Theorems 1 and 2, pp. 117-118], and of the later and more general results of Stoll [23, Theorems 1 and 2, pp. 301-302, 307]:

Theorem A: *If u is a function harmonic in $\mathbb{D}$ such that*

$$I(u) := \int_{\mathbb{D}} \mid u(z) \mid^p (1- \mid z \mid)^\beta \, dm(z) < +\infty, \tag{1}$$

where $p > 0$, $\beta > -1$, then

$$\lim_{r \to 1-} \mid u(re^{i\theta}) \mid^p (1-r)^{\beta+1} = 0 \tag{2}$$

for almost all $\theta \in [0, 2\pi)$.

Observe that Gehring, Hallenbeck and Stoll in fact considered subharmonic functions and that the limit in (2) was uniform in Stolz approach regions (in Stoll's result in even more general regions). For a more general result, see [19, Theorem, p. 31], [15, Theorem, p. 233], [10, Theorem 2, p. 73] and [18, Theorem 3.4.1, pp. 198-199].

With the aid of [12, Theorem A and Theorem 1, pp. 433-434], Pavlović showed that the convergence in (2) in Theorem A is dominated. At the same time he pointed out that whole Theorem A follows from his result:

Theorem B ([12, Theorem 1, pp. 433-434]) *If u is a function harmonic in $\mathbb{D}$ satisfying* (1), *where $p > 0$, $\beta > -1$, then*

$$J(u) := \int_0^{2\pi} \sup_{0<r<1} \mid u(re^{i\theta}) \mid^p (1-r)^{\beta+1} \, d\theta < +\infty.$$

Moreover, there is a constant $C = C_{p,\beta}$ such that $J(u) \leq C\, I(u)$.

The purpose of this note is to point out that, with the aid of [19, Theorem, p. 31], one can extend Theorem B considerably: Instead of absolute values of harmonic functions on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ we will consider nonnegative quasi-nearly subharmonic functions defined on rather general domains of $\mathbb{R}^n$, $n \geq 2$. See Theorems 1 and 2 below.

First the necessary notation and definitions.

1.2 Notation. Our notation is fairly standard, see e.g. [19, 21, 6]. However, for convenience of the reader we recall the following. The common convention $0 \cdot \infty = 0$ is used. The complex space $\mathbb{C}^n$ is identified with the real space $\mathbb{R}^{2n}$, $n \geq 1$. In the sequel D is an arbitrary domain in $\mathbb{R}^n$,

$n \geq 2$, $D \neq \mathbb{R}^n$, whereas Ω is a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is Ahlfors-regular with dimension d, $0 \leq d \leq n$ (for the definition of this see **1.6** below). The distance from $x \in D$ to ∂D is denoted by $\delta(x)$. If $\rho > 0$ write $D_\rho = \{\, x \in D : \delta(x) < \rho \,\}$. $B^n(x,r)$ is the Euclidean ball in $\mathbb{R}^n$, with center x and radius r, and $B(x) = B^n(x, \frac{1}{3}\delta(x))$. We write $B^n = B(0,1)$ and $S^{n-1} = \partial B^n$. m is the Lebesgue measure in $\mathbb{R}^n$, and $\nu_n = m(B^n)$. $\mathcal{L}^1_{\rm loc}(D)$ is the space of locally (Lebesgue) integrable functions on D. The d-dimensional Hausdorff (outer) measure in $\mathbb{R}^n$ is denoted by H^d, $0 \leq d \leq n$. Our constants C and K are always positive, mostly ≥ 1 and they may vary from line to line. (One exception: In the proof of Theorem 2 we write K for $\partial\Omega$, just in order to follow our previous notation in [19].) On the other hand, C_0 and r_0 are fixed constants which are involved with the used (and thus fixed) admissible function φ (see **1.5** (5) below). Similarly, if $\alpha > 0$ is given, $C_1 = C_1(C_0,\alpha)$, $C_2 = C_2(C_0,\alpha)$ and $C_3 = C_3(C_0,\alpha)$ are fixed constants, coming directly from [19, Lemma 2.3, pp. 32-33] or [15, Lemma 2.3, p. 234], and thus defined already there.

