CUBO A Mathematical Journal
Vol.11, Nº 04, (59–71). September 2009

Estimates for solutions to nonlinear degenerate elliptic equations

Fabrizio Cuccu,
Dipartimento di Matematica e Informatica,
Università di Cagliari,
Via Ospedale 72, 09124 Cagliari, Italy.
email: fcuccu@unica.it
Petar Popivanov
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
Sofia 1113, Bulgaria.
email: popivano@math.bas.bg
and
Giovanni Porru
Dipartimento di Matematica e Informatica,
Università di Cagliari,
Via Ospedale 72, 09124 Cagliari, Italy.
email: porru@unica.it

ABSTRACT

We find estimates of the L^∞ norm of solutions to special nonlinear degenerate elliptic partial differential equations in terms of norms of the data. We also discuss a special isoperimetric inequality involved in the definition of the ellipticity of the above equations.

RESUMEN

Encontramos estimaciones de la norma L^∞ de las soluciones de ecuaciones diferenciales parciales elípticas degeneradas nolineales en términos de la norma de los datos. Además, discutimos una desigualdad isoperimétrica especial involucrada en la definición de la elipticidad de las ecuaciones anteriormente descritas.

Key words and phrases: *Degenerate elliptic equations; Isoperimetric inequality.*

Math. Subj. Class.: *35B45; 35J70.*

1 Introduction.

We are interested in the estimation of the L^∞ norm of generalized solutions of a special class of nonlinear second order degenerate elliptic partial differential equations in divergent form. The Sobolev type spaces $W_0^{1,p}(\psi, D)$ to whom our solutions belong are defined below. Our main tools in the proposed investigation are a variant Fubini's theorem (see for example [7]) and a generalization of the famous isoperimetric inequality (see [2]).

The condition (3) (see below) is inspired by the main assumption of [8]. Unfortunately, in [8] there are no geometrical conditions leading to the fulfillment of such an assumption, and there are not nontrivial examples satisfying it. We propose here the proof of (3) in the two dimensional case and for functions $\phi(x) = \lambda|x|^\beta$, $\psi(x) = \Lambda|x|^\gamma$ with $0 < \lambda \le \Lambda$ and $\gamma + 1 > \beta \ge \gamma \ge 0$. Thus, the complete study of the generalized isoperimetric inequality (3) is an open and, we think, a difficult problem.

Our main result is the a priori estimate of the L^∞ norm of the solution by means of appropriate norms of ψ and $f\psi^{\frac{1}{s-1}}$, $s > s_0$, and relies heavily on the parameter α of condition (3). Here s_0 is a special number which depends on p and α and it is sharp. In fact, as simple examples show, our equation (1) even in the linear case can possess unbounded solutions for $s = s_0$. Our results are very precise when $\phi(x)$ and $\psi(x)$ are constants, as in this case the classical isoperimetric inequality holds. We propose a special example when the corresponding a priori estimate is sharp.

The paper is organized as follows: main results, special cases illustrating the main theorem, proof of (3) in dimension two and for special radially symmetric functions ϕ and ψ.

2 Main results

Let D be a bounded smooth domain in $\mathbb{R}^N$ and let $1 < p < N$. Consider the following non linear second order degenerate elliptic equation in divergent form

$$-\left((a_{ij}u_{x_i}u_{x_j})^{\frac{p-2}{2}}a_{ij}u_{x_i}\right)_{x_j} = f(x), \quad x \in D. \tag{1}$$

The summation convention from 1 to N over repeated indices is in effect. The matrix $[a_{ij}] = [a_{ij}(x, u, \nabla u)]$ is assumed to be symmetric and elliptic in the sense

$$0 \le \phi(x)|\xi|^p \le \left(a_{ij}\xi_i\xi_j\right)^{\frac{p}{2}} \le \psi(x)|\xi|^p \quad \forall \xi \in \mathbb{R}^N, \tag{2}$$

with $\phi \not\equiv 0$ and $\psi \in L^1(D)$. The functions ϕ and ψ are subject to the following condition: there exist two constants $C > 0$ and $\alpha \in (\frac{p-1}{p}, 1)$ such that, for each Borel set $E \subset D$ with smooth boundary ∂E we have

$$\int_{\partial E} \phi(x)d\sigma \geq C\Big(\int_E \psi(x)dx\Big)^{\alpha}. \tag{3}$$

Note that (1) is the Euler equation of the functional

$$\int_D \big[\big(a_{ij}u_{x_i}u_{x_j}\big)^{\frac{p}{2}} - p\ u\ f\big]dx.$$

We consider solutions $u \in W_0^{1,p}(\psi, D)$, where $W_0^{1,p}(\psi, D)$ is the completion of $C_0^1(D)$ with respect to the norm

$$\|u\| = \Big(\int_D \big(|u|^p + \psi(x)|\nabla u|^p\big)dx\Big)^{\frac{1}{p}}.$$

