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Projective Squares in $\mathbb{P}^2$ and Bott's Localization Formula[†]

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ABSTRACT

We give an explicit description of the Hilbert scheme that parametrizes the closed 0-dimensional subschemes of degree 4 in the projective plane that allows us to afford a natural embedding in a product of Grassmann varieties. We also use this description to explain how to apply Bott's localization formula (introduced in 1967 in Bott's work [2]) to give an answer for an enumerative question as used by the first time by Ellingsrud and Strømme in [8] to compute the number of twisted cubics on a general Calabi-Yau threefold which is a complete intersection in some projective space and used later by Kontsevich in [16] to count rational plane curves of degree d passing through $3d-1$ points in general position in the plane.

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RESUMEN

En este trabajo, damos una descripción explícita del esquema de Hilbert, que parametriza los subesquemas cerrados de dimensión cero y grado 4 del plano proyectivo, esto nos permite mapear este esquema en un producto de variedades de Grassmann. Usamos dicha construcción, para explicar como se utiliza la fórmula de localización de Bott (introducida en 1967 por Bott en [2]) para responder una pregunta de Geometria Enumerativa, tal como lo hicieron Ellingsrud y Strømme en [8], para calcular cuantas cúbicas torcidas existen en una variedad de Calabi-Yau tri-dimensional, que es una intersección completa en algún espacio proyectivo, y que fue usada posteriormente por Kontsevich en [16], para contar curvas planas racionales de grado d pasando por $3d-1$ puntos en posición general en el plano.

Key words and phrases: *Hilbert scheme, Bott's localization formula.*

Math. Subj. Class.: *14C05, 14N05.*

1 Introduction

Enumerative geometry has been an active and attractive research subject in math for a long time. A typical problem in enumerative geometry asks for the number of geometric objects of a certain type that satisfy a given set of conditions. For example:

1. *Very easy*: given two distinct points in the plane, how many lines go through all of them? (the answer - a result from Euclidean Geometry - is clearly one.)

2. *Easy*: given $2N$ general lines in the plane, how many N–gons are there with its set of vertices meeting all of them? (easy combinatorial answer: $\{2N-1\}!$ = factorial of odd's numbers between 1 and $2N-1$ (see Section 4).)

3. *Medium*: how many lines lie on a general cubic surface? (Famous answer: 27.) or how many lines lie on a general quintic threefold? (answer: 2875. Hermann Schubert determined this number explicitly at page 72 in [20], see also the computation at page 281 of Cox-Katz's book [4].) or in a more general way: how many lines lie on a general hypersurface of degree $2n-3$ in $\mathbb{P}^n$? (answer: see [11])

4. *Hard*: given $3d-1$ general points in the plane, how many plane rational curves of degree d pass through all of them? (Answer: $N(d)$. $N(d)$ denotes the Gromov-Witten invariants, they have their origins in physics, in the topological sigma models introduced by Witten in [22]. On the other hand, Kontsevich in [16] found a formula that expresses $N(d)$ in terms of $N(e)$ for $e<d$, so a single initial datum is required for the recursion, namely, the case $d=1$, which correspond to the fact that through two distinct points in the plane pass exactly one line. See Kock-Vainsencher [15] for an elementary introduction and chapter 9 in Cox-Katz's book [4].)

In the 19th century, geometers developed a powerful "calculus" for solving enumerative problems. Their method had no rigorous theoretical foundation, but it worked remarkably well. Justifying their results was the subject of Problem 15th on Hilbert's famous list. In the 20th century, enumerative geometry has been reconceptualized and made rigorous in terms of intersection theory on parameter spaces (see Fulton [10], Kleiman-Laksov [14] and Kleiman [13] for a survey).

So, in order to give a correct answer to an enumerative question, the key issue in the study of parameter spaces is to find a compactification. For example, the Kontsevich's moduli space of stable maps is used in [16] to calculate $N(d)$. In theorem 1.9 of Nakajima's book [19] is given an explicit description of the Hilbert scheme that parametrizes the closed 0-dimensional subschemes of degree n in $\mathbb{A}^2$. And second, what kind of techniques can be used on a given parameter space to solve an enumerative problem. Usually, the answer to an enumerative problem is reduced to compute Chern classes of some vector bundles. So, for example in [7] Ellingsrud-Göttsche study the Chern and Segre classes of tautological vector bundles on the Hilbert scheme that parametrizes the closed 0-dimensional subschemes of degree n over a smooth and projective surface over the complex numbers.

The purpose of this article is to explain how to apply Bott's localization formula to give an answer to question 2 above when $N = 4$ using an explicit and elementary description of a parameter space for squares in the plane, that is, the Hilbert scheme that parametrizes the closed 0-dimensional subschemes of degree 4 in $\mathbb{P}^2$

2 Notation and Convention

For any homogeneous ideal I in the ring $\mathbb{C}[x_0, x_1, x_2]$, let I_d denote the homogeneous part of degree d , that is, $I = \oplus_{d=0}^{\infty} I_d$. And when we refer to the Hilbert polynomial associated to the closed subscheme determined by the homogeneous ideal I we refer precisely to the Hilbert polynomial associated to the $\mathbb{C}[x_0, x_1, x_2]$-module $\mathbb{C}[x_0, x_1, x_2]/I$ (see pg. 51 in Hartshorne's book [12]).

Let $\overline{I} = \{f \in \mathbb{C}[x_0, x_1, x_2] \mid \text{ for each } i = 0, 1, 2, \text{ there is an } N_i \text{ such that } f \cdot x_i^{N_i} \in I\}$ be the saturation of the homogeneous ideal I in $\mathbb{C}[x_0, x_1, x_2]$. We say that I is saturated if $I = \overline{I}$.

Let $\mathcal{F}$ denote the vector space of linear forms in the variables x_0, x_1, x_2 and $\mathcal{F}_d$ the vector space of homogeneous forms of degree d. Let $f_1, ..., f_s \in \mathcal{F}_d$ ($f_1, ..., f_s \in \mathbb{C}[x_0, x_1, x_2]$), we denote by $[f_1, ..., f_s]$ ($\langle f_1, ..., f_s \rangle$) the $\mathbb{C}$-vector space generated by $f_1, ..., f_s$ in $\mathcal{F}_d$ (the ideal generated by $f_1, ..., f_s$ in $\mathbb{C}[x_0, x_1, x_2]$).

For each point $\mathrm{p} \in \mathbb{P}^2$, let $\mathcal{F}_d^{\mathrm{p}}$ denote the linear system of forms of degree d vanishing at the point p.

Let $\mathbb{G}_n(\mathcal{F}_d)$ denote the Grassmann variety parametrizing the n-dimensional vector subspaces of $\mathcal{F}_d$. Set $\mathbb{X} = \mathbb{G}_2(\mathcal{F}_2)$ be the Grassmannian of pencils of conics in $\mathbb{P}^2$, with tautological sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F}_2 \longrightarrow \overline{\mathcal{F}_2} \longrightarrow 0 \tag{2.1}$$

where $\mathcal{A} \subset \mathcal{F}_2$ denote a subbundle of rank 2 with fiber over $\pi \in \mathbb{G}_2(\mathcal{F}_2)$ given by the vector subspace $\pi \subset \mathcal{F}_2$.

3 An explicit description of $Hilb^4\mathbb{P}^2$

3.1 Hilbert scheme of points in $\mathbb{P}^2$

Let $Hilb^d\mathbb{P}^2$ be the Hilbert scheme that parametrizes the closed 0-dimensional subschemes of degree d in $\mathbb{P}^2$. As we have a 1-1 correspondence between saturated homogeneous ideals of $\mathbb{C}[x_0, x_1, x_2]$ and closed subschemes of $\mathbb{P}^2$ (see Ex. 5.10 of Chapter II in Hartshorne's book [12]) then we set.

$$Hilb^d\mathbb{P}^2 = \left\{ \begin{array}{l} I \subset \mathbb{C}[x_0, x_1, x_2] \mid I \text{ is saturated homogeneous ideal in } \mathbb{C}[x_0, x_1, x_2] \\ \qquad\qquad\qquad\qquad \text{such that the Hilbert polynomial of the} \\ \qquad\qquad\qquad\qquad \mathbb{C}[x_0, x_1, x_2]\text{-module } \mathbb{C}[x_0, x_1, x_2]/I \text{ is equal to } d. \end{array} \right\}. \tag{3.1}$$

It is known that $Hilb^d\mathbb{P}^2$ is nonsingular of dimension $2d$ (see [9]) and that for each positive integer d it embeds in the Grassmann variety of codimension d subspaces of $\mathcal{F}_d$ (see pg. 34 in [1], Lecture 15 in [18]).

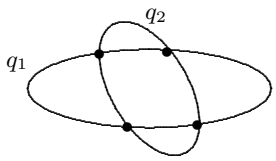
Having in mind (3.1) we are going to give an explicit description of all saturated homogeneous ideals in $Hilb^4\mathbb{P}^2$. For those who are interested in the scheme structure in more detail, we recommend the reading of [1] and [21]. Naturally as suggested by Bézout's Theorem we begin with a pair of conics in the plane.

3.2 Quadruplets determined by conics

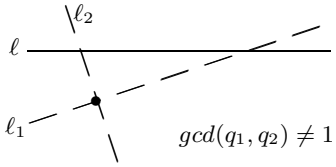
For each $\pi = [q_1, q_2] \in \mathbb{X}$ we can associate the ideal $I_\pi = \langle q_1, q_2 \rangle \subset \mathbb{C}[x_0, x_1, x_2]$ (I_π is a saturated ideal). The variety determined by I_π correspond to the intersection of two conics. Thus, we have the following two possibilities:

$$\left\{ \begin{array}{l} \text{If } gcd(q_1, q_2) = 1 \text{ then according to Bézout's Theorem the number of intersection points} \\ \qquad\qquad\qquad \text{between } q_1 \text{ and } q_2 \text{ should be 2x2=4 points counted with multiplicities.} \\ \text{If } gcd(q_1, q_2) \neq 1 \text{ then we have that } q_1 = \ell\ell_1,\ q_2 = \ell\ell_2 \text{ with } \ell,\ \ell_1 \in \mathbb{P}\big(\mathcal{F}\big) \text{ and } \ell_2 \in \mathbb{P}\big(\mathcal{F}/[\ell_1]\big). \end{array} \right.$$

So in the general case we have the following pictures:



$gcd(q_1, q_2) = 1$



$gcd(q_1, q_2) \neq 1$

Thus it is natural to consider the following subvariety of $\mathbb{X}$. Let

$$\mathbb{Y} = \Big\{ [q_1, q_2] \in \mathbb{X} \mid q_1 = \ell\ell_1,\ q_2 = \ell\ell_2 \text{ with } \ell,\ \ell_1 \in \mathbb{P}\big(\mathcal{F}\big) \text{ and } \ell_2 \in \mathbb{P}\big(\mathcal{F}/[\ell_1]\big) \Big\}. \tag{3.2}$$

Let p be the intersection point of two lines ℓ_1 and ℓ_2, then $\mathbb{Y}$ can be illustrated as follows:

$$\mathbb{Y} = \left\{ \begin{array}{c} \text{p} \bullet \\ \hline \\ \ell \end{array} \right\}$$

This figure suggest that we need a cubic form in order to obtain three points on the line ℓ. In the next section we are looking for that cubic form.

3.3 Quadruplets generated in degree three

Now, we will describe the cubic homogeneous polynomial $f \in \mathbb{C}[x_0, x_1, x_2]$, that we need to add to the ideal $I_\pi = \langle \ell\ell_1, \ell\ell_2 \rangle$ in order to get a quadruplets of points in the plane.