1.3 Nearly subharmonic functions. We recall that an upper semicontinuous function $u : D \to [-\infty, +\infty)$ is *subharmonic* if for all $\overline{B^n(x,r)} \subset D$,

$$u(x) \leq \frac{1}{\nu_n\, r^n} \int\limits_{B^n(x,r)} u(y)\, dm(y).$$

The function $u \equiv -\infty$ is considered subharmonic.

We say that a function $u : D \to [-\infty, +\infty)$ is *nearly subharmonic*, if u is Lebesgue measurable, $u^+ \in \mathcal{L}^1_{\rm loc}(D)$, and for all $\overline{B^n(x,r)} \subset D$,

$$u(x) \leq \frac{1}{\nu_n\, r^n} \int\limits_{B^n(x,r)} u(y)\, dm(y).$$

Observe that in the standard definition of nearly subharmonic functions one uses the slightly stronger assumption that $u \in \mathcal{L}^1_{\rm loc}(D)$, see e.g. [6, p. 14]. However, our above, slightly more general definition seems to be more useful, see [21, Proposition 2.1 (iii) and Proposition 2.2 (vi), (vii), pp. 54-55].

1.4 Quasi-nearly subharmonic functions. A Lebesgue measurable function $u : D \to [-\infty, +\infty)$ is *K-quasi-nearly subharmonic*, if $u^+ \in \mathcal{L}^1_{\rm loc}(D)$ and if there is a constant $K = K(n,u,D) \geq 1$ such that for all $\overline{B^n(x,r)} \subset D$,

$$u_M(x) \leq \frac{K}{\nu_n\, r^n} \int\limits_{B^n(x,r)} u_M(y)\, dm(y) \tag{3}$$

for all $M \geq 0$, where $u_M := \sup\{u, -M\} + M$. A function $u : D \to [-\infty, +\infty)$ is *quasi-nearly subharmonic*, if u is K-quasi-nearly subharmonic for some $K \geq 1$.

A Lebesgue measurable function $u : D \to [-\infty, +\infty)$ is *K-quasi-nearly subharmonic n.s. (in the narrow sense)*, if $u^+ \in \mathcal{L}^1_{\rm loc}(D)$ and if there is a constant $K = K(n,u,D) \geq 1$ such that for all

$\overline{B^n(x,r)} \subset D$,

$$u(x) \leq \frac{K}{\nu_n\, r^n} \int\limits_{B^n(x,r)} u(y)\, dm(y). \tag{4}$$

A function $u : D \to [-\infty, +\infty)$ is *quasi-nearly subharmonic n.s.*, if u is K-quasi-nearly subharmonic n.s. for some $K \geq 1$.

Quasi-nearly subharmonic functions (perhaps with a different terminology), or, essentially, perhaps just functions satisfying a certain generalized mean value inequality, more or less of the form (3) or (4) above, have previously been considered or used at least in [3, 25, 8, 14, 24, 5, 11, 9, 23, 15, 10, 16, 17, 18, 13, 19, 20, 21, 7]. We recall here only that this function class includes, among others, subharmonic functions, and, more generally, quasisubharmonic and nearly subharmonic functions (for the definitions of these, see above and e.g. [6]), also functions satisfying certain natural growth conditions, especially certain eigenfunctions, polyharmonic functions, subsolutions of certain general elliptic equations. Also, the class of Harnack functions is included, thus, among others, nonnegative harmonic functions as well as nonnegative solutions of some elliptic equations. In particular, the partial differential equations associated with quasiregular mappings belong to this family of elliptic equations, see Vuorinen [26]. Observe that already Domar [2] has pointed out the relevance of the class of (nonnegative) quasi-nearly subharmonic functions.

