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Entire Functions in Weighted L_2 and Zero Modes of the Pauli Operator with Non-Signdefinite Magnetic Field

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ABSTRACT

For a real non-signdefinite function $B(z)$, $z \in \mathbb{C}$, we investigate the dimension of the space of entire analytical functions square integrable with weight $e^{\pm 2F}$, where the function $F(z) = F(x_1, x_2)$ satisfies the Poisson equation $\Delta F = B$. The answer is known for the function B with constant sign. We discuss some classes of non-signdefinite positively homogeneous functions B, where both infinite and zero dimension may occur. In the former case we present a method of constructing entire functions with prescribed behavior at infinity in different directions. The topic is closely related with the question of the dimension of the zero energy subspace (zero modes) for the Pauli operator.

RESUMEN

Para una función no signo definida $B(z)$, $z \in \mathbb{C}$, investigamos la dimensión del espacio de funciones analíticas enteras de cuadrado integrable con peso $e^{\pm 2F}$, donde la función $F(z) = F(x_1, x_2)$ verifica la ecuación de Poisson $\Delta F = B$. La respuesta es conocida para la función B con signo constante. Discutimos algunas clases de funciones B no signo definida e positivamente homogéneas, donde dimensión zero y infinita pueden ocurrir. En el caso anterior nosotros presentamos un método de construir funciones enteras con un comportamiento en infinito prescrito en diferentes direcciones. El tópico es estrechamente relacionado con la cuestión de la dimensión del subespacio de energía zero para el operador de Pauli.

Key words and phrases: *Pauli operators, Zero modes, Entire functions.*

Math. Subj. Class.: *30D15, 81Q10, 47N50, 35Q40.*

1 Introduction

In 1979 Y. Aharonov and A. Cacher in [1] discovered that the Pauli operator in dimension 2 with a compactly supported bounded magnetic field $B(x)$, $x = (x_1, x_2)$, can possess zero modes, eigenfunctions with zero energy. The number of these zero modes (the dimension of the zero energy eigenspace) is finite and is determined by the total flux of the magnetic field. The zero modes problem has been investigated further on and the Aharonov-Casher formula was extended to rather singular and not compactly supported magnetic fields being signed measures with finite total variation ([2]). On the other hand, for sign-definite fields with infinite flux, the authors proved in [4] that the space of zero modes is infinite-dimensional, thus extending the Aharonov-Casher formula to this case. Moreover, the infiniteness of zero modes was established in [4] for a class of magnetic fields with variable sign, such that in certain sense the part having one direction is infinitesimal with respect to the part with another direction, while both parts have infinite flux, as well for weakly perturbed constant magnetic fields. On the other hand, in [2] an example was constructed of a magnetic field consisting of tiny islands, sparsely placed in the plane, carrying positive magnetic field, on the background of annuli with negative field, such that both positive and negative parts of the field have infinite total flux, so that no zero modes exist. So, it was, generally, unclear, what is the situation with zero modes for the case when neither of sign-parts of the magnetic field prevails over the other one. After having been acquainted with [4], B.Simon asked the first author (G.R.) about the number of zero modes for a very simple configuration of the field of this kind: some constant with one sign in one half-plane and a constant with different sign in another one. The answer (no zero modes at all) was found quite easily, but a more general question arose: how many zero modes are generated by the magnetic field which is constant in a sector in the plane and constant, with different sign, in the complement of this sector, or, more generally, by a non-signdefinite radial-homogeneous field. The present paper contains some results in this direction. For the sector case, it turns out that if one of the sectors is sufficiently small, the space of zero modes is infinite-dimensional. On the other hand, if the angles of the sectors are sufficiently close to π, zero modes are completely absent. Somewhat similar situation takes place for fields with some other degree of homogeneity.

Starting from the paper [1], it became clear that the progress in the zero modes problem depends heavily on the properties of solutions of the Poisson equation $\Delta F(x) = B(x)$ in terms of $B(x)$. The zero modes are generated by entire analytical (or anti-analytical) functions $u(z)$ of the variable $z = x_1 + ix_2$ such that $u \exp(\pm F) \in L_2(\mathbb{R}^2)$. If $B(x) = B > 0$ is a nonzero constant the equation has a solution of the form $F(x) = \frac{B}{2}|x|^2$, and this fact obviously leads to the infiniteness of the dimension of the space of analytical functions with $u \exp(-F) \in L_2(\mathbb{R}^2)$. However, generally, the boundedness of B does not guarantee by itself a quadratic estimate for F, moreover, it may happen that the Poisson equation has no semi-bounded solutions, and such a straightforward reasoning about zero modes fails. We need a deeper analysis of entire functions, square integrable with weight $\exp(\pm 2F)$, without the condition imposed, that F is subharmonic (the subharmonic case is investigated exhaustively in [3] and [5]).