We shall use the following formula ([7] pag. 37). If $g(x) \geq 0$ is measurable in the sense of Borel in an open set Ω and if $u \in C^{0,1}(\Omega)$ then

$$\int_\Omega g(x)|\nabla u|dx = \int_0^\infty d\tau \int_{F_\tau} g(x)d\sigma, \tag{4}$$

where

$$F_\tau = \{x \in \Omega : |u(x)| = \tau\},$$

and $d\sigma$ stands for the $(N-1)$-Hausdorff measure. The equality (4) has been extended to functions $u \in W_{loc}^{1,1}(D)$ (see for example [6], Theorem 1.1). A detailed theory of Sobolev spaces including $W_{loc}^{1,1}(D)$ can be found in [7].

If $u \in W_0^{1,p}(\psi, D)$ is a solution of equation (1) we shall denote

$$D_t = \{x \in D : u(x) > t\}.$$

By putting $\Omega = D_t$ with $t > 0$, equality (4) yields

$$\int_{D_t} g(x)|\nabla u|dx = \int_t^{\sup u} d\tau \int_{F_\tau} g(x)d\sigma, \tag{5}$$

with

$$\sup u = \sup_{x \in D} u(x), \quad F_\tau = \{x \in D : u(x) = \tau\}.$$

We note that $\partial D_t \subset F_t$.

Lemma 2.1. *If condition* (3) *holds and if* $\phi(x) \leq H(x) \leq \psi(x)$ *then for almost all* $t > 0$ *we have*

$$\int_{F_t} H(x)|\nabla u|^{p-1}d\sigma \geq C^p \frac{(V(t))^{\alpha p}}{\big(-V'(t)\big)^{p-1}},$$

where

$$V(t) = \int_{D_t} \psi(x)dx.$$

Proof. By using the Hölder inequality we find

$$\Big(\int_{F_t} H(x)d\sigma\Big)^p = \Big(\int_{F_t} \big(H(x)|\nabla u|^{p-1}\big)^{\frac{1}{p}} \Big(\frac{H(x)}{|\nabla u|}\Big)^{\frac{p-1}{p}} d\sigma\Big)^p$$

$$\le \int_{F_t} H(x)|\nabla u|^{p-1}d\sigma \Big(\int_{F_t} \frac{H(x)}{|\nabla u|}d\sigma\Big)^{p-1}.$$

It follows that

$$\int_{F_t} H(x)|\nabla u|^{p-1}d\sigma \ge \frac{\Big(\int_{F_t}\phi(x)d\sigma\Big)^p}{\Big(\int_{F_t}\frac{\psi(x)}{|\nabla u|}d\sigma\Big)^{p-1}}. \tag{6}$$

Equality (5) with $g(x) = \frac{\psi(x)}{|\nabla u|}$ yields

$$V(t) = \int_t^{\sup u} d\tau \int_{F_\tau} \frac{\psi(x)}{|\nabla u|}d\sigma.$$

Hence, for almost all $t > 0$ we have

$$V'(t) = -\int_{F_t} \frac{\psi(x)}{|\nabla u|}d\sigma.$$

By using the latter equation and condition (3), from inequality (6) we get the statement of the lemma.

Theorem 2.2. *Assume conditions* (2) *and* (3)*. Let* $s > \frac{1}{p(1-\alpha)}$*, and let* $f\psi^{-1+\frac{1}{s}} \in L^s(D)$*. If* $u \in W_0^{1,p}(\psi, D)$ *is a solution of equation* (1) *then we have*

$$\|u\|_{L^\infty(D)} \le \frac{1}{C^{\frac{p}{p-1}}} \frac{s(p-1)}{ps(1-\alpha)-1} \|f\psi^{\frac{1}{s}-1}\|_{L^s(D)}^{\frac{1}{p-1}} \Big(\int_D \psi(x)dx\Big)^{\frac{ps(1-\alpha)-1}{s(p-1)}}.$$

Proof. Putting

$$H(x) = \frac{\big(a_{ij}u_{x_i}u_{x_j}\big)^{\frac{p}{2}}}{|\nabla u|^p},$$

condition (2) implies

$$\phi(x) \le H(x) \le \psi(x).$$

According to the definition of weak solution of (1) we have that for every $v \in W_0^{1,p}(\psi, D)$

$$\int_D \big(a_{ij}u_{x_i}u_{x_j}\big)^{\frac{p-2}{2}} a_{ij}u_{x_i}v_{x_j}dx = \int_D f(x)v(x)dx.$$

Since $u \in W_0^{1,p}(\psi, D)$, for $t > 0$ we can take $v = (u(x)-t)^+$. We find

$$\int_{D_t} \big(a_{ij}u_{x_i}u_{x_j}\big)^{\frac{p}{2}}dx = \int_{D_t} f(x)(u(x)-t)dx,$$

where

$$D_t = \{x \in D : u(x) > t\}.$$

Recalling our definition of $H(x)$, by the latter equation we have

$$\int_{D_t} H(x)|\nabla u|^p dx = \int_{D_t} f(x)dx \int_t^u d\tau = \int_t^{\sup u} d\tau \int_{D_\tau} f(x)dx. \tag{7}$$

