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K-Theory of an Algebra of Pseudodifferential Operators on a Noncompact Manifold

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ABSTRACT

Let $\mathcal{A}$ denote the C*-algebra of bounded operators on $L^2(\mathbb{R}\times\mathbb{S}^1)$ generated by: all multiplications $a(M)$ by functions $a \in C^\infty(\mathbb{S}^1)$, all multiplications $b(M)$ by functions $b \in C([-\infty,+\infty])$, all multiplications by 2π-periodic continuous functions, $\Lambda = (1-\Delta_{\mathbb{R}\times\mathbb{S}^1})^{-1/2}$, where $\Delta_{\mathbb{R}\times\mathbb{S}^1}$ is the Laplacian operator on $L^2(\mathbb{R}\times\mathbb{S}^1)$, and $\partial_t\Lambda,\ \partial_x\Lambda$, for $t \in \mathbb{R}$ and $x \in \mathbb{S}^1$. We compute the K-theory of $\mathcal{A}$ and of its quotient by the ideal of compact operators.

RESUMEN

Denotemos $\mathcal{A}$ la C*-algebra de operadores acotados sobre $L^2(\mathbb{R}\times\mathbb{S}^1)$ gerados por todas las multiplicaciones $a(M)$ por funciones $a \in C^\infty(\mathbb{S}^1)$, todas las multiplicaciones $b(M)$ por funciones $b \in C([-\infty,+\infty])$, todas las multiplicaciones por funciones continuas 2π-periódicas, $\Lambda = (1-\Delta_{\mathbb{R}\times\mathbb{S}^1})^{-1/2}$, donde $\Delta_{\mathbb{R}\times\mathbb{S}^1}$ es el operador de Laplace sobre $L^2(\mathbb{R}\times\mathbb{S}^1)$, y $\partial_t\Lambda,\ \partial_x\Lambda$, para $t \in \mathbb{R}$ y $x \in \mathbb{S}^1$. Nosotros calculamos la K-teoria de $\mathcal{A}$ y su cuociente por el ideal de los operadores compactos.

Key words and phrases: *K-Theory, pseudodifferential operators.*

Math. Subj. Class.: *46L80, 47G30.*

1 Introduction

Let $\mathbb{B}$ denote an n-dimensiomal compact Riemannian manifold. Write $\Omega = \mathbb{R} \times \mathbb{B}^1$ and let Δ_Ω denote its Laplacian. Define $\mathcal{A}$ as the C*-algebra of bounded operators on $L^2(\Omega)$ generated by:

(i) all $a(M_x)$, operators of multiplication by $a \in C^\infty(\mathbb{B})$, $[a(M_x)f](t,x) = a(x)f(t,x)$;

(ii) all $b(M_t)$, operators of multiplication by $b \in C([-\infty,+\infty])$, $[b(M_t)f](t,x) = b(t)f(t,x)$;

(iii) every multiplication by 2π-periodic continuous functions;

(iv) $\Lambda = (1-\Delta_\Omega)^{-1/2}$;

(v) $\frac{1}{i}\frac{\partial}{\partial t}\Lambda,\ t \in \mathbb{R}$;

(vi) $L\Lambda$, where L ia a first order differential operator on $\mathbb{B}$.

$\mathcal{A}$ contains a class of classical pseudodifferential operators, including all $A = L\Lambda^N$, where L is a differential operator of order N with coefficients approaching periodic functins at infinity. $A : L^2(\Omega) \to L^2(\Omega)$ is Fredholm if and only if so is $L : H^N(\Omega) \to L^2(\Omega)$ and they have the same index.

The structure of $\mathcal{A}$ is given in [4], so we will mention some of those results. The principal symbol extends to a $*$-homomorphism $\sigma : \mathcal{A} \to C_b(S^*\Omega)$, where $C_b(S^*\Omega)$ denotes the algebra of continuous funtions on the co-sphere bundle of Ω. Let $\mathcal{E}$ denote the kernel of σ and $\mathcal{K}_\Omega := \mathcal{K}(L^2(\Omega))$ the ideal of compact operators on $L^2(\Omega)$.