We start in Section 2 by considering solutions of the Poisson equations for a radial homogeneous right-hand side. In particular, for our 'sector' configuration of the field we construct a special solution of the Poisson equation. This solution F is not semi-bounded, behaves at infinity as $C|x|^2 \log |x|$ in all directions but four, with C depending on the direction and having variable sign. Next, in Sect.3, we construct entire analytical functions u such that $u \exp(-F) \in L_2$. Such functions u must decay rather rapidly in directions where F is negative, and they may grow, but in a controllable way, in directions where F is positive. We present a method for constructing entire functions with such behavior. This construction can be put through provided the angle of the sector where B is negative is sufficiently small. So, it turns out that in this latter case there are infinitely many zero modes.

We consider the case of a sector with an astute angle in Sect.4. We show that in this case there are no zero modes at all, provided that the angle of the sector is sufficiently close to π. The reason for this is that, if an entire function is square integrable in a sector with sufficiently fast growing weight, it must be zero everywhere, disregarding its behavior outside the sector. This idea requires certain work for being implemented in our case, since for a sector type magnetic field the sets where the potential has fixed sign differ only slightly from the quarter-plane so that logarithmic effects must be taken into account.

In the final section we briefly consider magnetic fields, radial homogeneous of some negative order.

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2 General Constructions

2.1 Homogeneous solutions of the Poisson equation

We identify the real plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}^1$, setting $z = x_1 + ix_2$; by $d\mu$ we denote the Lebesgue measure on the plane. Let $B(x)$ be a real-valued function in $\mathbb{R}^2$ positively homogeneous of degree s. Then, as it is well known, the solution of the Poisson equation

$$\Delta F(x) = B(x) \tag{2.1}$$

can be looked for as a positively homogeneous function of degree $s+2$. In fact, if the function B has the form $b(\psi)r^s$ in polar coordinates (r,ψ), we can look for F in the form $\varphi(\psi)r^{s+2}$ and obtain an equation for φ

$$\varphi(\psi)'' + (s+2)^2\varphi(\psi) = b(\psi) \tag{2.2}$$

If s in not an integer, (2.2) has a unique solution, and thus $F = \varphi(\psi)r^{s+2}$ is a solution for (2.1). However, if s is an integer, the equation (2.2) is solvable only for those b which are orthogonal to $\exp(i(s+2)\psi)$. If this orthogonality condition is not satisfied, the solution of (2.1) must contain a logarithmic factor,

$$F(r,\psi) = A\sin((s+2)(\psi-\psi_0))r^{s+2}\log r + \varphi(\psi)r^{s+2}, \tag{2.3}$$

with properly selected A and ψ_0. Such case will be referred to as the resonance one; if the orthogonality condition is met as well as for a noninteger s we have the non-resonance case.

For $0 < \alpha < \pi$ we denote by Ω_1 the sector $\psi = \arg z \in (0,\alpha)$ and by Ω_2 the complementing sector in the plane. Having two numbers, $b_1 < 0 < b_2$, we set $B(x) \equiv B(x) = 2b_1$ for $x \in \Omega_1$ and $B(x) = 2b_2$ in Ω_2. By scaling, one can reduce the situation to the case $b_2 = 1$ and we always suppose that it is already done.

We are looking for a solution $F(x)$ of the equation (2.1). Since the homogeneity degree equals $s = 0$, the solution must contain a power-logarithmical term as in (2.3). It is convenient to write the solution of (2.1) in a somewhat different form.

2.2 A solution of the Poisson equation for the sector configuration

We are looking for an explicit formula for F. This function will be constructed step-wise. We start by elementary solutions separately in Ω_1 and Ω_2. These solutions do not fit together on the ray $\arg z = \alpha$. Then some correction terms will be introduced.