As it concerns the left hand side, we apply (5) with $g(x) = H(x)|\nabla u|^{p-1}$. We find

$$\int_{D_t} H(x)|\nabla u|^p dx = \int_t^{\sup u} d\tau \int_{F_\tau} H(x)|\nabla u|^{p-1} d\sigma.$$

The latter equation and (7) yield

$$\int_t^{\sup u} d\tau \int_{F_\tau} H(x)|\nabla u|^{p-1} d\sigma = \int_t^{\sup u} d\tau \int_{D_\tau} f(x)dx.$$

After a differentiation, for almost all t we get

$$\int_{F_t} H(x)|\nabla u|^{p-1} d\sigma = \int_{D_t} f(x)dx \leq \int_{D_t} |f(x)|dx. \tag{8}$$

Applying Lemma 2.1 we find

$$C^p \frac{(V(t))^{\alpha p}}{\big(-V'(t)\big)^{p-1}} \leq \int_{D_t} |f(x)|\psi^{\frac{1-s}{s}} \psi^{\frac{s-1}{s}} dx \leq \|f\psi^{\frac{1}{s}-1}\|_{L^s(D)} \Big(\int_{D_t} \psi(x)dx\Big)^{\frac{s-1}{s}}.$$

Rearranging we have

$$1 \leq \frac{1}{C^{\frac{p}{p-1}}} \|f\psi^{\frac{1}{s}-1}\|_{L^s(D)}^{\frac{1}{p-1}} (V(t))^{\frac{s(1-p\alpha)-1}{s(p-1)}} (-V'(t)).$$

Finally, integrating over $(0, \sup u)$ we get

$$\sup u \leq \frac{1}{C^{\frac{p}{p-1}}} \frac{s(p-1)}{ps(1-\alpha)-1} \|f\psi^{\frac{1}{s}-1}\|_{L^s(D)}^{\frac{1}{p-1}} (V(0))^{\frac{ps(1-\alpha)-1}{s(p-1)}}.$$

Being $V(0) \leq \int_D \psi(x)dx$ we find

$$\sup u \leq \frac{1}{C^{\frac{p}{p-1}}} \frac{s(p-1)}{ps(1-\alpha)-1} \|f\psi^{\frac{1}{s}-1}\|_{L^s(D)}^{\frac{1}{p-1}} \Big(\int_D \psi(x)dx\Big)^{\frac{ps(1-\alpha)-1}{s(p-1)}}. \tag{9}$$

If u is a solution of equation (1) then $-u$ is a solution the same equation with $-f$ in place of f. Therefore, the estimate (9) also holds for $-u$. The theorem follows.

3 Special cases illustrating Theorem 2.2

We shall begin this section with the case when $\phi(x) = \lambda$ and $\psi(x) = \Lambda$. Then condition (3) holds with

$$\alpha = 1 - \frac{1}{N}, \quad C = \frac{\lambda}{\Lambda^{\frac{N-1}{N}}} N\omega_N^{\frac{1}{N}},$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$. For $1 < p < N$, $s > \frac{N}{p}$, Theorem 2.2 yields

$$\|u\|_{L^\infty(D)} \le \frac{\Lambda}{\left(N\lambda\omega_N^{\frac{1}{N}}\right)^{\frac{p}{p-1}}} \frac{Ns(p-1)}{ps-N} \|f\|_{L^s(D)}^{\frac{1}{p-1}} |D|^{\frac{ps-N}{Ns(p-1)}}. \tag{10}$$

However, in this special case we can prove an inequality sharper than (10). Indeed, since $\lambda \le H(x)$, by the inequality (8) we find

$$\lambda \int_{F_t} |\nabla u|^{p-1} d\sigma \le \|f\|_{L^s(D)} |D_t|^{\frac{s-1}{s}}. \tag{11}$$

Instead of (6) we use the inequality

$$\int_{F_t} |\nabla u|^{p-1} d\sigma \ge \frac{\left(\int_{F_t} d\sigma\right)^p}{\left(\int_{F_t} \frac{1}{|\nabla u|} d\sigma\right)^{p-1}}.$$

Putting $\mu(t) = |D_t|$ we know that

$$\mu'(t) = -\int_{F_t} \frac{1}{|\nabla u|} d\sigma.$$

Using this equality and the familiar isoperimetric inequality (see, for example, [2])

$$\int_{F_t} d\sigma \ge N\omega_N^{\frac{1}{N}} (\mu(t))^{\frac{N-1}{N}}$$

we find

$$\int_{F_t} |\nabla u|^{p-1} d\sigma \ge \frac{N^p \omega_N^{\frac{p}{N}} (\mu(t))^{\frac{p(N-1)}{N}}}{(-\mu'(t))^{p-1}}.$$