Theorem 1. *There is a $*$-isomorphism*

$$\Psi : \frac{\mathcal{E}}{\mathcal{K}_\Omega} \longrightarrow C(S^1 \times \{-1,+1\}, \mathcal{K}_{\mathbb{Z}\times\mathbb{B}}). \tag{1}$$

In fact, $\mathcal{E}$ is the commutator ideal of $\mathcal{A}$. Composing the canonical projection $\mathcal{E} \to \mathcal{E}/\mathcal{K}_\Omega$ with Ψ, we can extend the map $\mathcal{E} \to C(S^1 \times \{-1,+1\}, \mathcal{K}_{\mathbb{Z}\times\mathbb{B}})$ and obtain a $*$-homomorphism $\gamma : \mathcal{A} \to C(S^1 \times \{-1,+1\}, \mathcal{L}(L^2(\mathbb{Z}\times\mathbb{B})))$.

From now on, we will just work in the case that the manifold $\mathbb{B}$ is the circle $\mathbb{S}^1$. Therefore, the class of generators $L\Lambda$ can be replaced by the operator $\frac{1}{i}\frac{\partial}{\partial\beta}\Lambda,\ \beta \in \mathbb{S}^1$, since $\mathbb{S}^1$ has trivial tangent bundle.

Definition 2. *Let $\mathcal{D}$ be a C*-algebra and $\mathcal{I}$ its commutator ideal. We call symbol space of $\mathcal{D}$ the compact space M such that we have $\mathcal{D}/\mathcal{I} \cong C(M)$ by the Gelfand map.*

With this definition, we observe that the map σ is the composition $\mathcal{A} \to \mathcal{A}/\mathcal{E} \to C(M_\mathcal{A})$, where $M_\mathcal{A}$ is the symbol space of $\mathcal{A}$. Given $A \in \mathcal{A}$, we say that the symbol of A is the continuous function $\sigma_A \in C(M_\mathcal{A})$ associated to the class A in $\mathcal{A}/\mathcal{E}$ by the Gelfand isomorphism.

Theorem 3. *The symbol space of $\mathcal{A}$ is the set*

$$\mathbf{M}_{\mathcal{A}} = \{(t, x, (\tau, \xi), e^{i\theta});\ t \in [-\infty, +\infty],\ x \in \mathbb{S}^1,\ (\tau, \xi) \in \mathbb{R}^2 : \tau^2 + \xi^2 = 1, \\ \theta \in \mathbb{R}\ e\ \theta = t\ se\ |t| < \infty\},$$

and the symbols of the generators (or classes of them) of $\mathcal{A}$ are given below as functions on $(t, x, (\tau, \xi), e^{i\theta})$ in the same order that they appear in the definition:

$$a(x), \quad b(t), \quad e^{ij\theta}, \quad 0, \quad \tau, \quad \xi. \tag{2}$$

For each $\varphi \in \mathbb{R}$, let U_φ be the operator on $L^2(S^1)$ given by $U_\varphi f(z) = z^{-\varphi} f(z),\ z \in S^1$, and let $Y_\varphi := F_d U_\varphi F_d^{-1}$, where $F_d : L^2(S^1) \to \ell^2(\mathbb{Z})$ is the discrete Fourier transform. So, Y_φ is an unitary operator such that, for all $k \in \mathbb{Z}$, $(Y_k u)_j = u_{j+k}$, and $Y_\varphi Y_\omega = Y_{\varphi+\omega}$.

Consider the following functions defined on $\mathbb{R}$ and taking values in the algebra of bounded operators on $L^2(\mathbb{S}^1)$:

$$B_4(\tau) = \tilde{\Lambda}(\tau)\ ,\quad B_5(\tau) = -\tau\tilde{\Lambda}(\tau)\ ,\quad B_6(\tau) = \frac{1}{i}\frac{\partial}{\partial\beta}\tilde{\Lambda}(\tau),\ \tau \in \mathbb{R},\ \beta \in \mathbb{S}^1, \tag{3}$$

where $\tilde{\Lambda}(\tau) = (1 + \tau^2 - \Delta_{\mathbb{S}^1})^{-1/2}$.