To motivate the reader still further, we recall here the following, see e.g. [13, Proposition 1, Theorem A, Theorem B, p. 91] and [21, Proposition 2.1 and Proposition 2.2, pp. 54-55]:

(i) *A K-quasi-nearly subharmonic function n.s. is K-quasi-nearly subharmonic, but not necessarily conversely.*

(ii) *A nonnegative Lebesgue measurable function is K-quasi-nearly subharmonic if and only if it is K-quasi-nearly subharmonic n.s.*

(iii) *A Lebesgue measurable function is* 1*-quasi-nearly subharmonic if and only if it is* 1*-quasi-nearly subharmonic n.s. and if and only if it is nearly subharmonic (in the sense defined above).*

(iv) *If $u : D \to [0, +\infty)$ is quasi-nearly subharmonic and $p > 0$, then u^p is quasi-nearly subharmonic. Especially, if $h : D \to \mathbb{R}$ is harmonic and $p > 0$, then $\mid h \mid^p$ is quasi-nearly subharmonic.*

(v) *If $u : D \to [-\infty, +\infty)$ is quasi-nearly subharmonic n.s., then either $u \equiv -\infty$ or u is finite almost everywhere in D, and $u \in \mathcal{L}^1_{\mathrm{loc}}(D)$.*

1.5 Admissible functions. A function $\varphi : [0, +\infty) \to [0, +\infty)$ is *admissible*, if it is strictly increasing, surjective and there are constants $C_0 = C_0(\varphi) \geq 1$ and $r_0 > 0$ such that

$$\varphi(2t) \leq C_0\, \varphi(t) \quad \text{and} \quad \varphi^{-1}(2s) \leq C_0\, \varphi^{-1}(s) \tag{5}$$

for all s, t, $0 \leq s$, $t \leq r_0$.

Functions $\varphi_1(t) = t^\tau$, $\tau > 0$, or, more generally, nonnegative, increasing surjective functions $\varphi_2(t)$ which satisfy the Δ_2-condition and for which the functions $t \mapsto \frac{\varphi_2(t)}{t}$ are increasing, are examples of admissible functions. Further examples are $\varphi_3(t) = c\,t^\alpha[\log(\delta + t^\gamma)]^\beta$, where $c > 0$, $\alpha > 0$, $\delta \geq 1$, and $\beta, \gamma \in \mathbb{R}$ are such that $\alpha + \beta\gamma > 0$. For more examples, see [15, 18].

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be an admissible function and let $\alpha > 0$. One says that $\zeta \in \partial D$ is (φ, α)-*accessible*, shortly *accessible*, if

$$\Gamma_\varphi(\zeta, \alpha) \cap B^n(\zeta, \rho) \neq \emptyset$$

for all $\rho > 0$. Here

$$\Gamma_\varphi(\zeta, \alpha) = \{\, x \in D \,:\, \varphi(|x - \zeta|) < \alpha\, \delta(x) \,\},$$

and it is called a (φ, α)-*approach region*, shortly an *approach region, in D at ζ*. Choosing $\varphi(t) = t$ (in the case of the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$) one gets the familiar Stolz approach region. Choosing $\varphi(t) = t^\tau$, $\tau \geq 1$, say, one gets more general approach regions, see [23].

1.6 Let $0 \leq d \leq n$. A set $E \subset \mathbb{R}^n$ is *Ahlfors-regular with dimension d* if it is closed and there is a constant $C_4 > 0$ so that

$$C_4^{-1} r^d \leq H^d(E \cap B^n(x, r)) \leq C_4 r^d$$

for all $x \in E$ and $r > 0$. The smallest constant C_4 is called the *regularity constant* for E. Simple examples of Ahlfors-regular sets include d-planes and d-dimensional Lipschitz graphs. Also certain Cantor sets and self-similar sets are Ahlfors-regular. For more details, see [1, pp. 9-10].