So, we start with

$$\phi(z) = b_1x_2^2, z \in \Omega_1; \phi(z) = x_2^2, z \in \Omega_2.$$

This function satisfies the equation (2.1) everywhere except the ray $L_\alpha = \{\arg z = \alpha\}$ where it is discontinuous. To compensate this jump, as well as the jump of its derivative, we will use the branch of logarithm, continuous in the domain $\mathbb{C}^{(\alpha)} = \mathbb{C} \setminus L_\alpha$. For the function $\xi(z) = \frac{(ze^{-i\alpha})^2}{2\pi} \log z$ the imaginary part has a jump on L_α while the real part is continuous but has a discontinuous derivative. Our first correction will make the whole solution continuous on L_α. The jump of ϕ at the point $z_0 = r_0 e^{i\alpha}$ equals

$$\phi(z_0)|_{\Omega_1} - \phi(z_0)|_{\Omega_2} = -c_0 r_0^2 \sin^2 \alpha,$$

$c_0 = 1 - b_1$, therefore we set

$$\phi_1(z) = \phi(z) + c_0 \sin^2 \alpha \, \mathrm{Im}\,(\xi(z)). \tag{2.4}$$

Since the jump of the second summand in (2.4) at the point $z_0 = r_0 e^{i\alpha}$ equals $c_0 r_0^2 \sin^2 \alpha$, the function ϕ_1 is continuous in $\mathbb{C}$.

We consider now the the derivatives of ϕ_1 at L_α. Obviously, the derivative along the ray is continuous. The derivative across the ray has a jump, and we will compensate this jump by subtracting the real part of $\xi(z)$ with a proper coefficient. We set $F(z) = \phi_1(z) - c_0 \sin \alpha \cos \alpha \, \mathrm{Re}\,(\xi(z))$. Since we added the real and imaginary parts of functions that are analytic in $\mathbb{C}^{(\alpha)}$, the Poisson equation (2.1) will be satisfied by F in $\mathbb{C}^{(\alpha)}$. The function F and its derivatives are continuous everywhere, thus the distributional Laplacian of F coincides with the classical Laplacian, and therefore F is the solution we need. To get a better understanding of F, we represent it in a little bit different way:

$$F(z) = \phi(z) + \frac{c_0 \sin^2 \alpha}{2\pi} \mathrm{Re}\left(\frac{1}{i}(ze^{i\alpha})^2 \log z\right) - \tag{2.5}$$
$$\frac{c_0 \sin \alpha \cos \alpha}{2\pi} \mathrm{Re}\left((ze^{i\alpha})^2 \log z\right) = \phi(z) - \frac{c_0 \sin \alpha}{2\pi} \mathrm{Re}\left(\left(ze^{-\frac{i\alpha}{2}}\right)^2 \log z\right).$$

Note that the behavior of $F(z)$ for large $|z|$ is determined by the second, power-logarithmic term in (2.5), except the directions where it vanishes, i.e., except the directions $\arg(z) = \frac{\alpha}{2} + k\frac{\pi}{4}$, $k = 0, 1, 2, 3$. These half-lines divide $\mathbb{C}$ in four quarters, in two of those the function F grows as $C|z|^2 \log |z|$, with some positive C (depending on the direction), in the other two this functions tends to $-\infty$, again like $C|z|^2 \log |z|$ but with a negative C this time.

In the next Section we will construct entire analytical functions $u(z)$ such that $u \exp(F) \in L_2$.

3 Existence of Zero Modes

The aim of this Section is to establish the following fact concerning the sector configuration, as in Subsection 2.2.

Theorem 3.1. *Suppose that the size α of the sector and b_1 are sufficiently small. Then the space of entire analytical functions $u(z)$ satisfying $u \exp(F) \in L_2$ is infinite-dimensional.*

3.1 Construction of a subharmonic function

In this subsection we construct a subharmonic function of a special form, to be used further on in the construction of analytical functions with prescribed behavior at infinity. We fix some positive ϵ, to be determined later. Consider two sectors $\Theta_j^\circ = \{z : |\arg z - \pi j| < \epsilon\}$, $j = 0, 1$, and set $\Theta_j = \Theta_j^\circ \cap \{|z| > 1\}$. For some fixed σ, we cut each of the sectors into strips by straight lines $\mathrm{Im}\, z = k\sigma$; $k = 0, \pm 1, \pm 2, \dots$. Starting from the boundary lying closest to the imaginary axis, we cut each such strip by lines parallel to the imaginary axis, into domains having area σ^2. Just a finite number of such domains are not polygons, a few of domains in each strip are triangles or trapezia, all the rest are unit squares. We will denote generically all these pieces of different form by Q and the set of these domains by $\mathcal{Q}$; by $\mathcal{Q}_j$ we denote the set of pieces in Θ_j. For each $Q \in \mathcal{Q}$ we select a point $a_Q \in Q$ in the following way. If Q is a square we take the center of Q as a_Q. Otherwise we choose a_Q so that $\int_Q (z - a_Q) d\mu = 0$. A simple geometrical consideration shows that the distance between such points is not less than $\sigma/2$.