Proposition 4. *For each generator A of $\mathcal{A}$, $\gamma_A(e^{2\pi i\varphi}, \pm 1)$ is given respectively by:*

$$a(M_x)\ ,\quad b(\pm\infty)\ ,\quad Y_{-j}\ ,\quad Y_\varphi B_i(\varphi - M_j) Y_{-\varphi}\ ,\ i = 4, 5, 6,$$

where $B_i(\varphi - M_j) \in \mathcal{L}(\ell^2(\mathbb{Z}, L^2(\mathbb{S}^1)))$, $i = 4, 5, 6$, are the operators of multiplicaion by the sequences $B_i(\varphi - j) \in L^2(\mathbb{S}^1)$, and B_i's are given in (3).

In this note, we announce results about the calculus of the K-theory of $\mathcal{A}$. Their proofs will appear in a forthcoming paper.

2 C*-subalgebras of $\mathcal{A}$

Now, we define two C*-subalgebras of $\mathcal{A}$ and mention some facts about their structure. With the results mentioned here, we can compute their K-theory in the next section.

Let $\mathcal{A}^\dagger$ the C*-algebra of bounded operators on $L^2(\Omega)$ generated by the class of operators of multiplication by $a \in C^\infty(\mathbb{S}^1)$, Λ, $\frac{1}{i}\partial_t\Lambda$ and $\frac{1}{i}\partial_\beta\Lambda$, and let $\mathcal{A}^\diamond$ denote the C*-algebra generated by the same operators of $\mathcal{A}^\dagger$ and operators of multiplication by 2π-periodic continuous functions.

Based on Cordes, [1], and Melo, [4], we can prove the following result.

Theorem 5. *Let $\mathcal{E}^\dagger$ and $\mathcal{E}^\diamond$ denote the commutator ideals of $\mathcal{A}^\dagger$ and $\mathcal{A}^\diamond$, respectively. Then $\mathcal{E}^\dagger \cong C_0(\mathbb{R}, \mathcal{K}(L^2(\mathbb{S}^1)))$ and $\mathcal{E}^\diamond \cong C(S^1, \mathcal{K}(L^2(\mathbb{Z} \times \mathbb{S}^1)))$.*

The $*$-isomorphism $\mathcal{E}^\diamond \cong C(S^1, \mathcal{K}(L^2(\mathbb{Z} \times \mathbb{S}^1)))$ can be extended to a $*$-homomorphism γ' : $\mathcal{A}^\diamond \to C(S^1, \mathcal{L}(L^2(\mathbb{Z} \times \mathbb{S}^1)))$ given by $\gamma'_A(z) = \gamma_A(z, \pm 1)$ for every $A \in \mathcal{A}^\diamond \subset \mathcal{A}$.

Knowing these isomorphisms, we are able to describe the symbol spaces of both C*-subalgebras of $\mathcal{A}$.

Theorem 6. *The symbol space of $\mathcal{A}^\dagger$ is homeomorphic to $\mathbb{S}^1 \times S^1$ and the symbol space of $\mathcal{A}^\diamond$ is homeomorphic to $\mathbb{S}^1 \times S^1 \times S^1$. Moreover, the symbol of an operator in $\mathcal{A}^\dagger$ or $\mathcal{A}^\diamond$ coincides with the symbol of same operator regarded as an element of $\mathcal{A}$.*

We can see $\mathcal{A}^\dagger$ as a C*-subalgebra of $\mathcal{L}(L^2(\mathbb{R} \times \mathbb{S}^1))$ as well. So, if we conjugate $\mathcal{A}^\dagger$ with $F \otimes I_{\mathbb{S}^1}$, where $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier transform and $I_{\mathbb{S}^1}$ is the identity on $L^2(\mathbb{S}^1)$, we obtain an algebra that can be viewed as a subalgebra of $C_b(\mathbb{R}, \mathcal{L}(L^2(\mathbb{S}^1)))$. Let $\mathcal{B}^\dagger$ be this algebra.