3.1. Lemma. *Let $I = \langle \ell\ell_1, \ell\ell_2, f \rangle \subset \mathbb{C}[x_0, x_1, x_2]$ be an ideal where ℓ, ℓ_1 and ℓ_2 are linear forms such that $[\ell_1, \ell_2] \in \mathbb{G}_2(\mathcal{F})$ and $f \notin \langle \ell\ell_1, \ell\ell_2 \rangle$ is a cubic homogeneous polynomial. Then we have that*

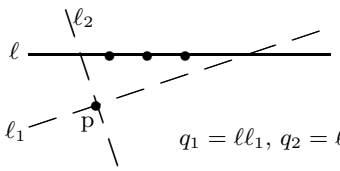
1. *If $f \notin \langle \ell \rangle$ and $f \in \langle \ell_1, \ell_2 \rangle$ then I is saturated and the Hilbert polynomial of the variety defined by I is 4.*

2. *If $f \notin \langle \ell \rangle$ and $f \notin \langle \ell_1, \ell_2 \rangle$ then I is saturated and the Hilbert polynomial of the variety defined by I is 3.*

3. *If $f \in \langle \ell \rangle$ then the saturation of I is $\overline{I} = \langle \ell \rangle$ and the Hilbert polynomial of the variety defined by $\overline{I}$ is $t + 1$.*

Proof. See [21]. ∎

In the general case we have the following pictures:



$q_1 = \ell\ell_1,\ q_2 = \ell\ell_2,$
$f_3 \in \langle \ell_1, \ell_2 \rangle \setminus \langle \ell \rangle.$

$q_1 = \ell\ell_1,\ q_2 = \ell\ell_2,$
$f_3 \in \langle \ell \rangle.$

We conclude from Lemma 3.1 that a good choice for a cubic form f such that the ideal $\langle \ell\ell_1, \ell\ell_2, f \rangle$ determines a quadruplets of points in the plane, will be to begin with $f \in \langle \ell_1, \ell_2 \rangle \setminus \langle \ell\ell_1, \ell\ell_2 \rangle$. Note that, the vector subspace of cubic forms $\ell\ell_1 \cdot \mathcal{F} + \ell\ell_2 \cdot \mathcal{F}$ is equal to the 5-dimensional vector space $\ell \cdot \mathcal{F}_2^{\mathrm{p}}$ of cubic forms that are multiple of the linear form ℓ times a conic passing through the point p ($\{\mathrm{p}\} = \ell_1 \cap \ell_2$). On the other hand, the vector space of cubic forms passing through the point p, $\mathcal{F}_3^{\mathrm{p}}$, have dimension 9, so $f \in \langle \ell_1, \ell_2 \rangle \setminus \langle \ell\ell_1, \ell\ell_2 \rangle$ is varying in a 4-dimensional vector space. Thus we have obtained a $\mathbb{P}^3$-bundle $\mathbb{E}^1$ over $\mathbb{Y}$ (cf. (3.2)). In fact, we can consider $\mathbb{E}^1$ embedded in $\mathbb{G}_2(\mathcal{F}_2) \times \mathbb{G}_6(\mathcal{F}_3)$ as follows:

$$\mathbb{E}^1 \ni ([\ell\ell_1, \ell\ell_2], \overline{f}) \longmapsto ([\ell\ell_1, \ell\ell_2], \ell \cdot \mathcal{F}_2^{\mathrm{p}} + [f]) \in \mathbb{G}_2(\mathcal{F}_2) \times \mathbb{G}_6(\mathcal{F}_3) \tag{3.3}$$

with $\overline{f} \in \mathbb{P}(\mathcal{F}_3^{\mathrm{p}}/(\ell \cdot \mathcal{F}_2^{\mathrm{p}}))$.

Note that, to each point $([\ell\ell_1, \ell\ell_2], \overline{f}) \in \mathbb{E}^1$, we can associate the homogeneous ideal $\langle \ell\ell_1, \ell\ell_2, f \rangle$ in $\mathbb{C}[x_0, x_1, x_2]$. Next, we will give a description of those points in $\mathbb{E}^1$ whose associated ideal define a quadruplets in the plane. Certainly, if $f \in \langle \ell \rangle$ we do not obtain a quadruplet in the plane (cf. Lemma 3.1). Thus the problem now it is to know when a cubic form $f \in \langle \ell_1, \ell_2 \rangle \setminus \langle \ell\ell_1, \ell\ell_2 \rangle$ will be a multiple of the line ℓ. In fact, we have the following result.

3.2. Lemma. *Let* $\mathbb{W} = \left\{ [\ell^2, \ell\ell_1] \in \mathbb{X} \mid [\ell, \ell_1] \in \mathbb{G}_2(\mathcal{F}) \right\} \subset \mathbb{Y}$, *which is illustrated as*

$$\mathbb{W} = \left\{ \quad \mathrm{p} \quad \ell \quad \right\}$$

where $\{\mathrm{p}\} = \ell \cap \ell_1$. *Then we have that*

1. *If* $[\ell\ell_1, \ell\ell_2] \in \mathbb{Y} \setminus \mathbb{W}$ *then* $\langle \ell_1, \ell_2 \rangle \cap \langle \ell \rangle = \langle \ell\ell_1, \ell\ell_2 \rangle$. *Therefore does not exist a cubic form* $f \in \langle \ell_1, \ell_2 \rangle \setminus \langle \ell\ell_1, \ell\ell_2 \rangle$ *being a multiple of* ℓ.

2. *If* $[\ell^2, \ell\ell_1] \in \mathbb{W}$ *then* $\langle \ell, \ell_1 \rangle \cap \langle \ell \rangle = \langle \ell \rangle$ *and* $\langle \ell \rangle_3 = \langle \ell^2, \ell\ell_1 \rangle_3 \oplus [\ell\varphi]$ *with* $\varphi(\mathrm{p}) \neq 0$, *that is,* $\varphi \in \mathcal{F}_2 \setminus \langle \ell, \ell_1 \rangle_2$. *Thus the fiber of* $\mathbb{E}^1$ *over* $[\ell^2, \ell\ell_1]$ *has exactly one point, does not define a quadruplet and all the others will do. In fact, the locus where* f *is a multiple of* ℓ *is given by the following section of* $\mathbb{E}^1{}_{|\mathbb{W}}$

$$\begin{array}{ccccc} \mathbb{E}^1{}_{|\mathbb{W}} \supset & \mathbb{W}^1 & \ni & ([\ell^2, \ell\ell_1], \overline{\ell\varphi}) & \longmapsto ([\ell^2, \ell\ell_1], \ell \cdot \mathcal{F}_2) \in \mathbb{G}_2(\mathcal{F}_2) \times \mathbb{G}_6(\mathcal{F}_3) \\ & \uparrow & & \uparrow & \\ & \mathbb{W} & \ni & [\ell^2, \ell\ell_1] & \end{array} \tag{3.4}$$

And the saturation of the ideal $\langle \ell^2, \ell\ell_1, \ell\varphi \rangle$ is equal to $\langle \ell \rangle$.

3.4 Quadruplets generated in degree four

It Follows from Lemma 3.1 (*2.*) that it does not help to add any other new generator to the ideal $\langle \ell\ell_1, \ell\ell_2, f \rangle$ in order to get a quadruplets of points in $\mathbb{P}^2$. And from (*3.*) and Bézout's Theorem that, it is sufficient to choose a degree four homogeneous polynomial $g \in \mathbb{C}[x_0, x_1, x_2]$ with $g \notin \langle \ell \rangle$. Thus we have obtained a $\mathbb{P}^4$-bundle $\mathbb{E}^2$ over $\mathbb{W}^1$ (cf. (3.4)). In fact, we can consider $\mathbb{E}^2$ embedded in $\mathbb{G}_2(\mathcal{F}_2) \times \mathbb{G}_6(\mathcal{F}_3) \times \mathbb{G}_{11}(\mathcal{F}_4)$ as follows:

$$\mathbb{E}^2 \ni ([\ell^2, \ell\ell_1], \overline{\ell\varphi}, \widetilde{g}) \longmapsto ([\ell^2, \ell\ell_1], \ell \cdot \mathcal{F}_2, \ell \cdot \mathcal{F}_3 + [g]) \in \mathbb{G}_2(\mathcal{F}_2,) \times \mathbb{G}_6(\mathcal{F}_3) \times \mathbb{G}_{11}(\mathcal{F}_4) \tag{3.5}$$

with $\overline{\ell\varphi} \in \mathbb{P}(\mathcal{F}_3^{\mathrm{p}}/(\ell \cdot \mathcal{F}_2^{\mathrm{p}}))$ and $\widetilde{g} \in \mathbb{P}(\mathcal{F}_4/(\ell \cdot \mathcal{F}_3))$. In fact, we have that.

3.3. Lemma. *Let $I = \langle \ell, g \rangle \subset \mathbb{C}[x_0, x_1, x_2]$ be an ideal where ℓ is a linear form and $g \notin \langle \ell \rangle$ is a quartic homogeneous polynomial. Then we have that I is saturated and the Hilbert polynomial of the variety defined by I is 4.*

Proof. See [21]. ∎

4 Enumerative Application

Now we are interested in giving an answer to the following enumerative question:

How many squares are there with its set of vertices meeting eight general lines? $\binom{\bullet\ \bullet}{\bullet\ \bullet}$ More generally, how many N-gons are there with its set of vertices meeting $2N$ general lines? Note that, each vertex in the N-gon is determined by the intersection of a pair of distinct lines. So, let $\ell_1, ..., \ell_{2N}$ be $2N$ general given lines in $\mathbb{P}^2$ and set

$$\mathcal{P}_N = \{N - \text{gons having its vertices in exactly one pair of these distinct lines}\}.$$

Now, fix $2N+2$ general lines $\ell_1, ..., \ell_{2N+2}$ in $\mathbb{P}^2$ and let

$$\mathcal{P}_{N+1,i} = \{(N+1) - \text{gons having one vertex over } \ell_{2N+2} \text{ and } \ell_i\} \text{ for } i = 1, ..., 2N+1.$$

Note that:

- $\mathcal{P}_{N+1,i} \cap \mathcal{P}_{N+1,j} = \emptyset$ for $i \neq j$;
- $\mathcal{P}_{N+1,i}$ are in bijection with $\mathcal{P}_N$ for $i = 1, ..., 2N+1$;

- $\mathcal{P}_{N+1} = \bigcup_{i=1}^{2N+1} \mathcal{P}_{N+1,i}$.

Thus, we have that $\#(\mathcal{P}_{N+1}) = \sum_{i=1}^{2N+1} \#(\mathcal{P}_{N+1,i}) = (2N+1) \cdot \#(\mathcal{P}_N)$. Using induction, we see that $\#(\mathcal{P}_{N+1}) = (2N+1) \cdot (2N-1) \cdot ... \cdot 5 \cdot 3 \cdot 1 = \{2N+1\}! =$ the factorial of odd's numbers between 1 and $2N+1$. Therefore, $\#(\mathcal{P}_4) = \{7\}! = 7 \cdot 5 \cdot 3 \cdot 1 = 105$.

Next we will use Bott's localization formula to find the answer to the enumerative problem $\binom{\bullet\ \bullet}{\bullet\ \bullet}$ on an appropriate parameter space. The Bott's localization formula that we will apply express the integral of a homogeneous polynomial in the Chern classes of a bundle on a smooth, compact variety with a $\mathbb{C}^*$-action in terms of data given by the induced linear actions on the fiber of the bundle and the tangent bundle in the (isolated) fixed points of the action. In fact, Bott's residues formula said that.

4.1. Theorem. *Let T be a torus and X be a smooth, complete variety with a T-action. Let $E_1, \ldots, E_s$ be $T-$ equivariant vector bundles. Then we have that.*

$$\int_X p(E) \cap [X] = \sum_{F \subset X^T} (\pi_F)_* \left(\frac{p^T(E_{|F}) \cap [F]_T}{c^T_{d_F}(\mathcal{N}_F X)} \right) \tag{4.1}$$

where

- *F is a $(dimX - d_F)$-dimensional component of X^T;*
- *X^T is the fixed point locus;*
- *$p(E) = p(E_1, ..., E_s)$ is a homogeneous polynomial of degree $dimX$ in the Chern classes of the bundles $E_j's$. In fact, $p(E)$ is a weighted homogeneous polynomial in the variables $x^i_j = c_i(E_j)$, where x^i_j has degree i;*
- *$\mathcal{N}_F X$ denoted the normal bundle of F in X;*
- *$[F]_T$ is the T-equivariant fundamental class of F;*
- *$c^T_{d_F}(\mathcal{N}_F X)$ denoted the top T-equivariant Chern class of the normal bundle $\mathcal{N}_F X$;*
- *$p^T(E_{|F}) = p(E_{1T}, ..., E_{sT})$, where E_{iT} denoted the quotient bundles associated to E_i;*
- *$(\pi_F)_*$ denoted the proper pushforward of the morphism $\underbrace{F \overset{i_F}{\hookrightarrow} X \overset{\pi_X}{\longrightarrow} pt}_{\pi_F}$.*

In spite of the possibly awe-inspiring appearance of (4.1) at first (in part because we do not explain what means each ingredient in the formula), we hope to convince the reader that it is rather simple to apply in practice. See [3], [5], [6] and the elementary exposition in [17] for details. See [8] and chapter 9 in [4] for applications. See also [11] for a computational improvement to Bott's application that have a close connection with Cauchy's residue formula.