2 The results

2.1 First a partial generalization to Pavlović's result [12, Theorem 1, pp. 433-434] or Theorem B above. Observe that though the constant C below in (6) does depend on K, it is, nevertheless, otherwise independent of the (K-)quasi-nearly subharmonic function u.

Theorem 1 *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, $\Omega \neq \mathbb{R}^n$, such that its boundary $\partial\Omega$ is Ahlfors-regular with dimension d, $0 \leq d \leq n$. Let $u : \Omega \to [0, +\infty)$ be a K-quasi-nearly subharmonic function. Let $\varphi : [0, +\infty) \to [0, +\infty)$ be an admissible function, with constants r_0 and C_0. Let $\alpha > 0$ be arbitrary. Let $\rho_0 := \min\{\, r_0/2^{1+\alpha}, r_0/2^{3\alpha C_0}, \varphi(r_0)/\alpha \,\}$. Let $\gamma \in \mathbb{R}$ be such that*

$$\int_\Omega \delta(x)^\gamma\, u(x)\, dm(x) < +\infty.$$

Then there is a constant $C = C(n, \Omega, d, \varphi, \alpha, \gamma, K)$ such that for all $\rho \leq \rho_0$,

$$\int_{\partial\Omega} \sup_{x \in \Gamma_{\varphi,\rho}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma} [\varphi^{-1}(\delta(x))]^{-d} u(x) \,\}\, dH^d(\zeta) \leq C \int_{\Omega_{\rho'}} \delta(x)^\gamma\, u(x)\, dm(x),$$

where $\rho' = \frac{4}{3}\rho$ *and*

$$\Gamma_{\varphi,\rho}(\zeta,\alpha) = \{\, x \in \Gamma_\varphi(\zeta,\alpha) : \delta(x) < \rho \,\}.$$

Proof. Proceeding as in [19, proof of Theorem (with $\psi = id$), pp. 31-35] (cf. [15, proof of Theorem, pp. 235-237]) and choosing $K = \partial\Omega$, one obtains

$$\int\limits_{\partial\Omega} M_\rho^{\partial\Omega}(\zeta)\, dH^d(\zeta) \le C \int\limits_{\Omega_{\rho'}} \delta(x)^\gamma\, u(x)\, dm(x)$$

where $\rho' = \frac{4}{3}\rho$ and $M_\rho^{\partial\Omega} : \partial\Omega \to [0,+\infty]$,

$$M_\rho^{\partial\Omega}(\zeta) \sup_{x\in\Gamma_{\varphi,\rho}(\zeta,\alpha)} \frac{\delta(x)^{n+\gamma} u(x)}{[\varphi^{-1}(\delta(x))]^d + H^d(B^n(x, C_1C_2\,\varphi^{-1}(\delta(x))) \cap \partial\Omega)}.$$

Here and below the constants $C_1 = C_1(C_0,\alpha)$, $C_2 = C_2(C_0,\alpha)$ and $C_3 = C_3(C_0,\alpha)$ are, as pointed out above, directly from [19, proof of Lemma 2.3, pp. 32-33] or [15, proof of Lemma 2.3, pp. 234-235]. By this lemma one has, for each $\zeta \in \partial\Omega$ and for each $x \in \Gamma_{\varphi,\rho}(\zeta,\alpha)$, $B^n(x, C_1C_2\varphi^{-1}(\delta(x))) \subset B^n(\zeta, C_1C_2C_3\varphi^{-1}(\delta(x)))$. Since $\partial\Omega$ is Ahlfors-regular with dimension d, we have

$$H^d(B^n(\zeta, C_1C_2C_3\varphi^{-1}(\delta(x))) \cap \partial\Omega) \le C_4[C_1C_2C_3\varphi^{-1}(\delta(x))]^d$$