Hence, by (11) we get

$$\frac{\lambda N^p \omega_N^{\frac{p}{N}} (\mu(t))^{\frac{p(N-1)}{N}}}{(-\mu'(t))^{p-1}} \le \|f\|_{L^s(D)} (\mu(t))^{\frac{s-1}{s}},$$

and

$$1 \le \frac{\mu^{-1+\frac{1}{p-1}\left(\frac{p}{N}-\frac{1}{s}\right)}(t)}{\left(\lambda N^p \omega_N^{\frac{p}{N}}\right)^{\frac{1}{p-1}}} (-\mu'(t)) \|f\|_{L^s(D)}^{\frac{1}{p-1}}. \tag{12}$$

Integration over $(0, \sup u)$ yields

$$\sup u \le \frac{1}{(N\lambda)^{\frac{1}{p-1}} \omega_N^{\frac{p}{N(p-1)}}} \frac{s(p-1)}{ps-N} \|f\|_{L^s(D)}^{\frac{1}{p-1}} |D|^{\frac{ps-N}{Ns(p-1)}}.$$

The same estimate can be found for $-u$. Therefore,

$$\|u\|_{L^\infty(D)} \le \frac{1}{(N\lambda)^{\frac{1}{p-1}} \omega_N^{\frac{p}{N(p-1)}}} \frac{s(p-1)}{ps-N} \|f\|_{L^s(D)}^{\frac{1}{p-1}} |D|^{\frac{ps-N}{Ns(p-1)}}.$$

The latter inequality improves (10) by the factor λ/Λ.

In case f is bounded in D then we can take $s \to \infty$ and we find

$$\|u\|_{L^\infty(D)} \le \frac{1}{(N\lambda)^{\frac{1}{p-1}} \omega_N^{\frac{p}{N(p-1)}}} \frac{p-1}{p} (\sup_{x\in D} |f(x)|)^{\frac{1}{p-1}} |D|^{\frac{p}{N(p-1)}}. \tag{13}$$

In the case $\lambda = \Lambda$, f is a constant and D is a ball, the inequality (13) is sharp. Indeed, if R is the radius of the ball and $f = A > 0$, with $u(x) = u(r)$ for $|x| = r$, the equation reads as

$$\lambda\big(r^{N-1}|u'|^{p-1}\big)' = r^{N-1}A.$$

Integrating we find

$$-u' = \frac{r^{\frac{1}{p-1}}}{(N\lambda)^{\frac{1}{p-1}}} A^{\frac{1}{p-1}},$$

$$u(r) = \frac{(p-1)A^{\frac{1}{p-1}}}{p(N\lambda)^{\frac{1}{p-1}}} \big[R^{\frac{p}{p-1}} - r^{\frac{p}{p-1}}\big],$$

and

$$\|u\|_{L^\infty(D)} = u(0) = \frac{(p-1)A^{\frac{1}{p-1}}}{p(N\lambda)^{\frac{1}{p-1}}} R^{\frac{p}{p-1}}. \tag{14}$$

Equation (14) yields (13) with the equality sign.

We shall consider now the case when $a_{ij} = |x|^\beta \delta_{ij}$, $\beta \ge 0$, δ_{ij} being the Kronecker symbol. In this case condition (2) holds with $\phi(x) = \psi(x) = |x|^{\frac{\beta p}{2}}$. If we look for condition (3) when E are balls centered in the origin we find $\alpha = 1 - \frac{1}{N+\beta}$. We think that this value of α is correct for all Borel sets E, but we can prove this fact in case of $N = 2$ only (see the next section).

Let us show that the conclusion of Theorem 2.2 is generally false if $f\psi^{\frac{1}{s}-1} \in L^s(D)$ with $s = \frac{N+\beta}{p}$ and $p \ge 2$.

We have

$$\big((a_{ij}u_{x_i}u_{x_j})^{\frac{p-2}{2}} a_{ij}u_{x_i}\big)_{x_j} = \big((|x|^\beta|\nabla u|^2)^{\frac{p-2}{2}} |x|^\beta u_{x_i}\big)_{x_i}.$$

If $u(x)$ is a radial function and $u(x) = v(r)$ for $|x| = r$, we have $u_{x_i} = v'\frac{x_i}{r}$ and

$$\big((|x|^\beta|\nabla u|^2)^{\frac{p-2}{2}} |x|^\beta u_{x_i}\big)_{x_i} = r^{1-N}\Big(r^{N-1+\frac{\beta p}{2}}|v'|^{p-2}v'\Big)'.$$