4.1 Parameter space for squares

Let us consider the following two closed subvarieties of $\mathbb{G}_2(\mathcal{F}_2)\times\mathbb{G}_6(\mathcal{F}_3)$ and $\mathbb{G}_2(\mathcal{F}_2)\times\mathbb{G}_6(\mathcal{F}_3)\times\mathbb{G}_{11}(\mathcal{F}_4)$ respectively.

For each pencil of conics $[q_1,q_2]\in\mathbb{G}_2(\mathcal{F}_2)$, let $\mathbf{C}\in\mathbb{G}_6(\mathcal{F}_3)$ be the linear system defined as follows

$$\mathbf{C}=\begin{cases} q_1\cdot\mathcal{F}+q_2\cdot\mathcal{F} & \text{if } gcd(q_1,q_2)=1,\\ \ell\cdot\mathcal{F}_2^{\mathrm{p}}+[f] & \text{if } q_1=\ell\ell_1,\ q_2=\ell\ell_2 \text{ with } \ell\in\mathbb{P}(\mathcal{F}),[\ell_1,\ell_2]\in\mathbb{G}_2(\mathcal{F})\\ & \quad\text{and } \overline{f}\in\mathbb{P}(\mathcal{F}_3^{\mathrm{p}}/(\ell\cdot\mathcal{F}_2^{\mathrm{p}})) \text{ where } \{\mathrm{p}\}=\ell_1\cap\ell_2. \end{cases} \tag{4.2}$$

Let

$$\mathbb{X}^1=\Big\{([q_1,q_2],\mathbf{C})\in\mathbb{G}_2(\mathcal{F}_2)\times\mathbb{G}_6(\mathcal{F}_3)\mid \mathbf{C}\text{ is defined as in (4.2)}\Big\}. \tag{4.3}$$

Now for each $([q_1,q_2],\mathbf{C})\in\mathbb{X}^1$, let $\mathbf{Q}\in\mathbb{G}_{11}(\mathcal{F}_4)$ be the linear system defined as follows:

$$\mathbf{Q}=\begin{cases} q_1\cdot\mathcal{F}_2+q_2\cdot\mathcal{F}_2 & \text{if } gcd(q_1,q_2)=1,\\ \ell\cdot\mathcal{F}_3^{\mathrm{p}}+f\cdot\mathcal{F} & \text{if } q_1=\ell\ell_1,\ q_2=\ell\ell_2,\ \ell\in\mathbb{P}(\mathcal{F}),[\ell_1,\ell_2]\in\mathbb{G}_2(\mathcal{F})\\ & \quad\text{and } \overline{f}\in\mathbb{P}(\mathcal{F}_3^{\mathrm{p}}/(\ell\cdot\mathcal{F}_2^{\mathrm{p}})) \text{ with } \{\mathrm{p}\}=\ell_1\cap\ell_2 \text{ such that } f\notin\langle\ell\rangle,\\ \ell\cdot\mathcal{F}_3+[g] & \text{if } q_1=\ell\ell_1,\ q_2=\ell\ell_2,\ f=\ell\varphi \text{ where } \varphi\in\mathcal{F}_2\setminus\mathcal{F}_2^{\mathrm{p}}\\ & \quad\text{and } \widetilde{g}\in\mathbb{P}(\mathcal{F}_4/(\ell\cdot\mathcal{F}_3)). \end{cases} \tag{4.4}$$

Let

$$\mathbb{X}^2=\Big\{([q_1,q_2],\mathbf{C},\mathbf{Q})\in\mathbb{X}^1\times\mathbb{G}_{11}(\mathcal{F}_4)\mid \mathbf{Q}\text{ is defined as in (4.4)}\Big\}. \tag{4.5}$$

Follows from (3.3), (4.2) and (4.3) that $\mathbb{E}^1$ is a subvariety of $\mathbb{X}^1$. In the same way follows from (3.5), (4.4) and (4.5) that $\mathbb{E}^2$ is a subvariety of $\mathbb{X}^2$. Therefore, we have the following diagram for our parameter space $\mathbb{X}^2$.

$$\begin{array}{ccccc} & & \mathbb{E}^2 & \hookrightarrow & \mathbb{X}^2 & & \\ & \swarrow & & & & \searrow & \\ \mathbb{W}^1 & \hookrightarrow & \mathbb{E}^1 & \hookrightarrow & \mathbb{X}^1 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{W} & \hookrightarrow & \mathbb{Y} & \hookrightarrow & \mathbb{X} & & \end{array} \tag{4.6}$$

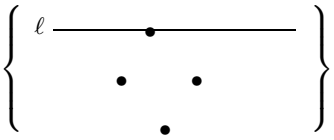
In fact, it is verified that $\mathbb{X}^1$ is the blowup of $\mathbb{X}$ along $\mathbb{Y}$ with $\mathbb{E}^1$ being the exceptional divisor and also that $\mathbb{X}^2$ is the blowup of $\mathbb{X}^1$ along $\mathbb{W}^1$ with $\mathbb{E}^2$ being the exceptional divisor (see [1], [21]).

On the other hand, for $a\in\mathbb{C}$, $([x_0^2,x_0(x_1+ax_2)],x_0\cdot\mathcal{F}_2,x_0\cdot\mathcal{F}_3+[x_1^4])$ are distinct points in $\mathbb{X}^2$, but its image in $Hilb^4\mathbb{P}^2$ is equal to the ideal $\langle x_0,x_1^4\rangle$. Therefore $\mathbb{X}^2$ is not isomorphic to $Hilb^4\mathbb{P}^2$. Nevertheless, can be verified that $\mathbb{X}^2$ is isomorphic to the the blowup of $Hilb^4\mathbb{P}^2$ along the 6-dimensional subvariety of aligned quadruplets (see [1]) ($\langle x_0,x_1^4\rangle$ is an aligned quadruplets).

5 Divisor of Incidence to a Line

Let ℓ be a line in $\mathbb{P}^2$ and D_ℓ be the hypersurface in $\mathbb{X} = \mathbb{G}_2(\mathcal{F}_2)$ defined by the condition

$$\ell \cap q_1 \cap q_2 \neq \emptyset \quad \text{for } [q_1, q_2] \in \mathbb{X}.$$

$D_\ell =$ 

Let $\widetilde{D_\ell}$ be the subvariety of $\ell \times \mathbb{X}$ defined by

$$\widetilde{D_\ell} = \Big\{(\mathrm{q}, \pi) \in \ell \times \mathbb{X} \mid \mathrm{q} \in \text{ base locus of the pencil } \pi\Big\}.$$

Note that:

- $\widetilde{D_\ell}$ is a codimension two subvariety of $\ell \times \mathbb{X}$.
- The image of $\widetilde{D_\ell}$ under $p_2 : \ell \times \mathbb{X} \longrightarrow \mathbb{X}$, the projection in the second coordinate, is equal to D_ℓ.

5.1 Class of D_ℓ

Let $\mathcal{A}$ be the tautological subbundle of $\mathbb{G}_2(\mathcal{F}_2)$ as in (2.1). Let us consider the diagram of natural maps of vector bundles over $\ell \times \mathbb{X}$,

$$\begin{array}{ccc} \mathcal{A} \hookrightarrow & \mathcal{F}_2 & \\ \searrow & \downarrow & \\ & \mathcal{F}_2/\mathcal{F}_2^\bullet \cong \mathcal{O}_\ell(2) & \end{array}$$

here the fiber $\mathcal{F}_2/\mathcal{F}_{2\,(\mathrm{q},\pi)}^\bullet$ is equal to $\mathcal{F}_2/\mathcal{F}_2{}^{\mathrm{q}}$.

Note that the slant arrow vanishes at $(\mathrm{q}, \pi) \in \ell \times \mathbb{X}$ if and only if $(\mathrm{q}, \pi) \in \widetilde{D}_\ell$.

Hence we have

$$[\widetilde{D}_\ell] = (c_2(\mathcal{A}^\vee \otimes \mathcal{O}_\ell(2))) \cap [\ell \times \mathbb{X}] = (c_2(\mathcal{A}) - 2h \cdot c_1(\mathcal{A})) \cap [\ell \times \mathbb{X}],$$

where $h = c_1(\mathcal{O}_\ell(1))$. Pushing forward via $p_2 : \ell \times \mathbb{X} \longrightarrow \mathbb{X}$, it follows that

$$\boxed{[D_\ell] = -2c_1(\mathcal{A}) \cap [\mathbb{X}].}$$

In fact,

$$p_{2\star}(c_2(p_2^\star \mathcal{A}) \cap [\ell \times \mathbb{X}]) = c_2(\mathcal{A}) \cap p_{2\star}[\ell \times \mathbb{X}] = 0.$$

$$\begin{aligned} c_1(p_1^\star\mathcal{O}_\ell(1)) \cap [\ell \times \mathbb{X}] &= c_1(p_1^\star\mathcal{O}_\ell(1)) \cap [p_1^\star(\ell)], \\ &= p_1^\star(c_1(\mathcal{O}_\ell(1)) \cap [\ell]), \\ &= p_1^\star([pt]), \\ &= [pt \times \mathbb{X}]. \end{aligned}$$

then

$$\begin{aligned} p_{2\star}(c_1(p_2^\star\mathcal{A}) \cdot c_1(p_1^\star\mathcal{O}_\ell(1)) \cap [\ell \times \mathbb{X}]) &= p_{2\star}(c_1(p_2^\star\mathcal{A}) \cap [pt \times \mathbb{X}]), \\ &= c_1(\mathcal{A}) \cap [\mathbb{X}]. \end{aligned}$$

A local coordinate check shows that D_ℓ contains the blowup center $\mathbb{Y}$ (see (3.2)) with multiplicity one. Hence we find the formula for the class of the strict transform in $\mathbb{X}^1$,

$$[D_\ell^{(1)}] = -2c_1(\mathcal{A}) \cap [\mathbb{X}^1] - [\mathbb{E}^1].$$

Similarly, (omitting pullbacks) we get for the succeeding strict transform,

$$[D_\ell^{(2)}] = -2c_1(\mathcal{A}) \cap [\mathbb{X}^2] - [\mathbb{E}^{2,1}] - [\mathbb{E}^2].$$

Here we have omitted the pull-back in $\mathcal{A}$ and $\mathbb{E}^{2,1}$ denote the strict transform of $\mathbb{E}^1$. Now a solution to the question (⋮ ⋮ ⋮) in Section 4 asks us to compute the degree of the self-intersection $[D_\ell^{(2)}]^8$. Thus from Bott's formula (cf. (4.1)) we have that.

$$\int_{[\mathbb{X}^2]} [D_\ell^{(2)}]^8 = \sum_F \int_{[F]_T} \frac{[2c_1^T(\mathcal{A}_F) + c_1^T(\mathcal{O}(\mathbb{E}^{2,1})_F) + c_1^T(\mathcal{O}(\mathbb{E}^2)_F)]^8}{c_{d_F}^T(\mathcal{N}_F\mathbb{X}^2)}, \tag{5.1}$$

where d_F denotes the codimension of the component F in $\mathbb{X}$. F is a component of $\mathbb{X}^T$ the locus of fixed points for a suitable torus action, starting at $\mathbb{X}$ and following all the way up to $\mathbb{X}^2$.