where also C_4 is a fixed constant. Therefore

$$\begin{aligned} M_\rho^{\partial\Omega}(\zeta) & \sup_{x\in\Gamma_{\varphi,\rho}(\zeta,\alpha)} \frac{\delta(x)^{n+\gamma} u(x)}{[\varphi^{-1}(\delta(x))]^d + H^d(B^n(x, C_1C_2\,\varphi^{-1}(\delta(x))) \cap \partial\Omega)} \\ & \ge \sup_{x\in\Gamma_{\varphi,\rho}(\zeta,\alpha)} \frac{\delta(x)^{n+\gamma} u(x)}{[\varphi^{-1}(\delta(x))]^d + H^d(B^n(\zeta, C_1C_2C_3\,\varphi^{-1}(\delta(x))) \cap \partial\Omega)} \\ & \ge \sup_{x\in\Gamma_{\varphi,\rho}(\zeta,\alpha)} \frac{\delta(x)^{n+\gamma} u(x)}{[\varphi^{-1}(\delta(x))]^d + C_4(C_1C_2C_3)^d[\varphi^{-1}(\delta(x))]^d} \\ & \ge \frac{1}{1+(C_1C_2C_3)^dC_4} \sup_{x\in\Gamma_{\varphi,\rho}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x) \,\}. \end{aligned}$$

Hence

$$\int\limits_{\partial\Omega} \sup_{x\in\Gamma_{\varphi,\rho}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x) \,\}\, dH^d(\zeta) \le C \int\limits_{\Omega_{\rho'}} \delta(x)^\gamma\, u(x)\, dm(x),$$

concluding the proof. □

2.2 Theorem 1 seems to be useful in many situations. For example, with the aid of it one gets the following improvements to Pavlović's result [12, Theorem 1, pp. 433-434] or Theorem B above:

Theorem 2 *Let* Ω, d, u, φ, α, γ *and* ρ_0 *be as above in Theorem 1. Suppose moreover that* $H^d(\partial\Omega) < +\infty$. *Then there is a constant* $C = C(n,\Omega,d,\varphi,\alpha,\gamma,K)$ *such that*

$$\int\limits_{\partial\Omega} \sup_{x\in\Gamma_\varphi(\zeta,\alpha)} \{\delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x)\}\, dH^d(\zeta) \le C \int\limits_{\Omega} \delta(x)^\gamma\, u(x)\, dm(x).$$

Proof. By Theorem 1 (we may clearly assume that $\int_\Omega \delta(x)^\gamma\, u(x)\, dm(x) < +\infty$),

$$\int\limits_{\partial\Omega} \sup_{x\in\Gamma_{\varphi,\rho_0}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x)\,\}\, dH^d(\zeta) \le C \int\limits_{\Omega_{\rho_0'}} \delta(x)^\gamma\, u(x)\, dm(x).$$

Write

$$\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha) := \{\, x \in \Gamma_\varphi(\zeta,\alpha) :\ \delta(x) \ge \rho_0\}.$$

Since

$$\begin{aligned}\sup_{x\in\Gamma_\varphi(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x)\,\} \le{}& \sup_{x\in\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}\varphi^{-1}(\delta(x))]^{-d} u(x)\,\}\\ &+ \sup_{x\in\Gamma_{\varphi,\rho_0}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}\varphi^{-1}(\delta(x))]^{-d} u(x)\,\},\end{aligned}$$

we obtain:

$$\begin{aligned}&\int\limits_{\partial\Omega} \sup_{x\in\Gamma_\varphi(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x)\,\}\, dH^d(\zeta)\\ &\le \int\limits_{\partial\Omega} \sup_{x\in\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x)\,\}\, dH^d(\zeta)\\ &+ \int\limits_{\partial\Omega} \sup_{x\in\Gamma_{\varphi,\rho_0}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x)\,\}\, dH^d(\zeta)\\ &\le \int\limits_{\partial\Omega} \sup_{x\in\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x)\,\}\, dH^d(\zeta) + C \int\limits_{\Omega_{\rho_0'}} \delta(x)^\gamma u(x)\, dm(x)\\ &\le \int\limits_{\partial\Omega} \sup_{x\in\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x)\,\}\, dH^d(\zeta) + C \int\limits_{\Omega} \delta(x)^\gamma u(x)\, dm(x).\end{aligned}$$