Consider problem (1) when Ω is a ball B centered in the origin and $f = f(r)$ is a radial function. Then the solution v is radial and satisfies the equation

$$-r^{1-N}\Big(r^{N-1+\frac{\beta p}{2}}|v'|^{p-2}v'\Big)' = f(r). \tag{15}$$

Let B be the ball with radius $1/e$. Consider the unbounded function

$$v(r) = \log(-\log r).$$

We find

$$v' = \frac{1}{r\log r}$$

and

$$-r^{1-N}\left(r^{N-1+\frac{\beta p}{2}}|v'|^{p-2}v'\right)' = r^{1-N}\left(r^{N+\frac{\beta p}{2}-p}(-\log r)^{1-p}\right)'$$

$$= r^{\frac{\beta p}{2}-p}(-\log r)^{1-p}\Big[N+\frac{\beta p}{2}-p+(1-p)\frac{1}{\log r}\Big].$$

Hence, our function v satisfies equation (15) with

$$f(r) = C(r)r^{\frac{\beta p}{2}-p}(-\log r)^{1-p},$$

where $C(r)$ is a bounded function for $r < \frac{1}{e}$.

If $s = \frac{N+\beta}{p}$ we have

$$\left(f(r)\psi^{(\frac{1}{s}-1)}\right)^s = \tilde{C}(r)r^{\frac{\beta p}{2}-N-\beta}(-\log r)^{(1-p)\frac{N+\beta}{p}}.$$

If $\beta > 0$ it is easy to see that $f(r)\psi^{(\frac{1}{s}-1)} \in L^s(B)$ for $s \leq \frac{N+\beta}{p}$ and $p \geq 2$.

The same computations with $\beta = 0$ show that when $f \in L^s(D)$ with $s = \frac{N}{p}$ and $p > \frac{N}{N-1}$ we may have unbounded solutions of equation (1).

4 Proof of the isoperimetric inequality in dimension two and for radially symmetric functions ϕ and ψ

We shall deal with $\phi(x) = \lambda|x|^\beta$, $\psi(x) = \Lambda|x|^\gamma$, $0 < \lambda \leq \Lambda$, $\gamma + 1 > \beta \geq \gamma \geq 0$ in this section.

We use polar coordinates (ρ, θ). If E is a given set, we define a new set E' according to the following rule. For every $\rho > 0$, if $F_\rho = \{x \in \mathbb{R}^2 : |x| = \rho\}$ we replace $E \cap F_\rho$ with the arc with radius ρ, having the same 1-dimensional measure as $E \cap F_\rho$, centered in $(\rho, 0)$. The sets E' are situated symmetrically with respect to $\theta = 0$. We have

$$\int_E |x|^\gamma dx = \int_{E'} |x|^\gamma dx.$$

Indeed, if we integrate from ρ to $\rho + d\rho$ we find $\rho^\gamma l d\rho$, where l is the 1-dimensional measure of $E \cap F_\rho$ (which is equal to the 1-dimensional measure of $E' \cap F_\rho$). Moreover, we have

$$\int_{\partial E} |x|^\beta ds \geq \int_{\partial E'} |x|^\beta ds'.$$

Indeed, if $\beta = 0$ this is the classical isoperimetric inequality (see, for example, [5]). If $\beta > 0$ we can apply this inequality to the region of E enclosed between ρ and $\rho + d\rho$. The boundary integral of this elementary part of E is $\rho^\beta(ds + 2l)$ (l has the same meaning as before). Similarly, the boundary integral of the part of E' enclosed between ρ and $\rho + d\rho$ is $\rho^\beta(ds' + 2l)$. Hence, $ds \geq ds'$ for each value of ρ. The inequality follows.

Therefore, in order to prove condition (3) we can confine ourselves to sets E having the representation

$$E = \{r \leq \rho \leq R, \quad -h(\rho) \leq \theta \leq h(\rho)\}$$

with $r \geq 0$. Of course, $0 \leq h(\rho) \leq \pi$.

Consider first the case $h(\rho) = \pi$ for $0 \leq \rho \leq R$, that is the disc centered in the origin and of radius R. We find

$$\int_{\partial E} \lambda |x|^\beta ds = 2\pi\lambda R^{\beta+1}, \qquad \int_E \Lambda |x|^\gamma dx = \frac{2\pi\Lambda}{\gamma+2} R^{\gamma+2}.$$

Therefore, with

$$\alpha = \frac{\beta+1}{\gamma+2} \tag{20}$$

we have

$$\frac{\int_{\partial E} \lambda |x|^\beta ds}{\left(\int_E \Lambda |x|^\gamma dx\right)^\alpha} = \frac{2\pi\lambda}{\left(\frac{2\pi\Lambda}{\gamma+2}\right)^\alpha}.$$