6 Fixed Points at $\mathbb{X}^2$

Let $\mathbb{V}$ be an n-dimensional complex vector space. Then a general action of $\mathbb{C}^*$ on $\mathbb{V}$ is diagonalized, so there is a basis $\{v_1, ..., v_n\}$ of $\mathbb{V}$ such that $t \cdot v_i = \lambda(t)v_i$ for all $t \in \mathbb{C}$. In fact, λ is a character of the group $\mathbb{C}^*$. So $\lambda(t) = t^{w_i}$ for some integer w_i. We also have an induced action on $\mathbb{G}_k(V)$, the Grassmann variety of k-planes in $\mathbb{V}$, given by

$$t \cdot W = [t \cdot w_1, ..., t \cdot w_k] \quad \text{for any } W = [w_1, ..., w_k] \in \mathbb{G}_k(V).$$

And the fixed points are given by:

$$W_{i_1,i_2,...,i_k} = [v_{i_1}, v_{i_2}, ..., v_{i_k}] \text{ where } (i_1 i_2 ... i_k) \text{ is a } k\text{-cicle in } S_n,$$

so we have at all $\binom{n}{k}$ fixed points in $\mathbb{G}_k(V)$.

Consider now the action of $\mathbb{C}^*$ over $\mathcal{F}_d$ given by $t \circ x_0^{i_0} x_1^{i_1} x_2^{i_2} = t^{i_0 w_0 + i_1 w_1 + i_2 w_2} x_0^{i_0} x_1^{i_1} x_2^{i_2}$ with $i_0 + i_1 + i_2 = d$ and extend it by linearity. We also have an induced action on $\mathbb{G}_n(\mathcal{F}_d)$, $\mathbb{X}^1$ and $\mathbb{X}^2$ respectively.

According to (4.6) the image of $\mathbb{E}^2$ and $\mathbb{E}^1$ in $\mathbb{X}$ are respectively $\mathbb{W}$ and $\mathbb{Y}$. And we also have that $\mathbb{X}^2 \setminus \mathbb{E}^2 \cong \mathbb{X}^1 \setminus \mathbb{W}^1$ and $\mathbb{X}^1 \setminus \mathbb{E}^1 \cong \mathbb{X} \setminus \mathbb{Y}$. Let $\mathbb{E}^{2,1} \subset \mathbb{X}^2$ be the strict transform of $\mathbb{E}^1$, then we have that:

Fixed points in	are in correspondence with fixed points in	are in correspondence with fixed points in
$\mathbb{X}^2 \setminus (\mathbb{E}^{2,1} \cup \mathbb{E}^2)$	$\mathbb{X}^1 \setminus \mathbb{E}^1$	$\mathbb{X} \setminus \mathbb{Y}$
Fixed points in	are mapped on fixed points in	are mapped on fixed points in
$\mathbb{E}^{2,1} \setminus \mathbb{E}^2$	$\mathbb{E}^1 \setminus \mathbb{W}^1$	$\mathbb{Y} \setminus \mathbb{W}$
$\mathbb{E}^2$	$\mathbb{W}^1$	$\mathbb{W}$

(6.1)

So we will look for fixed points having in mind (6.1).

6.1 Fixed points in $\mathbb{X}^2 \setminus (\mathbb{E}^{2,1} \cup \mathbb{E}^2)$

If the weights (w_0, w_1, w_2) are sufficiently general, we find the following 6 fixed points in $\mathbb{X} \setminus \mathbb{Y}$:

$$\pi_1 = [x_0^2, x_1^2], \pi_2 = [x_0^2, x_1 x_2], \pi_3 = [x_0^2, x_2^2], \pi_4 = [x_0 x_1, x_2^2], \pi_5 = [x_0 x_2, x_1^2], \pi_6 = [x_1^2, x_2^2].$$

Since this 6 fixed points lie off $\mathbb{Y}$ then they lift (isomorphically) all the way up to $\mathbb{X}^2$. So their contribution can be obtained at once, down on $\mathbb{X}$. Of course the exceptional divisors give no contribution here. On the numerator of (5.1) we have for $2c_1^T(\mathcal{A}_{\pi_i})$ $i = 1, ..., 6$,

Fixed points in $\mathbb{X} \setminus \mathbb{Y}$	$\mathcal{A}_{\pi_i}$	Decomposition of $\mathcal{A}_{\pi_i}$ into eigenspaces	$2c_1^T(\mathcal{A}_{\pi_i})$
$\pi_1 = [x_0^2, x_1^2]$	$[x_0^2, x_1^2]$	$t^{2w_0} + t^{2w_1}$	$2(2w_0 + 2w_1)$
$\pi_2 = [x_0^2, x_1 x_2]$	$[x_0^2, x_1 x_2]$	$t^{2w_0} + t^{w_1 + w_2}$	$2(2w_0 + w_1 + w_2)$
$\pi_3 = [x_0^2, x_2^2]$	$[x_0^2, x_2^2]$	$t^{2w_0} + t^{2w_2}$	$2(2w_0 + 2w_2)$
$\pi_4 = [x_0 x_1, x_2^2]$	$[x_0 x_1, x_2^2]$	$t^{w_0 + w_1} + t^{2w_2}$	$2(w_0 + w_1 + 2w_2)$
$\pi_5 = [x_0 x_2, x_1^2]$	$[x_0 x_2, x_1^2]$	$t^{w_0 + w_2} + t^{2w_1}$	$2(w_0 + w_2 + 2w_1)$
$\pi_6 = [x_1^2, x_2^2]$	$[x_1^2, x_2^2]$	$t^{2w_1} + t^{2w_2}$	$2(2w_1 + 2w_2)$

On the denominator of (5.1) we get $\mathcal{N}_{\pi_i}\mathbb{X} = \mathcal{T}_{\pi_i}\mathbb{X}^2 = \mathcal{T}_{\pi_i}\mathbb{X} = \mathcal{F}_2/\mathcal{A}_{\pi_i} \otimes \mathcal{A}_{\pi_i}^{\vee}$.

Note that the eigen-decomposition of $\mathcal{F}_2$ is given by

$$\mathcal{F}_2 = \sum_{0\le i\le j\le 2} t^{w_i+w_j} = t^{2w_0} + t^{w_0+w_1} + t^{w_0+w_2} + t^{2w_1} + t^{w_1+w_2} + t^{2w_2}.$$

Thus for $\boxed{\pi_1 = [x_0^2, x_1^2]}$ we have that

$\mathcal{T}_{\pi_1}\mathbb{X} = \mathcal{F}_2/\mathcal{A}_{\pi_1}\otimes\mathcal{A}_{\pi_1}^{\vee} = \mathcal{F}_2/[x_0^2, x_1^2]\otimes[x_0^2, x_1^2]^{\vee} = (t^{w_0+w_1} + t^{w_0+w_2} + t^{w_1+w_2} + t^{2w_2})(t^{-2w_0} + t^{-2w_1})$.

Next we give the eigen-decomposition of $\mathcal{T}_{\pi_i}\mathbb{X}$ and $c_8^T(\mathcal{T}_{\pi_i}\mathbb{X})$, for $i = 1, ..., 6$:

$$\begin{aligned}
\pi_1 = [x_0^2, x_1^2] \quad &\leftrightarrow \quad (t^{w_0+w_1} + t^{w_0+w_2} + t^{w_1+w_2} + t^{2w_2})(t^{-2w_0} + t^{-2w_1}),\\
&\leftrightarrow \quad (w_1-w_0)(w_0-w_1)(w_2-w_0)(w_0+w_2-2w_1)(w_1+w_2-2w_0)(w_2-w_1)(2w_2-2w_0)\\
&\qquad (2w_2-2w_1),\\
\pi_2 = [x_0^2, x_1x_2] \quad &\leftrightarrow \quad (t^{w_0+w_1} + t^{w_0+w_2} + t^{2w_1} + t^{2w_2})(t^{-2w_0} + t^{-(w_1+w_2)}),\\
&\leftrightarrow \quad (w_1-w_0)(w_0-w_2)(w_2-w_0)(w_0-w_1)(2w_1-2w_0)(w_1-w_2)(2w_2-2w_0)(w_2-w_1),\\
\pi_3 = [x_0^2, x_2^2] \quad &\leftrightarrow \quad (t^{w_0+w_1} + t^{w_0+w_2} + t^{2w_1} + t^{w_1+w_2})(t^{-2w_0} + t^{-2w_2}),\\
&\leftrightarrow \quad (w_2-w_0)(w_0-w_2)(w_1-w_0)(w_0+w_1-2w_2)(w_1+w_2-2w_0)(w_1-w_2)(2w_1-2w_0)\\
&\qquad (2w_1-2w_2),\\
\pi_4 = [x_0x_1, x_2^2] \quad &\leftrightarrow \quad (t^{2w_0} + t^{w_0+w_2} + t^{2w_1} + t^{w_1+w_2})(t^{-2w_2} + t^{-(w_0+w_1)}),\\
&\leftrightarrow \quad (w_1-w_2)(w_2-w_0)(w_0-w_2)(w_2-w_1)(2w_1-2w_2)(w_1-w_0)(2w_0-2w_2)(w_0-w_1),\\
\pi_5 = [x_0x_2, x_1^2] \quad &\leftrightarrow \quad (t^{2w_0} + t^{w_0+w_1} + t^{w_1+w_2} + t^{2w_2})(t^{-2w_1} + t^{-(w_0+w_2)}),\\
&\leftrightarrow \quad (w_2-w_1)(w_1-w_0)(w_0-w_1)(w_1-w_2)(2w_2-2w_1)(w_2-w_0)(2w_0-2w_1)(w_0-w_2),\\
\pi_6 = [x_1^2, x_2^2] \quad &\leftrightarrow \quad (t^{2w_0} + t^{w_0+w_1} + t^{w_0+w_2} + t^{w_1+w_2})(t^{-2w_1} + t^{-2w_2}),\\
&\leftrightarrow \quad (w_1-w_2)(w_2-w_1)(w_0-w_2)(w_0+w_2-2w_1)(w_1+w_0-2w_2)(w_0-w_1)(2w_0-2w_2)\\
&\qquad (2w_0-2w_1).
\end{aligned}$$

So the first six contributions to (5.1) are:

$$\left\{\begin{array}{l}
\frac{2^8(2w_0+2w_1)^8}{(w_1-w_0)(w_0-w_1)(w_2-w_0)(w_0+w_2-2w_1)(w_1+w_2-2w_0)(w_2-w_1)(2w_2-2w_0)(2w_2-2w_1)} +\\
\frac{2^8(2w_0+w_1+w_2)^8}{(w_1-w_0)(w_0-w_2)(w_2-w_0)(w_0-w_1)(2w_1-2w_0)(w_1-w_2)(2w_2-2w_0)(w_2-w_1)} +\\
\frac{2^8(2w_0+2w_2)^8}{(w_2-w_0)(w_0-w_2)(w_1-w_0)(w_0+w_1-2w_2)(w_1+w_2-2w_0)(w_1-w_2)(2w_1-2w_0)(2w_1-2w_2)} +\\
\frac{2^8(w_0+w_1+2w_2)^8}{(w_1-w_2)(w_2-w_0)(w_0-w_2)(w_2-w_1)(2w_1-2w_2)(w_1-w_0)(2w_0-2w_2)(w_0-w_1)} +\\
\frac{2^8(w_0+w_2+2w_1)^8}{(w_2-w_1)(w_1-w_0)(w_0-w_1)(w_1-w_2)(2w_2-2w_1)(w_2-w_0)(2w_0-2w_1)(w_0-w_2)} +\\
\frac{2^8(2w_1+2w_2)^8}{(w_1-w_2)(w_2-w_1)(w_0-w_2)(w_0+w_2-2w_1)(w_1+w_0-2w_2)(w_0-w_1)(2w_0-2w_2)(2w_0-2w_1)}.
\end{array}\right.$$

6.2 Fixed points in $\mathbb{E}^{2,1}\setminus\mathbb{E}^2$

Since $\mathbb{E}^{2,1}\setminus\mathbb{E}^2$ is isomorphic to $\mathbb{E}^1\setminus\mathbb{W}^1$. Then, we have to look for fixed points on $\mathbb{Y}\setminus\mathbb{W}$ (cf. (6.1)). We find after some computation the following 3 fixed points in $\mathbb{Y}\setminus\mathbb{W}$.

$$\pi_7 = [x_0x_1, x_0x_2], \pi_8 = [x_1x_0, x_1x_2], \pi_9 = [x_2x_0, x_2x_1]. \tag{6.2}$$

Thus to determine the contributions to (5.1) in this case, we only have to calculate $c_8^T(\mathcal{T}_{\mathrm{y}^1}\mathbb{X}^1)$, $c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}^1})$ for those fixed points $\mathrm{y}^1 \in \mathbb{E}^1$ lying over π_i and $2c_1^T(\mathcal{A}_{\pi_i})$ for $i = 7, 8, 9$.