It remains to show that

$$\int\limits_{\partial\Omega} \sup_{x\in\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)} \{\delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} u(x)\}\, dH^d(\zeta) \le C \int\limits_{\Omega} \delta(x)^\gamma\, u(x)\, dm(x)$$

for some $C = C(n, \Omega, d, \varphi, \alpha, \gamma, K)$. For all $x \in \Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)$ we have

$$u(x) \le \frac{K}{\nu_n(\frac{\delta(x)}{3})^n} \int\limits_{B(x)} u(y)\, dm(y).$$

Using also the facts that $\frac{2}{3}\delta(x) \leq \delta(y) \leq \frac{4}{3}\delta(x)$ for all $y \in B(x)$, one gets easily:

$$\begin{aligned}
&\int_{\partial\Omega} \sup_{x\in\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)} \{\delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d}u(x)\}\, dH^d(\zeta) \\
&\leq \int_{\partial\Omega} \sup_{x\in\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)} \{\delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} \frac{K}{\nu_n(\frac{\delta(x)}{3})^n} \int_{B(x)} u(y)\, dm(y)\} dH^d(\zeta) \\
&\leq \frac{3^n K}{\nu_n} \int_{\partial\Omega} \sup_{x\in\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)} \{\delta(x)^{\gamma}[\varphi^{-1}(\delta(x))]^{-d} \int_{B(x)} u(y)\, dm(y)\} dH^d(\zeta) \\
&\leq \left(\frac{3}{2}\right)^{|\gamma|} \frac{3^n K}{\nu_n} \int_{\partial\Omega} \sup_{x\in\Gamma^c_{\varphi,\rho_0}(\zeta,\alpha)} \{[\varphi^{-1}(\delta(x))]^{-d} \int_{B(x)} \delta(y)^{\gamma} u(y)\, dm(y)\} dH^d(\zeta) \\
&\leq \frac{3^{|\gamma|+n}K}{2^{|\gamma|}\nu_n} [\varphi^{-1}(\rho_0)]^{-d} H^d(\partial\Omega) \int_{\Omega} \delta(y)^{\gamma} u(y)\, dm(y).
\end{aligned}$$

Thus

$$\int_{\partial\Omega} \sup_{x\in\Gamma_{\varphi}(\zeta,\alpha)} \{\, \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d}u(x)\,\}\, dH^d(\zeta) \leq C \int_{\Omega} \delta(x)^{\gamma}\, u(x)\, dm(x),$$

concluding the proof. □

Corollary *Let $u : B^n \to [0, +\infty)$ be a subharmonic function and let $p > 0$, $\alpha > 1$ and $\gamma > -1 - \max\{\,(n-1)(1-p), 0\,\}$. Then there is a constant $C = C(n, \gamma, p, \alpha)$ such that*

$$\int_{S^{n-1}} \sup_{x\in\Gamma_{id}(\zeta,\alpha)} \{(1-\mid x \mid)^{\gamma+1} u(x)^p\}\, d\sigma(\zeta) \leq C \int_{B^n} (1-\mid x \mid)^{\gamma} u(x)^p\, dm(x).$$

Here id is the identity mapping of $\mathbb{R}^n$ and σ is the spherical (Lebesgue) measure in S^{n-1}.

Remark Observe that Suzuki [24, Theorem 2, pp. 272-273] has shown the following: If $p > 0$ and $\gamma \leq -1 - \max\{\,(n-1)(1-p), 0\,\}$, then the only nonnegative subharmonic function on a bounded domain D of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary satisfying

$$\int_D \delta(x)^{\gamma}\, u(x)^p\, dm(x) < +\infty \tag{6}$$

is the zero function. On the other hand, if $p > 0$ and $\gamma > -1 - \max\{\,(n-1)(1-p), 0\,\}$, then there exist nonnegative non-zero subharmonic functions on $D = B^n$ satisfying (6).

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