Let $h(\rho) < \pi$ for $0 \leq \rho \leq R$. Define the new set

$$D_\tau = \{0 \leq \rho \leq R, \quad -\tau \leq \theta \leq \tau\},$$

with τ such that

$$\int_E |x|^\gamma dx = \int_{D_\tau} |x|^\gamma dx.$$

This value of $\tau \in (0, \pi)$ exists because D_π is the disc with radius R and D_0 is the segment $(0, R)$, $E \subset D_\pi$ but $E \neq D_\pi$ and

$$F(\tau) = \int_{D_\tau} |x|^\gamma dx$$

is a continuous monotonically increasing function for $0 < \tau < \pi$. Let us show that

$$\int_{\partial E} |x|^\beta ds \geq 2 \int_L |x|^\beta ds' = \frac{2}{\beta+1} R^{\beta+1}, \tag{21}$$

where L is the segment $\theta = \tau$, $0 \leq \rho \leq R$. Indeed, if ds is the length of the part of the arc ∂E between ρ and $\rho + d\rho$, and if ds' is the length of the part of the segment L between ρ and $\rho + d\rho$, we have $ds \geq 2ds'$. We notice that the segment L is situated at $\theta = \tau$, whereas ∂E has one part situated at $\theta \geq 0$, and the symmetric part situated at $\theta \leq 0$.

Easy computations yield

$$\int_{D_\tau} |x|^\gamma dx = \frac{2\tau}{\gamma+2} R^{\gamma+2}.$$

Therefore, with α as in (20) we have

$$\frac{\int_{\partial E}\lambda|x|^\beta ds}{\left(\int_E \Lambda|x|^\gamma dx\right)^\alpha} \ge \frac{\frac{2\lambda}{\beta+1}}{\left(\frac{2\tau\Lambda}{\gamma+2}\right)^\alpha} > \frac{\frac{2\lambda}{\beta+1}}{\left(\frac{2\pi\Lambda}{\gamma+2}\right)^\alpha}.$$

Now we consider $r > 0$. If $h(r) = \pi$ for $r \le \rho \le R$ we can replace E by the ball $h(r) = \pi$ for $0 \le \rho \le R$. The integral of $\Lambda|x|^\gamma$ over the ball is greater than the integral over E, whereas, the integral of $\lambda|x|^\beta$ over the boundary of the ball is smaller than the integral over ∂E. Hence, in this situation we find

$$\frac{\int_{\partial E}\lambda|x|^\beta ds}{\left(\int_E \Lambda|x|^\gamma dx\right)^\alpha} \ge \frac{2\pi\lambda}{\left(\frac{2\pi\Lambda}{\gamma+2}\right)^\alpha}.$$

Let $h(r) < \pi$ for $r \le \rho \le R$. Define the set

$$G_\tau = \{r \le \rho \le R, \quad -\tau \le \theta \le \tau\},$$

with τ such that

$$\int_E |x|^\gamma dx = \int_{G_\tau} |x|^\gamma dx.$$

We have $0 < \tau < \pi$. Arguing as in the proof of (21), now we find

$$\int_{\partial E} |x|^\beta ds \ge 2\int_L |x|^\beta ds' = \frac{2}{\beta+1}(R^{\beta+1} - r^{\beta+1}), \tag{22}$$

where L is the segment $\theta = \tau$, $r \le \rho \le R$.

Let us show that we also have

$$\int_{\partial E} |x|^\beta ds \ge 2\int_\Gamma |x|^\beta ds' = 2\tau\, r^{\beta+1}, \tag{23}$$

where Γ is the arc $\rho = r$, $0 < \theta < \tau$. Indeed, if ds is the length of the part of the arc ∂E between θ and $\theta + d\theta$, $\theta > 0$, and if ds' is the length of the part of the arc Γ between θ and $\theta + d\theta$, we have $ds \ge ds'$. Recall that ∂E is symmetric with respect to $\theta = 0$.

If we add (22) and (23) we get

$$\int_{\partial E} |x|^\beta ds \ge \frac{1}{\beta+1}(R^{\beta+1} - r^{\beta+1}) + \tau\, r^{\beta+1}.$$

On the other side, direct computation yields

$$\int_E |x|^\gamma dx = \int_{G_\tau} |x|^\gamma dx = 2\tau\int_r^R \rho^{\gamma+1} d\rho = \frac{2\tau}{\gamma+2}(R^{\gamma+2} - r^{\gamma+2}).$$

Therefore, with α as in (20) we find

$$\frac{\int_{\partial E}\lambda|x|^\beta ds}{\left(\int_E \Lambda|x|^\gamma dx\right)^\alpha} \ge \frac{\frac{\lambda}{\beta+1}(R^{\beta+1} - r^{\beta+1}) + \tau\,\lambda\, r^{\beta+1}}{\left(\frac{2\tau\Lambda}{\gamma+2}\right)^\alpha (R^{\gamma+2} - r^{\gamma+2})^\alpha}.$$