According to (3.3) the fiber of $\mathbb{E}^1$ over $[\ell\ell_1, \ell\ell_2] \in \mathbb{Y} \setminus \mathbb{W}$ is given by

$$\mathbb{E}^1_{[\ell\ell_1,\ell\ell_2]} = \mathbb{P}(\langle \overline{\ell_1^3}, \overline{\ell_1^2\ell_2}, \overline{\ell_1\ell_2^2}, \overline{\ell_2^3} \rangle) \tag{6.3}$$

where $\overline{f}$ indicates classes of $f \in \mathcal{F}_3^{\mathrm{p}}$ modulo $\ell \cdot \mathcal{F}_2^{\mathrm{p}}$ with $\{\mathrm{p}\} = \ell_1 \cap \ell_2$. Note that, $\ell = x_0, \ell_1 = x_1, \ell_2 = x_2$ for π_7 and so on. And can be verified that $([\ell\ell_1, \ell\ell_2], \ell \cdot \mathcal{F}^{\mathrm{p}} + [f]) \in \mathbb{E}^1 \subset \mathbb{X}^1$ with $f \in \{\ell_1^3, \ell_1^2\ell_2, \ell_1\ell_2^2, \ell_2^3\}$ are fixed points for the induced action of $T = \mathbb{C}^*$ on $\mathbb{X}^1$. Thus we obtain $3 \times 4 = 12$ fixed points lying in $\mathbb{E}^1 \setminus \mathbb{W}^1$. In order to compute the contributions coming from this 12 fixed points to (5.1) we need to determine tangent and normal spaces.

Since the exact sequence of $\mathbb{C}^*$–representations $0 \to \mathcal{T}_\pi\mathbb{Y} \to \mathcal{T}_\pi\mathbb{X} \to (\mathcal{N}_{\mathbb{Y}}\mathbb{X})_\pi \to 0$ splits, we may write the following decomposition into eigen spaces for $[\ell\ell_1, \ell\ell_2] \in \mathbb{Y} \setminus \mathbb{W}$,

$$\begin{aligned}
(\mathcal{N}_{\mathbb{Y}}\mathbb{X})_{[\ell\ell_1,\ell\ell_2]} &= \mathcal{T}_{[\ell\ell_1,\ell\ell_2]}\mathbb{X} - \mathcal{T}_{[\ell\ell_1,\ell\ell_2]}\mathbb{Y} \\
&= \mathcal{T}_{[\ell\ell_1,\ell\ell_2]}\mathbb{G}_2(\mathcal{F}_2) - \mathcal{T}_{[\ell\ell_1,\ell\ell_2]}\overbrace{(\mathbb{P}(\mathcal{F}) \times \mathbb{G}_2(\mathcal{F}))}^{\cong \mathbb{Y}} \\
&= (\ell\ell_1 + \ell\ell_2)^\vee \otimes \overbrace{(\ell^2 + \ell_1^2 + \ell_1\ell_2 + \ell_2^2)}^{\mathcal{F}_2 - (\ell\ell_1 + \ell\ell_2)} - \overbrace{\Big(\frac{\ell_1}{\ell} + \frac{\ell_2}{\ell} + \frac{\ell}{\ell_1} + \frac{\ell}{\ell_2}\Big)}^{\mathcal{T}_{[\ell\ell_1,\ell\ell_2]}\mathbb{Y}} \\
&= \tfrac{\ell_2^2}{\ell\ell_1} + \tfrac{\ell_1^2}{\ell\ell_2} + \tfrac{\ell_1}{\ell} + \tfrac{\ell_2}{\ell}.
\end{aligned} \tag{6.4}$$

Note that from (6.3) and (6.4) we have the two descriptions,

$$\mathbb{E}^1_{[\ell\ell_1,\ell\ell_2]} = \mathbb{P}((\mathcal{N}_{\mathbb{Y}}\mathbb{X})_{[\ell\ell_1,\ell\ell_2]}) = \mathbb{P}(\langle \overline{\ell_1^3}, \overline{\ell_1^2\ell_2}, \overline{\ell_1\ell_2^2}, \overline{\ell_2^3} \rangle) \text{ and } (\mathcal{N}_{\mathbb{Y}}\mathbb{X})_{[\ell\ell_1,\ell\ell_2]} = \frac{\ell_2^2}{\ell\ell_1} + \frac{\ell_1^2}{\ell\ell_2} + \frac{\ell_1}{\ell} + \frac{\ell_2}{\ell}.$$

We can reconcile this two descriptions noting that to any normal vector ξ as in (6.4) we can associated a curve γ_t in $\mathbb{X}$ with tangent ξ at $t = 0$ such that $\gamma_t \in \mathbb{X} \setminus \mathbb{Y}$ for $t \neq 0$, so it lifts to a curve γ_t^1 in $\mathbb{X}^1$ whose tangent at $t = 0$ give a monomial in $\{\ell_1^3, \ell_1^2\ell_2, \ell_1\ell_2^2, \ell_2^3\}$ associated to the normal direction corresponding to ξ as described in the following table:

Normal vector ξ	Curve with tangent ξ	Lifts to a curve in $\mathbb{X}^1$
$\frac{\ell_2^2}{\ell\ell_1} = (\ell\ell_1)^\vee \otimes \ell_2^2$	$\gamma_t = [\ell\ell_1 + t\ell_2^2, \ell\ell_2]$	$\gamma_t^1 = (\gamma_t, \ell\ell_2 \cdot \mathcal{F} + (\ell\ell_1 + t\ell_2^2)\ell + (\ell\ell_1 + t\ell_2^2)\ell_1 + t\ell_2^3)$
$\frac{\ell_1^2}{\ell\ell_2} = (\ell\ell_2)^\vee \otimes \ell_1^2$	$\gamma_t = [\ell\ell_1, \ell\ell_2 + t\ell_1^2]$	$\gamma_t^1 = (\gamma_t, \ell\ell_1 \cdot \mathcal{F} + (\ell\ell_2 + t\ell_1^2)\ell + (\ell\ell_2 + t\ell_1^2)\ell_2 + t\ell_1^3)$
$\frac{\ell_1}{\ell} = (\ell)^\vee \otimes \ell_1$	$\gamma_t = [\ell\ell_1, \ell\ell_2 + t\ell_1\ell_2]$	$\gamma_t^1 = (\gamma_t, \ell\ell_1 \cdot \mathcal{F} + (\ell\ell_2 + t\ell_1\ell_2)\ell + (\ell\ell_2 + t\ell_1\ell_2)\ell_2 + t\ell_1^2\ell_2)$
$\frac{\ell_2}{\ell} = (\ell)^\vee \otimes \ell_2$	$\gamma_t = [\ell\ell_1, \ell\ell_2 + t\ell_2^2]$	$\gamma_t^1 = (\gamma_t, \ell\ell_1 \cdot \mathcal{F} + (\ell\ell_2 + t\ell_2^2)\ell + (\ell\ell_2 + t\ell_2^2)\ell_2 + t\ell_1\ell_2^2)$

(6.5)

Determination of $c_8^T(\mathcal{T}_{\mathrm{y}^1}\mathbb{X}^1)$, $c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}^1})$ for those fixed points $\mathrm{y}^1 \in \mathbb{E}^1$ lying over π_i and $2c_1^T(\mathcal{A}_{\pi_i})$ for $i = 7, 8, 9$.

On the other hand, for any $\pi^1 \in \mathbb{E}^1$ lying over $\pi \in \mathbb{Y}$, we have that

$$\begin{aligned} \mathcal{T}_{\pi^1}\mathbb{X}^1 &= \mathcal{T}_{\pi^1}\mathbb{E}^1 + (\mathcal{N}_{\mathbb{E}^1}\mathbb{X}^1)_{\pi^1} \\ &= \mathcal{T}_{\pi^1}\mathbb{E}^1_\pi + \mathcal{T}_\pi\mathbb{Y} + [\pi^1]. \qquad \text{Note that } [\pi^1] = \mathcal{O}_{\mathbb{E}^1}(-1)_{\pi^1} = \mathcal{O}(\mathbb{E}^1)_{\pi^1}. \end{aligned} \tag{6.6}$$

Let $\mathrm{y} = [\ell\ell_1, \ell\ell_2] \in \mathbb{Y}\setminus\mathbb{W}$ and $\mathrm{y}^1_i = (\mathrm{y}, \overline{f_i}) \in \mathbb{E}^1$ with $f_i \in \{\ell_1^3, \ell_1^2\ell_2, \ell_1\ell_2^2, \ell_2^3\}$. So for $\mathrm{y}^1_1 = (\mathrm{y}, \overline{\ell_1^3}) \in \mathbb{E}^1$ we have that:

$$\begin{aligned} \mathcal{T}_{\mathrm{y}^1_1}\mathbb{X}^1 &= \mathcal{T}_{\mathrm{y}^1_1}\mathbb{E}^1_{\mathrm{y}} + \mathcal{T}_{\mathrm{y}}\mathbb{Y} + [\mathrm{y}^1_1] \\ &= \mathcal{T}_{\mathrm{y}^1_1}\overbrace{\mathbb{P}([\overline{\ell_1^3}, \overline{\ell_1^2\ell_2}, \overline{\ell_1\ell_2^2}, \overline{\ell_2^3}])}^{\mathbb{E}^1_{\mathrm{y}}} + \mathcal{T}_{\mathrm{y}}\overbrace{(\mathbb{P}(\mathcal{F}) \times \mathbb{G}_{\in}(\mathcal{F}))}^{\cong \mathbb{Y}} + [\mathrm{y}^1_1] \\ &= \underbrace{\frac{\ell_2}{\ell_1} + \frac{\ell_2^2}{\ell_1^2} + \frac{\ell_2^3}{\ell_1^3}}_{\mathcal{T}_{\mathrm{y}^1_1}\mathbb{E}^1_{\mathrm{y}}} + \underbrace{\frac{\ell_1}{\ell} + \frac{\ell_2}{\ell} + \frac{\ell}{\ell_1} + \frac{\ell}{\ell_2}}_{\mathcal{T}_{\mathrm{y}}\mathbb{Y}} + \underbrace{\frac{\ell_1^2}{\ell\ell_2}}_{\mathcal{O}_{\mathbb{E}^1}(-1)_{\mathrm{y}^1_1}}. \end{aligned}$$

$$\mathcal{A}_{\mathrm{y}} = \ell\ell_1 + \ell\ell_2 \quad \text{and} \quad \mathcal{O}(\mathbb{E}^1)_{(\mathrm{y},\overline{\ell_1^3})} = \mathcal{O}_{\mathbb{E}^1}(-1)_{(\mathrm{y},\overline{\ell_1^3})} = \ell_1^3 \overset{from\,(6.5)}{\longleftrightarrow} \frac{\ell_1^2}{\ell\ell_2}.$$

We listed below, the eigen-decomposition of the tangent and first exceptional divisor at each fixed point $\mathrm{y}^1_i \in \mathbb{E}^1$, following the description above.

Fixed point type for $\ell = x_0$, $\ell_1 = x_1$ and $\ell_2 = x_2$.	Tangent and first exceptional divisor
$\mathrm{y}^1_1 = ([x_0x_1, x_0x_2], \overline{x_1^3})$	$\mathcal{T}_{\mathrm{y}^1_1}\mathbb{X}^1 = t^{(w_2-w_1)} + t^{(2w_2-2w_1)} + t^{(3w_2-3w_1)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_0-w_1)} + t^{(w_0-w_2)} + t^{(2w_1-w_0-w_2)}$, $\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}^1_1} = t^{(2w_1-w_0-w_2)}$.
$\mathrm{y}^1_2 = ([x_0x_1, x_0x_2], \overline{x_1^2x_2})$	$\mathcal{T}_{\mathrm{y}^1_2}\mathbb{X}^1 = t^{(w_1-w_2)} + t^{(w_2-w_1)} + t^{(2w_2-2w_1)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_0-w_1)} + t^{(w_0-w_2)} + t^{(w_1-w_0)}$, $\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}^1_2} = t^{(w_1-w_0)}$.
$\mathrm{y}^1_3 = ([x_0x_1, x_0x_2], \overline{x_1x_2^2})$	$\mathcal{T}_{\mathrm{y}^1_3}\mathbb{X}^1 =$ permute w_1 and w_2 in $\mathcal{T}_{\mathrm{y}^1_2}\mathbb{X}^1$, $\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}^1_3} = t^{(w_2-w_0)}$.
$\mathrm{y}^1_4 = ([x_0x_1, x_0x_2], \overline{x_2^3})$	$\mathcal{T}_{\mathrm{y}^1_4}\mathbb{X}^1 =$ permute w_1 and w_2 in $\mathcal{T}_{\mathrm{y}^1_1}\mathbb{X}^1$, $\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}^1_4} = t^{(2w_2-w_0-w_1)}$.