The idea is to define an action, and then, to show that that $\mathcal{A}^\diamond$ has a crossed product structure. Consider α the following translation-by-one automorphism:

$$[\alpha(B)](\tau) = B(\tau - 1),\ \tau \in \mathbb{R},\ \mathbb{L} \in \mathcal{B}^\dagger.$$

Let $\mathcal{B}^\diamond$ the algebra $\mathcal{A}^\diamond$ conjugated with the Fourier transform. Analogously to [5, Theorem 8], we obtain the next result.

Theorem 7. *Let α be as described above. We have:*

$$\mathcal{B}^\diamond \cong \mathcal{B}^\dagger \rtimes_\alpha \mathbb{Z}$$

where $\mathcal{B}^\dagger \rtimes_\alpha \mathbb{Z}$ is the envelopping C-algebra([2], 2.7.7) of the Banach algebra with involution $\ell^1(\mathbb{Z}, \mathcal{B}^\dagger)$ of all summable $\mathbb{Z}$-sequences in $\mathcal{B}^\dagger$, equipped with the product*

$$(\mathbf{A} \cdot \mathbf{B})(n) = \sum_{k \in \mathbb{Z}} A_k \alpha^k (B_{n-k}),\ n \in \mathbb{Z},\ \mathbf{A} = (A_k)_{k \in \mathbb{Z}},\ \mathbf{B} = (B_k)_{k \in \mathbb{Z}},$$

and involution $\mathbf{A}^*(n) = \alpha^n(A^*_{-n})$.

3 K-Theory

In this section, we compute the K-groups of the algebras that we were defined in this work.

The main idea is to compute the connecting maps in the K-theory six-term exact sequence associated to the C*-algebra short exact sequence induced by the principal symbol

$$0 \longrightarrow \mathcal{E} = ker\sigma \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \frac{\mathcal{A}}{\mathcal{E}} \cong Im\sigma \longrightarrow 0, \tag{4}$$

where i is the inclusion and π is the canonical projection.

But before we begin the computation involving the largest algebra, we start with $\mathcal{A}^\dagger$ and $\mathcal{A}^\diamond$using the same idea as above.

Studying the K-theory six-term exact sequence induced by the sequence

$$0 \longrightarrow \mathcal{E}^\dagger \xrightarrow{i} \mathcal{A}^\dagger \xrightarrow{\pi} \frac{\mathcal{A}^\dagger}{\mathcal{E}^\dagger} \longrightarrow 0,$$

we just obtain partial results about the K-theory of $\mathcal{A}^\dagger$, because we do not have a explicit description of the range of the exponential map.

The same thing happens with the exponential map in the sequence involving $\mathcal{A}^\diamond$. To compute the index map $\delta_1^\diamond : K_1(\mathcal{A}^\diamond/\mathcal{E}^\diamond) \to K_0(\mathcal{E}^\diamond)$, it was necessary to use the Fedosov index formula [3].

Theorem 8. *$\delta_1^\diamond$ is surjective.*

From Theorem 7, we have $\mathcal{B}^\diamond \cong \mathcal{B}^\dagger \rtimes_\alpha \mathbb{Z}$. Then the K-groups of $\mathcal{B}^\dagger$ and $\mathcal{B}^\diamond$ fit in the Pimsner-Voiculescu exact sequence, [6, theorem 2.4]:

$$\begin{array}{ccccc} K_0(\mathcal{B}^\dagger) & \stackrel{id_*-\alpha_*^{-1}}{\longrightarrow} & K_0(\mathcal{B}^\dagger) & \stackrel{\varphi_*\circ i_*}{\longrightarrow} & K_0(\mathcal{B}^\diamond) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{B}^\diamond) & \stackrel{\varphi_*\circ i_*}{\longleftarrow} & K_1(\mathcal{B}^\dagger) & \stackrel{id_*-\alpha_*^{-1}}{\longleftarrow} & K_1(\mathcal{B}^\dagger) \end{array} \tag{5}$$