If $\tau \geq \frac{1}{\beta+1}$ we get

$$\frac{\frac{\lambda}{\beta+1}(R^{\beta+1}-r^{\beta+1})+\tau\,\lambda\,r^{\beta+1}}{\left(\frac{2\tau\Lambda}{\gamma+2}\right)^{\alpha}(R^{\gamma+2}-r^{\gamma+2})^{\alpha}} \geq \frac{\frac{\lambda}{\beta+1}R^{\beta+1}}{\left(\frac{2\pi\Lambda}{\gamma+2}\right)^{\alpha}(R^{\gamma+2}-r^{\gamma+2})^{\alpha}} \geq \frac{\frac{\lambda}{\beta+1}}{\left(\frac{2\pi\Lambda}{\gamma+2}\right)^{\alpha}}.$$

To discuss the case $\tau < \frac{1}{\beta+1}$ we consider the function

$$F(r,t)=\frac{R^{\beta+1}-r^{\beta+1}+t\,r^{\beta+1}}{\Big(t(R^{\gamma+2}-r^{\gamma+2})\Big)^{\alpha}}$$

for $0<t<1$ and $0<r<R$. It is easy to see that $\frac{\partial F}{\partial t}=0$ for

$$\bar{t}=\frac{\alpha}{1-\alpha}\frac{R^{\beta+1}-r^{\beta+1}}{r^{\beta+1}}.$$

With $t\in(0,1)$, the function $F(r,t)$ is positive, strictly decreasing for $t<\bar{t}$ and strictly increasing for $t>\bar{t}$. Then two cases are considered.

a) $0\leq r\leq R\alpha^{\frac{1}{\beta+1}}$. Then we have $\bar{t}\geq 1$. Therefore, recalling that α is given by equation (20) we find

$$F(r,t)\geq F(r,1)=\frac{R^{\beta+1}}{\big(R^{\gamma+2}-r^{\gamma+2}\big)^{\alpha}}\geq 1, \qquad \forall t\in(0,1).$$

b) $R\alpha^{\frac{1}{\beta+1}}<r<R$.

(i) From $\gamma+1>\beta\geq\gamma\geq 0$ we find

$$\frac{1}{2}\leq\frac{\gamma+1}{\gamma+2}\leq\alpha=\frac{\beta+1}{\gamma+2}<1,$$

which implies $2\alpha-1\geq 0$ and $\alpha(\gamma+1)-\beta(1-\alpha)>0$.

(ii) Since $\frac{\alpha}{r^{\beta+1}}<\frac{1}{R^{\beta+1}}$ we have

$$\begin{aligned}
F(r,t)\geq F(r,\bar{t}) &= \frac{1}{1-\alpha}\frac{R^{\beta+1}-r^{\beta+1}}{\Big(\frac{\alpha}{1-\alpha}\frac{R^{\beta+1}-r^{\beta+1}}{r^{\beta+1}}(R^{\gamma+2}-r^{\gamma+2})\Big)^{\alpha}}\\
&\geq \frac{1}{1-\alpha}\frac{R^{\beta+1}-r^{\beta+1}}{\Big(\frac{1}{1-\alpha}\frac{R^{\beta+1}-r^{\beta+1}}{R^{\beta+1}}(R^{\gamma+2}-r^{\gamma+2})\Big)^{\alpha}}\\
&= (1-\alpha)^{\alpha-1}\frac{(R^{\beta+1}-r^{\beta+1})^{1-\alpha}R^{\alpha(\beta+1)}}{(R^{\gamma+2}-r^{\gamma+2})^{\alpha}}.
\end{aligned}$$

(iii) As we know from the theory of homogeneous functions,

$$R^{\beta+1}-r^{\beta+1}\geq C_{\beta}(R-r)(R+r)^{\beta}$$

and

$$R^{\gamma+2} - r^{\gamma+2} \le C_\gamma (R-r)(R+r)^{\gamma+1},$$

for suitable positive constants C_β and C_γ. This implies

$$\frac{(R^{\beta+1} - r^{\beta+1})^{1-\alpha} R^{\alpha(\beta+1)}}{(R^{\gamma+2} - r^{\gamma+2})^\alpha} \ge \frac{C_\beta^{1-\alpha}}{C_\gamma^\alpha} \frac{R^{\alpha(\beta+1)}}{(R-r)^{2\alpha-1}(R+r)^{\alpha(\gamma+1)-\beta(1-\alpha)}}$$

$$\ge \frac{C_\beta^{1-\alpha}}{C_\gamma^\alpha} \frac{R^{\alpha(\beta+1)}}{R^{2\alpha-1}(2R)^{\alpha(\gamma+1)-\beta(1-\alpha)}} = \frac{C_\beta^{1-\alpha}}{C_\gamma^\alpha 2^{\alpha(\gamma+1)-\beta(1-\alpha)}}.$$

Therefore, for this kind of sets E there is $C_{\beta,\gamma} > 0$ such that

$$\frac{\int_{\partial E} \lambda |x|^\beta ds}{\left(\int_E \Lambda |x|^\gamma dx\right)^\alpha} \ge C_{\beta,\gamma} \frac{\lambda}{\Lambda^\alpha}.$$