Thus the contribution to (5.1) at each $\mathrm{y}^1_i \in \mathbb{E}^1$ lying over $\mathrm{y} = [\ell\ell_1, \ell\ell_2] \in \mathbb{Y}\setminus\mathbb{W}$ is given by:

Fixed point for $\ell = x_0$, $\ell_1 = x_1$ and $\ell_2 = x_2$.	Contribution to the numerator in (5.1)	Contribution to the denominator in (5.1)
$\mathrm{y}_1^1 = (\mathrm{y}, \overline{x_1^3})$	$2c_1^T(\mathcal{A}_\mathrm{y}) = 2(2w_0 + w_1 + w_2) +$ $c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}_1^1}) = 2w_1 - w_0 - w_2.$	$c_8^T(\mathcal{T}_{\mathrm{y}_1^1}\mathbb{X}^1) =$ $(w_2 - w_1)(2w_2 - 2w_1)(3w_2 - 3w_1)\cdot$ $(w_1 - w_0)(w_2 - w_0)(w_0 - w_1)\cdot$ $(w_0 - w_2)(2w_1 - w_0 - w_2).$
$\mathrm{y}_2^1 = (\mathrm{y}, \overline{x_1^2 x_2})$	$2c_1^T(\mathcal{A}_\mathrm{y}) = 2(2w_0 + w_1 + w_2) +$ $c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}_2^1}) = w_1 - w_0.$	$c_8^T(\mathcal{T}_{\mathrm{y}_2^1}\mathbb{X}^1) =$ $(w_1 - w_2)w_2 - w_1)(2w_2 - 2w_1)(w_1 - w_0)\cdot$ $(w_2 - w_0)(w_0 - w_1)(w_0 - w_2)(w_1 - w_0).$
$\mathrm{y}_3^1 = (\mathrm{y}, \overline{x_1 x_2^2})$	permute w_1 and w_2 in $2c_2^T(\mathcal{A}_\mathrm{y}) + c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}_2^1})$.	$c_8^T(\mathcal{T}_{\mathrm{y}_3^1}\mathbb{X}^1) =$ permute w_1 and w_2 in $c_8^T(\mathcal{T}_{\mathrm{y}_2^1}\mathbb{X}^1)$.
$\mathrm{y}_4^1 = (\mathrm{y}, \overline{x_2^3})$	permute w_1 and w_2 in $2c_2^T(\mathcal{A}_\mathrm{y}) + c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{y}_1^1})$.	$c_8^T(\mathcal{T}_{\mathrm{y}_4^1}\mathbb{X}^1) =$ permute w_1 and w_2 in $c_8^T(\mathcal{T}_{\mathrm{y}_1^1}\mathbb{X}^1)$.

In fact, if we make a cyclic permutation of $x_i's$ in the table above, we will obtain all the 12 contributions to (5.1) determined by the 3 fixed points $\pi_i \in \mathbb{Y} \setminus \mathbb{W}$ for $i = 7, 8, 9$ (cf. (6.2)).

6.3 Fixed points in $\mathbb{E}^2$

According to (6.1) we have to look for fixed points in $\mathbb{W}$. After some computation we find the following 6 fixed points in $\mathbb{W}$:

$$\pi_{10} = [x_0^2, x_0x_1], \pi_{11} = [x_0^2, x_0x_2], \pi_{12} = [x_1^2, x_1x_0], \pi_{13} = [x_1^2, x_1x_2], \pi_{14} = [x_2^2, x_2x_0], \pi_{15} = [x_2^2, x_2x_1]. \tag{6.7}$$

In order to determine the contributions to (5.1) in this case we have to calculate $c_8^T(\mathcal{T}_{\pi^2}\mathbb{X}^2)$, $c_1^T(\mathcal{O}(\mathbb{E}^2)_{\pi^2})$ for those fixed points $\pi^2 \in \mathbb{E}^2$ lying over fixed points $\pi^1 \in \mathbb{W}^1 \subset \mathbb{E}^1$, $c_1^T(\mathcal{O}(\mathbb{E}^1)_{\pi^1})$ for those fixed points $\pi^1 \in \mathbb{E}^1$ lying over π_i and $2c_1^T(\mathcal{A}_{\pi_i})$ for $i = 10, ..., 15$.

We have from (3.3) that the fiber of $\mathbb{E}^1$ over $[\ell^2, \ell\ell_1] \in \mathbb{W}$ is given by:

$$\mathbb{E}^1_{[\ell^2, \ell\ell_1]} = \mathbb{P}(\langle \overline{\ell\ell_2^2}, \overline{\ell_1^3}, \overline{\ell_1^2\ell_2}, \overline{\ell_1\ell_2^2} \rangle) \tag{6.8}$$

where $\overline{f}$ indicates classes of $f \in \mathcal{F}_3^{\mathrm{p}}$ modulo $\ell \cdot \mathcal{F}_2^{\mathrm{p}}$ with $\{\mathrm{p}\} = \ell \cap \ell_1$. Note that, $\ell = x_0, \ell_1 = x_1$ for π_{10} and so on. And can be verified that $([\ell^2, \ell\ell_1], \ell \cdot \mathcal{F}^{\mathrm{p}} + [f]) \in \mathbb{E}^1 \subset \mathbb{X}^1$ with $f \in \{\ell\ell_2^2, \ell_1^3, \ell_1^2\ell_2, \ell_1\ell_2^2\}$ are fixed points for the induced action of $T = \mathbb{C}^*$ on $\mathbb{X}^1$. Thus we obtain:

$$\begin{cases} 6 \times 3 = 18 & \text{fixed points lying in } \mathbb{E}^1 \setminus \mathbb{W}^1 \text{ if } f \in \{\ell_1^3, \ell_1^2\ell_2, \ell_1\ell_2^2\}, \\ 6 \times 1 = 6 & \text{fixed points lying in } \mathbb{W}^1 \text{ if } f = \ell\ell_2^2. \end{cases} \tag{6.9}$$

In order to compute the contribution of this fixed points to (5.1) we need to determine tangent and normal spaces as we did in (6.4).

We may write the following decomposition into eigen spaces for $[\ell^2, \ell\ell_1] \in \mathbb{W}$,

$$\begin{aligned}(\mathcal{N}_{\mathbb{Y}}\mathbb{X})_{[\ell^2,\ell\ell_1]} &= \mathcal{T}_{[\ell^2,\ell\ell_1]}\mathbb{X} - \mathcal{T}_{[\ell^2,\ell\ell_1]}\mathbb{Y} \\ &= \mathcal{T}_{[\ell^2,\ell\ell_1]}\mathbb{G}_2(\mathcal{F}_2) - \mathcal{T}_{[\ell^2,\ell\ell_1]}(\overbrace{\mathbb{P}(\mathcal{F}) \times \mathbb{G}_2(\mathcal{F})}^{\cong \mathbb{Y}}) \\ &= (\ell^2+\ell\ell_1)^\vee \otimes \overbrace{(\ell\ell_2+\ell_1^2+\ell_1\ell_2+\ell_2^2)}^{\mathcal{F}_2-(\ell^2+\ell\ell_1)} - \overbrace{\left(\frac{\ell_1}{\ell}+\frac{\ell_2}{\ell}+\frac{\ell_2}{\ell}+\frac{\ell_2}{\ell_1}\right)}^{\mathcal{T}_{[\ell^2,\ell\ell_1]}\mathbb{Y}} \\ &= \tfrac{\ell_1^2}{\ell^2}+\tfrac{\ell_1\ell_2}{\ell^2}+\tfrac{\ell_2^2}{\ell^2}+\tfrac{\ell_2^2}{\ell\ell_1}.\end{aligned} \tag{6.10}$$

Note that from (6.8) and (6.10) we have the two descriptions,

$$\mathbb{E}^1_{[\ell^2,\ell\ell_1]} = \mathbb{P}((\mathcal{N}_{\mathbb{Y}}\mathbb{X})_{[\ell^2,\ell\ell_1]}) = \mathbb{P}(\langle \overline{\ell\ell_2^2}, \overline{\ell_1^3}, \overline{\ell_1^2\ell_2}, \overline{\ell_1\ell_2^2}\rangle) \text{ and } (\mathcal{N}_{\mathbb{Y}}\mathbb{X})_{[\ell^2,\ell\ell_1]} = \frac{\ell_1^2}{\ell^2}+\frac{\ell_1\ell_2}{\ell^2}+\frac{\ell_2^2}{\ell^2}+\frac{\ell_2^2}{\ell\ell_1}.$$

Again we can reconcile this two descriptions as we did in (6.5). In this case the correspondence is given by:

$$\begin{array}{cccc} \frac{\ell_1^2}{\ell^2} & \frac{\ell_1\ell_2}{\ell^2} & \frac{\ell_2^2}{\ell^2} & \frac{\ell_2^2}{\ell\ell_1} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \ell_1^3 & \ell_1^2\ell_2 & \ell_1\ell_2^2 & \ell\ell_2^2 \end{array} \tag{6.11}$$

Now, let $\mathrm{w}_i^1 = ([\ell^2, \ell\ell_1], \overline{f_i}) \in \mathbb{E}^1$ with $f_i \in \{\ell\ell_2^2, \ell_1^3, \ell_1^2\ell_2, \ell_1\ell_2^2\}$.

Contributions to (5.1) coming from w_i^1 for $i = 2, 3, 4$

Note that the three points w_i^1 for $i = 2, 3, 4$ lift (isomorphically) all the way up to $\mathbb{X}^2$ since $\mathbb{X}^2 \setminus \mathbb{E}^2 \cong \mathbb{X}^1 \setminus \mathbb{W}^1$. So their contribution can be obtained at once on $\mathbb{X}^1$.

So for $\mathrm{w}_2^1 = (\mathrm{w}, \overline{\ell_1^3}) \in \mathbb{E}^1$ lying over $\mathrm{w} = [\ell^2, \ell\ell_1]$ we have from (6.6) that

$$\begin{aligned}\mathcal{T}_{\mathrm{w}_2^1}\mathbb{X}^1 &= \mathcal{T}_{\mathrm{w}_2^1}\mathbb{E}^1_{\mathrm{w}} + \mathcal{T}_{\mathrm{w}}\mathbb{Y} + [\mathrm{w}_2^1] \\ &= \mathcal{T}_{\mathrm{w}_2^1}\overbrace{\mathbb{P}([\overline{\ell\ell_2^2}, \overline{\ell_1^3}, \overline{\ell_1^2\ell_2}, \overline{\ell_1\ell_2^2}])}^{\mathbb{E}^1_{\mathrm{w}}} + \mathcal{T}_{\mathrm{w}}(\overbrace{\mathbb{P}(\mathcal{F}) \times \mathbb{G}_{\in}(\mathcal{F})}^{\cong \mathbb{Y}}) + [\mathrm{w}_2^1] \\ &= \underbrace{\tfrac{\ell\ell_2^2}{\ell_1^3}+\tfrac{\ell_2}{\ell_1}+\tfrac{\ell_2^2}{\ell_1^2}}_{\mathcal{T}_{\mathrm{w}_2^1}\mathbb{E}^1_{\mathrm{w}}} + \underbrace{\tfrac{\ell_1}{\ell}+\tfrac{\ell_2}{\ell}+\tfrac{\ell_2}{\ell}+\tfrac{\ell_2}{\ell_1}}_{\mathcal{T}_{\mathrm{w}}\mathbb{Y}} + \underbrace{\tfrac{\ell_1^2}{\ell^2}}_{\mathcal{O}_{\mathbb{E}^1}(-1)_{\mathrm{w}_2^1}}.\end{aligned} \tag{6.12}$$

$$\mathcal{A}_{[\ell^2,\ell\ell_1]} = \ell^2+\ell\ell_1 \quad \text{and} \quad \mathcal{O}(\mathbb{E}^1)_{([\ell^2,\ell\ell_1],\overline{\ell_1^3})} = \mathcal{O}_{\mathbb{E}^1}(-1)_{([\ell^2,\ell\ell_1],\overline{\ell_1^3})} = \ell_1^3 \overset{from\,(6.11)}{\longleftrightarrow} \tfrac{\ell_1^2}{\ell^2}.$$

We listed below, the eigen-decomposition of the tangent and first exceptional divisor at each fixed point $\mathrm{w}_i^1 \in \mathbb{E}^1$ for $i = 2, 3, 4$, following the description above.