Now, we have enough information to conclude that

$$K_0(\mathcal{A}^\dagger) \cong \mathbb{Z}\ ,\quad K_1(\mathcal{A}^\dagger) \cong \mathbb{Z}^2\ ,\quad K_i(\mathcal{A}^\diamond) \cong \mathbb{Z}^3\ ,\quad i = 0, 1.$$

The next step is to work with $\mathcal{A}$. In the begining of this section the sequence induced by the symbol was mentioned. We will rewrite it, quotiented by the ideal of compact operators $\mathcal{K}_\Omega$:

$$0 \longrightarrow \frac{\mathcal{E}}{\mathcal{K}_\Omega} \xrightarrow{i} \frac{\mathcal{A}}{\mathcal{K}_\Omega} \xrightarrow{\pi} \frac{\mathcal{A}}{\mathcal{E}} \longrightarrow 0, \tag{6}$$

where i is the inclusion and π is the canonical projection.

By the isomorphism (1), we know that $\mathcal{E}/\mathcal{K}_\Omega$ can be viewed as two copies of $C(S^1, \mathcal{K}(L^2(\mathbb{Z}\times \mathbb{S}^1)))$. Then,

$$K_i\left(\frac{\mathcal{E}}{\mathcal{K}_\Omega}\right) \cong \mathbb{Z}\oplus\mathbb{Z},\ i = 0, 1.$$

We know the range of σ by Theorem 3. With the help of [5, Proposition 3], we can state the next result.

Proposition 9. $K_0(\mathcal{A}/\mathcal{E}) \cong K_1(\mathcal{A}/\mathcal{E}) \cong \mathbb{Z}^6$.

From sequence (6), we have:

$$\begin{array}{ccccc} \mathbb{Z}^2 \cong K_0(\mathcal{E}/\mathcal{K}_\Omega) & \stackrel{i_*}{\longrightarrow} & K_0(\mathcal{A}/\mathcal{K}_\Omega) & \stackrel{\pi_*}{\longrightarrow} & K_0(\mathcal{A}/\mathcal{E}) \cong \mathbb{Z}^6 \\ \delta_1\ \uparrow & & & & \downarrow\ \delta_0 \\ \mathbb{Z}^6 \cong K_1(\mathcal{A}/\mathcal{E}) & \stackrel{\pi_*}{\longleftarrow} & K_1(\mathcal{A}/\mathcal{K}_\Omega) & \stackrel{i_*}{\longleftarrow} & K_1(\mathcal{E}/\mathcal{K}_\Omega) \cong \mathbb{Z}^2 \end{array} \tag{7}$$

The image of δ_1 is determined the same way as the image of $\delta_1^{\diamond}$.

Theorem 10. *δ_1 is surjective.*

Corollary 11. *$K_0(\mathcal{A}/\mathcal{K}_\Omega)$ has no torsion.*

Theorem 12. *Given the short exact sequence*

$$0 \longrightarrow \mathcal{K}_\Omega \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \frac{\mathcal{A}}{\mathcal{K}_\Omega} \longrightarrow 0,$$

where i is the inclusion and π is the canonical projection, the following statements hold:

(i) $\delta_1 : K_1(\frac{\mathcal{A}}{\mathcal{K}_\Omega}) \to \mathbb{Z}$ *is surjective (there exists a matrix of operators in $\mathcal{A}$ with Fredholm index equal to one);*

(ii) $K_0(\mathcal{A}) \cong K_0(\mathcal{A}/\mathcal{K}_\Omega) \cong \mathbb{Z}^5$;

(iii) $K_1(\mathcal{A}) \cong \mathbb{Z}^4$ *and* $K_1(\mathcal{A}/\mathcal{K}_\Omega) \cong \mathbb{Z}^5$.

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