Let $h(\rho) = \pi$ for $0 \le \rho \le r$ and $h(\rho) < \pi$ for $r < \rho \le R$. If there is a set $G_\tau = \{r \le \rho \le R, \ \ -\tau \le \theta \le \tau\}$, with $\tau < \pi$ such that

$$\int_E |x|^\gamma dx = 2 \int_{G_\tau} |x|^\gamma dx,$$

then we can argue as in the previous case and find

$$\frac{\int_{\partial E} \lambda |x|^\beta ds}{\left(\int_E \Lambda |x|^\gamma dx\right)^\alpha} \ge C_{\beta,\gamma} \frac{\lambda}{(2\Lambda)^\alpha}.$$

Otherwise we must have

$$\int_E |x|^\gamma dx \ge 2 \int_{G_\pi} |x|^\gamma dx.$$

Since $\int_{B_R} |x|^\gamma dx \ge \int_E |x|^\gamma dx$ and $G_\pi = B_R \setminus B_r$, this implies that

$$\int_{B_R} |x|^\gamma dx \ge 2 \int_{B_R} |x|^\gamma dx - 2 \int_{B_r} |x|^\gamma dx,$$

from which we get

$$\left(\frac{r}{R}\right)^{\gamma+2} \ge \frac{1}{2}. \tag{24}$$

On the other side, since $B_r \subset E \subset B_R$ we have

$$\int_E |x|^\gamma dx \le \int_{B_R} |x|^\gamma dx,$$

$$\int_{\partial E} \lambda |x|^\beta ds \ge \int_{\partial B_r} \lambda |x|^\beta ds.$$

Hence,

$$\frac{\int_{\partial E} \lambda |x|^\beta ds}{\left(\int_E \Lambda |x|^\gamma dx\right)^\alpha} \ge \frac{\int_{\partial B_r} \lambda |x|^\beta ds}{\left(\int_{B_R} \Lambda |x|^\gamma dx\right)^\alpha} = \frac{\frac{2\pi\lambda}{\beta+1}}{\left(\frac{2\pi\Lambda}{\gamma+2}\right)^\alpha} \left(\frac{r}{R}\right)^{\beta+1}. \tag{25}$$

Estimates (24) and (25) yield

$$\frac{\int_{\partial E}\lambda|x|^{\beta}ds}{\left(\int_{E}\Lambda|x|^{\gamma}dx\right)^{\alpha}} \geq \frac{\frac{2\pi\lambda}{\beta+1}}{\left(\frac{4\pi\Lambda}{\gamma+2}\right)^{\alpha}}.$$

The case E has the representation $h(\rho) \leq \pi$ for $0 \leq \rho \leq R$, can be reduced to one of the previous cases. Indeed, let $b = \sup\{\rho : h(\rho) = \pi\}$. If $b = R$ we replace E with the ball $\tilde{E}$ with radius R. If $b < R$ we replace E with the set $\tilde{E}$ having the representation $\tilde{h}(\rho) = \pi$ for $0 \leq \rho \leq b$, and $\tilde{h}(\rho) = h(\rho)$ for $b < \rho \leq R$. The integral of $\Lambda|x|^{\gamma}$ over $\tilde{E}$ is greater than the corresponding integral over E, whereas, the integral of $\lambda|x|^{\beta}$ over $\partial\tilde{E}$ is smaller than the corresponding integral over ∂E.

The case E has the representation $h(\rho) \leq \pi$ for $r \leq \rho \leq R$ can be treated similarly. If b is as before and $b = R$ we replace E with the ball $\tilde{E}$ with radius R. If $b < R$ we replace E with the set $\tilde{E}$ having the representation $\tilde{h}(\rho) = \pi$ for $0 \leq \rho \leq b$, and $\tilde{h}(\rho) = h(\rho)$ for $b < \rho \leq R$.

This way we have completed the proof of (3) in our special case.

Received: May 2008. Revised: September 2008.

References

[1] Agmon, S. Douglis, A. and Nirenberg, L., *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* I, Commun. Pure Appl. Math., **12** (1959), 623–727.

[2] Evans, L. C. and Gariepy, R. F., *Measure theory and fine properties of functions*, CRC Press 1992.

[3] Federer, H., *Curvature measures* Transaction AMS, Vol. 93 (1959), 418–451.

[4] Gilbarg, D. and Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin, 1977.

[5] Kawohl, B., *Rearrangements and Convexity of Level Sets in PDE,* Lectures Notes in Mathematics, 1150, Berlin, 1985.

[6] Malý, J. Swanson, D. and Ziemer, W. P., *The coarea formula for Sobolev mappings*, arXiv:math.CA/0112008 v1, 1 Dec. 2001, 1–16.

[7] Mazya, V., *Sobolev spaces,* Springer, Berlin, 1985.

[8] Novruzov, A. A., *On the maximum principle of elliptic equations of the second order with non-negative characteristic form,* Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., 20 (2000), 91–96.