Fixed point for $\ell = x_0$ and $\ell_1 = x_1$.	Tangent and first exceptional divisor
$\mathrm{w}_2^1 = ([x_0^2, x_0x_1], \overline{x_1^3})$	$\mathcal{T}_{\mathrm{w}_2^1}\mathbb{X}^1 = t^{(w_0+2w_2-3w_1)} + t^{(w_2-w_1)} + t^{(2w_2-2w_1)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_1)} + t^{(2w_1-2w_0)}$, $\mathcal{O}(\mathbb{E}^1)_{\mathrm{w}_2^1} = t^{(2w_1-2w_0)}$.
$\mathrm{w}_3^1 = ([x_0^2, x_0x_1], \overline{x_1^2x_2})$	$\mathcal{T}_{\mathrm{w}_3^1}\mathbb{X}^1 = t^{(w_0+w_2-2w_1)} + t^{(w_1-w_2)} + t^{(w_2-w_1)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_1)} + t^{(w_1+w_2-2w_0)}$, $\mathcal{O}(\mathbb{E}^1)_{\mathrm{w}_3^1} = t^{(w_1+w_2-2w_0)}$.
$\mathrm{w}_4^1 = ([x_0^2, x_0x_1], \overline{x_1x_2^2})$	$\mathcal{T}_{\mathrm{w}_4^1}\mathbb{X}^1 = t^{(w_0-w_1)} + t^{(2w_1-2w_2)} + t^{(w_1-w_2)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_1)} + t^{(2w_2-2w_0)}$, $\mathcal{O}(\mathbb{E}^1)_{\mathrm{w}_4^1} = t^{(2w_2-2w_0)}$.

Thus the contribution to (5.1) at each $\mathrm{w}_i^1 \in \mathbb{E}^1$ for $i = 2, 3, 4$ lying over $\mathrm{w} = [\ell^2, \ell\ell_1]$ is given by:

Fixed point for $\ell = x_0$, $\ell_1 = x_1$.	Contribution to the numerator in (5.1) $\underbrace{2c_1^T(\mathcal{A}_\mathrm{w})}_{2(3w_0+w_1)} + c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{w}_i^1})$	Contribution to the denominator in (5.1) $c_8^T(\mathcal{T}_{\mathrm{w}_i^1}\mathbb{X}^1)$
$\mathrm{w}_2^1 = (\mathrm{w}, \overline{x_1^3})$	$2(3w_0 + w_1) + (2w_1 - 2w_0)$	$4(w_0 + 2w_2 - 3w_1)(w_2 - w_1)^3(w_1 - w_0)^2(w_2 - w_0)^2$
$\mathrm{w}_3^1 = (\mathrm{w}, \overline{x_1^2x_2})$	$2(3w_0 + w_1) + (w_1 + w_2 - 2w_0)$	$-(w_0 + w_2 - 2w_1)(w_2 - w_1)^3(w_1 - w_0)(w_2 - w_0)^2 (w_1 + w_2 - 2w_0)$
$\mathrm{w}_4^1 = (\mathrm{w}, \overline{x_1x_2^2})$	$2(3w_0 + w_1) + (2w_2 - 2w_0)$	$4(w_0 - w_1)^2(w_1 - w_2)^3(w_2 - w_0)^3$

Making a cyclic permutation of $x_i's$ in the table above, we will obtain all the 18 contributions to (5.1) determined by the 6 fixed points $\pi_i \in \mathbb{W}$ for $i = 10, ..., 15$ (cf. (6.7) and (6.9)).

Contributions to (5.1) coming from $\mathrm{w}_1^1 = ([\ell^2, \ell\ell_1], \overline{\ell\ell_2^2}) \in \mathbb{W}^1$

In order to determine the contributions to (5.1) in this case we have to calculate $c_8^T(\mathcal{T}_{\mathrm{w}^2}\mathbb{X}^2)$, $c_1^T(\mathcal{O}(\mathbb{E}^2)_{\mathrm{w}^2})$ for those fixed points $\mathrm{w}^2 \in \mathbb{E}^2$ lying over the fixed point $\mathrm{w}_1^1 \in \mathbb{W}^1 \subset \mathbb{E}^1$, $c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{w}_1^1})$ and $2c_1^T(\mathcal{A}_\mathrm{w})$.

Consider now the fiber of $\mathbb{E}^2$ over $\mathrm{w}_1^1 = ([\ell^2, \ell\ell_1], \overline{\ell\ell_2^2})$. According to (3.5), it is just

$$\mathbb{E}^2_{([\ell^2,\ell\ell_1],\widetilde{\ell\ell_2^2})} = \mathbb{P}(\langle\, \widetilde{\ell_1^4}, \widetilde{\ell_1^3\ell_2}, \widetilde{\ell_1^2\ell_2^2}, \widetilde{\ell_1\ell_2^3}, \widetilde{\ell_2^4}\,\rangle) \tag{6.13}$$

where $\widetilde{g}$ indicates classes of $g \in \mathcal{F}_4$ modulo $\ell\cdot\mathcal{F}_3$. And can be verified that $([\ell^2, \ell\ell_1], \ell\cdot\mathcal{F}_2, \ell\cdot\mathcal{F}_2 + [g]) \in \mathbb{E}^2 \subset \mathbb{X}^2$ with $g \in \{\ell_1^4, \ell_1^3\ell_2, \ell_1^2\ell_2^2, \ell_1\ell_2^3, \ell_2^4\}$ are fixed points for the induced action of $T = \mathbb{C}^*$ on $\mathbb{X}^2$. Thus we obtain $6 \times 5 = 30$ fixed points lying in $\mathbb{E}^2$. In order to compute the contribution of this 30 fixed points to (5.1) we need to determine tangent and normal spaces as we did in (6.4).

We may write the following decomposition into eigen spaces for $\mathrm{w}_1^1=(\mathrm{w},\overline{\ell\ell_2^2})\in\mathbb{W}^1$ lying over $\mathrm{w}=[\ell^2,\ell\ell_1]$,

$$\begin{aligned}
(\mathcal{N}_{\mathbb{W}^1}\mathbb{X}^1)_{\mathrm{w}_1^1} &= \mathcal{T}_{\mathrm{w}_1^1}\mathbb{X}^1-\mathcal{T}_{\mathrm{w}_1^1}\mathbb{W}^1\\
&= \mathcal{T}_{\mathrm{w}_1^1}\mathbb{E}^1_{\mathrm{w}}+\mathcal{T}_{\mathrm{w}}\mathbb{Y}+[\mathrm{w}_1^1]-\mathcal{T}_{\mathrm{w}_1^1}\mathbb{W}^1\\
&= \mathcal{T}_{\mathrm{w}_1^1}\overbrace{\mathbb{P}([\overline{\ell\ell_2^2},\overline{\ell_1^3},\overline{\ell_1^2\ell_2},\overline{\ell_1\ell_2^2}])}^{\mathbb{E}^1_{\mathrm{w}}}+\mathcal{T}_{\mathrm{w}}\overbrace{(\mathbb{P}(\mathcal{F})\times\mathbb{G}_{\in}(\mathcal{F}))}^{\cong\mathbb{Y}}+[\mathrm{w}_1^1]-\mathcal{T}_{\mathrm{w}_1^1}\mathbb{W}^1\\
&= \underbrace{\frac{\ell_1^3}{\ell\ell_2^2}+\frac{\ell_1^2}{\ell\ell_2}+\frac{\ell_1}{\ell}}_{\mathcal{T}_{\mathrm{w}_1^1}\mathbb{E}^1_{\mathrm{w}}}+\underbrace{\frac{\ell_1}{\ell}+\frac{\ell_2}{\ell}+\frac{\ell_2}{\ell}+\frac{\ell_2}{\ell_1}}_{\mathcal{T}_{\mathrm{w}}\mathbb{Y}}+\underbrace{\frac{\ell_2^2}{\ell\ell_1}}_{\mathcal{O}_{\mathbb{E}^1}(-1)_{\mathrm{w}_1^1}}-\underbrace{\left(\frac{\ell_1}{\ell}+\frac{\ell_2}{\ell}+\frac{\ell_2}{\ell_1}\right)}_{\mathcal{T}_{\mathrm{w}_1^1}\mathbb{W}^1}\\
&= \frac{\ell_1^3}{\ell\ell_2^2}+\frac{\ell_1^2}{\ell\ell_2}+\frac{\ell_1}{\ell}+\frac{\ell_2}{\ell}+\frac{\ell_2^2}{\ell\ell_1}.
\end{aligned}\tag{6.14}$$

Note that from (6.13) and (6.14) we have the two descriptions at $\mathrm{w}_1^1=([\ell^2,\ell\ell_1],\overline{\ell\ell_2^2})$,

$$\mathbb{E}^2_{\mathrm{w}_1^1}=\mathbb{P}((\mathcal{N}_{\mathbb{W}^1}\mathbb{X}^1)_{\mathrm{w}_1^1})=\mathbb{P}(\langle\widetilde{\ell_1^4},\widetilde{\ell_1^3\ell_2},\widetilde{\ell_1^2\ell_2^2},\widetilde{\ell_1\ell_2^3},\widetilde{\ell_2^4}\rangle)\text{ and }(\mathcal{N}_{\mathbb{W}^1}\mathbb{X}^1)_{\mathrm{w}_1^1}=\frac{\ell_1^3}{\ell\ell_2^2}+\frac{\ell_1^2}{\ell\ell_2}+\frac{\ell_1}{\ell}+\frac{\ell_2}{\ell}+\frac{\ell_2^2}{\ell\ell_1}.$$

We can reconcile this two descriptions as we did in (6.5) and (6.11). In this case the correspondence is given by:

$$\begin{array}{ccccc}
\frac{\ell_1^3}{\ell\ell_2^2} & \frac{\ell_1^2}{\ell\ell_2} & \frac{\ell_1}{\ell} & \frac{\ell_2}{\ell} & \frac{\ell_2^2}{\ell\ell_1}\\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow\\
\ell_1^4 & \ell_1^3\ell_2 & \ell_1^2\ell_2^2 & \ell_1\ell_2^3 & \ell_2^4
\end{array}\tag{6.15}$$

Now let $\mathrm{w}_i^2=(\mathrm{w}_1^1,\widetilde{g_i})\in\mathbb{E}^2$ with $g_i\in\{\ell_1^4,\ell_1^3\ell_2,\ell_1^2\ell_2^2,\ell_1\ell_2^3,\ell_2^4\}$. For $\mathrm{w}_1^2=(\mathrm{w}_1^1,\widetilde{\ell_1^4})\in\mathbb{E}^2$ lying over $\mathrm{w}_1^1=([\ell^2,\ell\ell_1],\overline{\ell\ell_2^2})$ we have from (6.6) changing 1 by 2 that:

$$\begin{aligned}
\mathcal{T}_{\mathrm{w}_1^2}\mathbb{X}^2 &= \mathcal{T}_{\mathrm{w}_1^2}\mathbb{E}^2_{\mathrm{w}_1^1}+\mathcal{T}_{\mathrm{w}_1^1}\mathbb{W}^1+[\mathrm{w}_1^2]\\
&= \underbrace{\frac{\ell_1^3\ell_2}{\ell_1^4}+\frac{\ell_2^2}{\ell_1^2}+\frac{\ell_2^3}{\ell_1^3}+\frac{\ell_2^4}{\ell_1^4}}_{\mathcal{T}_{\mathrm{w}_1^2}\mathbb{E}^2_{\mathrm{w}_1^1}}+\underbrace{\frac{\ell_1}{\ell}+\frac{\ell_2}{\ell}+\frac{\ell_2}{\ell_1}}_{\mathcal{T}_{\mathrm{w}_1^1}\mathbb{W}^1}+\underbrace{\frac{\ell_1^3}{\ell\ell_2^2}}_{\mathcal{O}_{\mathbb{E}^2}(-1)_{\mathrm{w}_1^2}}.
\end{aligned}$$

$$\mathcal{O}(\mathbb{E}^2)_{\mathrm{w}_1^2}=\mathcal{O}_{\mathbb{E}^2}(-1)_{(\mathrm{w}_1^1,\widetilde{\ell_1^4})}\quad=\ell_1^4\ (\overset{from\,(6.15)}{\longleftrightarrow}\frac{\ell_1^3}{\ell\ell_2^2}).$$

We listed below, the eigen-decomposition of the tangent and first exceptional divisor at each fixed point $\mathrm{w}_i^2\in\mathbb{E}^2$ for $i=1,...,5$, following the description above.

Fixed point for $\ell = x_0$, $\ell_1 = x_1$ and $\ell_2 = x_2$.	Tangent and second exceptional divisor
$\mathrm{w}_1^2 = (\mathrm{w}_1^1, \widetilde{x_1^4})$	$\mathcal{T}_{\mathrm{w}_1^2}\mathbb{X}^2 = t^{(w_2-w_1)} + t^{(2w_2-2w_1)} + t^{(3w_2-3w_1)} + t^{(4w_2-4w_1)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_1)} + t^{(3w_1-w_0-2w_2)}$, $\mathcal{O}(\mathbb{E}^2)_{\mathrm{w}_1^2} = t^{(3w_1-w_0-2w_2)}$.
$\mathrm{w}_2^2 = (\mathrm{w}_1^1, \widetilde{x_1^3x_2})$	$\mathcal{T}_{\mathrm{w}_2^2}\mathbb{X}^2 = t^{(w_1-w_2)} + t^{(w_2-w_1)} + t^{(2w_2-2w_1)} + t^{(3w_2-3w_1)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_1)} + t^{(2w_1-w_0-w_2)}$, $\mathcal{O}(\mathbb{E}^2)_{\mathrm{w}_2^2} = t^{(2w_1-w_0-w_2)}$.
$\mathrm{w}_3^2 = (\mathrm{w}_1^1, \widetilde{x_1^2x_2^2})$	$\mathcal{T}_{\mathrm{w}_3^2}\mathbb{X}^2 = t^{(2w_1-2w_2)} + t^{(w_1-w_2)} + t^{(w_2-w_1)} + t^{(2w_2-2w_1)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_1)} + t^{(w_1-w_0)}$, $\mathcal{O}(\mathbb{E}^2)_{\mathrm{w}_3^2} = t^{(w_1-w_0)}$.
$\mathrm{w}_4^2 = (\mathrm{w}_1^1, \widetilde{x_1x_2^3})$	$\mathcal{T}_{\mathrm{w}_4^2}\mathbb{X}^2 = t^{(3w_1-3w_2)} + t^{(2w_1-2w_2)} + t^{(w_1-w_2)} + t^{(w_2-w_1)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_1)} + t^{(w_2-w_0)}$, $\mathcal{O}(\mathbb{E}^2)_{\mathrm{w}_4^2} = t^{(w_2-w_0)}$.
$\mathrm{w}_5^2 = (\mathrm{w}_1^1, \widetilde{x_2^4})$	$\mathcal{T}_{\mathrm{w}_5^2}\mathbb{X}^2 = t^{(4w_1-4w_2)} + t^{(3w_1-3w_2)} + t^{(2w_1-2w_2)} + t^{(w_1-w_2)} + t^{(w_1-w_0)} + t^{(w_2-w_0)} + t^{(w_2-w_1)} + t^{(2w_2-w_0-w1)}$, $\mathcal{O}(\mathbb{E}^2)_{\mathrm{w}_5^2} = t^{(2w_2-w_0-w1)}$.

Following the description given in (6.12), we obtain the following eigen-decomposition for the tangent space of $\mathbb{X}^1$ at w_1^1.

$$\begin{aligned}\mathcal{T}_{\mathrm{w}_1^1}\mathbb{X}^1 &= \mathcal{T}_{\mathrm{w}_1^1}\mathbb{E}^1_{\mathrm{w}} + \mathcal{T}_{\mathrm{w}}\mathbb{Y} + [\mathrm{w}_1^1] \\ &= \mathcal{T}_{\mathrm{w}_1^1}\overbrace{\mathbb{P}([\overline{\ell\ell_2^2}, \overline{\ell_1^3}, \overline{\ell_1^2\ell_2}, \overline{\ell_1\ell_2^2}])}^{\mathbb{E}^1_{\mathrm{w}}} + \mathcal{T}_{\mathrm{w}}\overbrace{(\mathbb{P}(\mathcal{F}) \times \mathbb{G}_{\in}(\mathcal{F}))}^{\cong \mathbb{Y}} + [\mathrm{w}_1^1] \\ &= \underbrace{\frac{\ell_1^3}{\ell\ell_2^2} + \frac{\ell_1^2}{\ell\ell_2} + \frac{\ell_1}{\ell}}_{\mathcal{T}_{\mathrm{w}_1^1}\mathbb{E}^1_{\mathrm{w}}} + \underbrace{\frac{\ell_1}{\ell} + \frac{\ell_2}{\ell} + \frac{\ell_2}{\ell} + \frac{\ell_2}{\ell_1}}_{\mathcal{T}_{\mathrm{w}}\mathbb{Y}} + \underbrace{\frac{\ell_2^2}{\ell\ell_1}}_{\mathcal{O}_{\mathbb{E}^1}(-1)_{\mathrm{w}_1^1}}.\end{aligned} \tag{6.16}$$

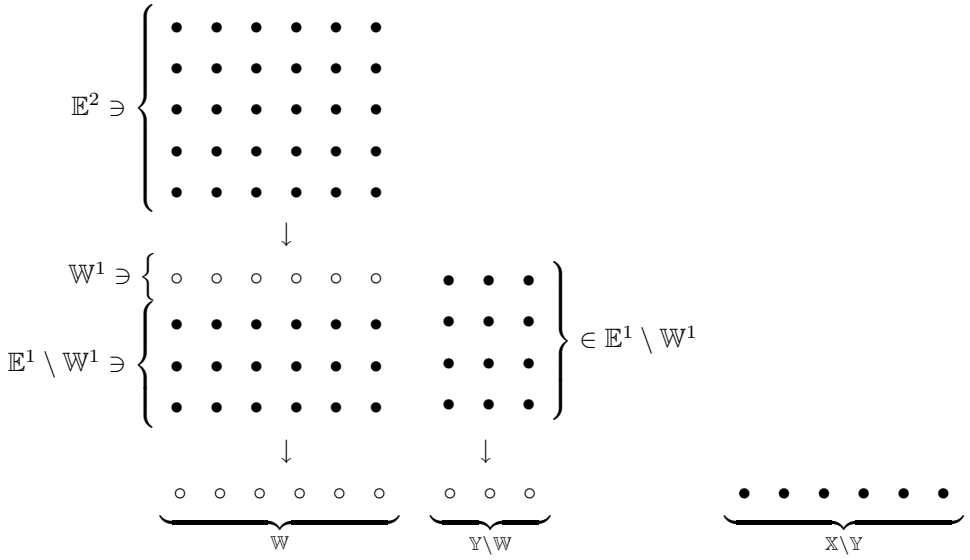
and $\mathcal{O}(\mathbb{E}^1)_{([\ell^2,\ell\ell_1],\overline{\ell\ell_2^2})} = \mathcal{O}_{\mathbb{E}^1}(-1)_{([\ell^2,\ell\ell_1],\overline{\ell\ell_2^2})} = \ell\ell_2^2 \overset{from\,(6.11)}{\longleftrightarrow} \frac{\ell_2^2}{\ell\ell_1}$, then $c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{w}_1^1}) = 2w_2 - w_0 - w_1$ doing $\ell = x_0, \ell_1 = x_1$ and $\ell_2 = x_2$.

Thus the contribution to (5.1) at each $\mathrm{w}_i^2 \in \mathbb{E}^2$ for $i = 1, ..., 5$ lying over w_1^1 is given by:

Fixed point for $\ell = x_0$, $\ell_1 = x_1$ and $\ell_2 = x_2$.	Contribution to the numerator in (5.1) $\underbrace{2c_1^T(\mathcal{A}_{\mathrm{w}})}_{2(3w_0+w_1)} + \underbrace{c_1^T(\mathcal{O}(\mathbb{E}^1)_{\mathrm{w}_1^1})}_{2w_2-w_0-w_1} + c_1^T(\mathcal{O}(\mathbb{E}^2)_{\mathrm{w}_i^2})$	Contribution to the denominator in (5.1) $c_8^T(\mathcal{T}_{\mathrm{w}_i^2}\mathbb{X}^2)$
$\mathrm{w}_1^2 = (\mathrm{w}_1^1, \widetilde{x_1^4})$	$(5w_0+w_1+2w_2)+(3w_1-w_0-2w_2)$	$24(w_2-w_1)^5(w_1-w_0)(w_2-w_0)(3w_1-w_0-2w_2)$
$\mathrm{w}_2^2 = (\mathrm{w}_1^1, \widetilde{x_1^3x_2})$	$(5w_0+w_1+2w_2)+(2w_1-w_0-w_2)$	$-6(w_2-w_1)^5(w_1-w_0)(w_2-w_0)(2w_1-w_0-w_2)$
$\mathrm{w}_3^2 = (\mathrm{w}_1^1, \widetilde{x_1^2x_2^2})$	$(5w_0+w_1+2w_2)+(w_1-w_0)$	$4(w_2-w_1)^5(w_1-w_0)^2(w_2-w_0)$
$\mathrm{w}_4^2 = (\mathrm{w}_1^1, \widetilde{x_1x_2^3})$	$(5w_0+w_1+2w_2)+(w_2-w_0)$	$-6(w_2-w_1)^5(w_1-w_0)(w_2-w_0)^2$
$\mathrm{w}_5^2 = (\mathrm{w}_1^1, \widetilde{x_2^4})$	$(5w_0+w_1+2w_2)+(2w_2-w_0-w_1)$	$24(w_2-w_1)^5(w_1-w_0)(w_2-w_0)(2w_2-w_0-w_1)$

Making a cyclic permutation of $x_i's$ in the table above, we will obtain all the 30 contributions to (5.1) determined by the 6 fixed points $\mathrm{w}_1^1 = ([\ell^2, \ell\ell_1], \overline{\ell\ell_2^2}) \in \mathbb{W}^1$ (cf. (6.7) and (6.9)).

So, there are altogether 66 fixed points as indicated bellow by the bold points. In fact, consider the diagram below, where we use "$\bullet$" to indicate the terminal fixed points and "$\circ$" to indicate the non-terminal ones.



In the first line in the bottom we put the $15 = \underbrace{6 \text{ (in } \mathbb{X}\setminus\mathbb{Y})}_{\text{terminal}} + \underbrace{3 \text{ (in } \mathbb{Y}\setminus\mathbb{W})}_{\text{non-terminal}} + \underbrace{6 \text{ (in } \mathbb{W})}_{\text{non-terminal}}$ fixed points in $\mathbb{X}$.

In the middle, we have 12 (respectively 18) terminal fixed points in $\mathbb{E}^1 \setminus \mathbb{W}^1$ that are mapped to the 3 (respectively 6) fixed points in $\mathbb{Y}\setminus\mathbb{W}$ (respectively $\mathbb{W}$) by the the first blow-up map, and we also have 6 non-terminal fixed points in $\mathbb{W}^1$ that are mapped to the 6 fixed points in $\mathbb{W}$ ($\mathbb{W}^1$ is the second blow-up center and $\mathbb{W}^1 \cong \mathbb{W}$).

At the top, we have 30 terminal fixed points in $\mathbb{E}^2$ that are mapped to the 6 fixed points in $\mathbb{W}^1$ by the the second blow-up map (this last 6 fixed points in $\mathbb{W}^1$ are mapped isomorphically to the 6 fixed points in $\mathbb{W}$ by the first blow-up map).

Finally using a MAPLE script, we find one more time that there exist 105 squares whose vertices lie over 8 lines in general position in $\mathbb{P}^2$.

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