Now we define

$$V_\epsilon(z) = \mathrm{Re}\left[\sigma^2 \sum_{Q \in \mathcal{Q}} \left(\log\left(1 - \frac{z}{a_Q}\right) + \frac{z}{a_Q} + \frac{1}{2}\frac{z^2}{a_Q^2}\right)\right]. \tag{3.1}$$

It is clear that the series in (3.1) converges uniformly on compacts not containing the points a_Q and thus (3.1) defines a harmonic function in the plane, with these points removed. Due to symmetry, we can express $V_\epsilon(z)$ via the sum only over the domains Q belonging to $\mathcal{Q}_1$, i.e., lying in the right half-plane,

$$V_\epsilon(z) = \mathrm{Re}\left[\sigma^2 \sum_{Q \in \mathcal{Q}_1} \left(\log\left(1 - \frac{z^2}{a_Q^2}\right) + \frac{z^2}{a_Q^2}\right)\right]. \tag{3.2}$$

The function $V_\epsilon(z)$ will be approximated by the real part of the integral

$$W_\epsilon(z) = \int_{-\epsilon}^{\epsilon} d\theta \int_1^\infty \left(\log\left(1 - \frac{z^2}{\tau^2}e^{-2i\theta}\right) + \frac{z^2}{\tau^2}e^{-2i\theta}\right)\tau d\tau. \tag{3.3}$$

The behavior of $W_\epsilon(z)$ is studied in the Appendix. Let us estimate the difference $V_\epsilon(z) - \mathrm{Re}\, W_\epsilon(z)$ for $2\epsilon < |\arg z| < \pi - 2\epsilon$, i.e. outside some sectorial neighborhood of Θ_j (assuming $\epsilon < \pi/8$).

$$V_\epsilon(z) - \mathrm{Re}\, W_\epsilon(z) = \tag{3.4}$$
$$\sum_{Q \in \mathcal{Q}_1} \mathrm{Re}\left[\iint_Q \left(\log\left(1 - \frac{z^2}{a_Q^2}\right) + \frac{z^2}{a_Q^2} - \log\left(1 - \frac{z^2}{w^2}\right) - \frac{z^2}{w^2}\right) d\mu(w)\right].$$

To estimate a single term in (3.4), consider the function $\beta(w) = \log(1 - \frac{z^2}{w^2}) + \frac{z^2}{w^2}$, for $2\epsilon <$

$|\arg z| < \pi - 2\epsilon$. We have

$$\iint\limits_Q \beta(w)d\mu(w) =$$

$$\beta(a_Q) + \beta'(a_Q)\iint\limits_Q (w - a_Q)d\mu(w) + O\left(\iint\limits_Q |\beta''(w)|d\mu(w)\right).$$

Since $\iint\limits_Q (w - a_Q)d\mu(w) = 0$, we get the estimate

$$|V_\epsilon(z) - \text{Re } W_\epsilon(z)| \leq C \sum_{Q\in\mathcal{Q}_1} \int\limits_Q |\beta''(w)|d\mu(w). \tag{3.5}$$

Next,

$$\beta''(w) = \frac{2z^2}{w^2}\frac{3w^2 - z^2}{(w^2 - z^2)^2} + 6z^2w^{-4},$$

therefore, $|\beta''(w)| \leq C|z|^2|w|^{-4}$, so, finally,

$$|V_\epsilon(z) - \text{Re } W_\epsilon(z)| \leq C|z|^2 \iint\limits_{|w|\geq 1, |\arg w|<\epsilon} |w|^{-4}d\mu(w) \leq \epsilon|z|^2. \tag{3.6}$$

Now we are going to estimate (3.4) in the sectors around x_1-axis, $|\arg z - j\pi| \leq 2\epsilon$ for $j = 0$ or $j = 1$. Of course, V_ϵ has logarithmic singularities at all points $z = \pm a_Q$ and W_ϵ has not. We surround each point by a small disk, $|z \pm a_Q| \leq \frac{\sigma}{4}$and consider first this difference for z lying outside all these disks. We cut the angles into three parts,

$$\Gamma_1 : |w| \geq 2|z|;\ \Gamma_2 : |w| \leq \frac{1}{2}|z|;\ \Gamma_3 : |w| \in \left(\frac{1}{2}|z|, 2|z|\right).$$

Correspondingly, we denote by $\mathcal{Q}_{1,j}$ the set of those domains $Q \in \mathcal{Q}_1$ for which $a_Q \in \Gamma_j$, $j = 1, 2, 3$. We suppress the z-dependence of these sets in notations.

For $Q \in \mathcal{Q}_{1,1}$, $w \in Q$ we have $5|w| \geq |z - w|$, $|z + w| \geq \frac{|w|}{2}$, therefore the quantity $w^2(3w^2 - z^2)/(w^2 - z^2)^2$ is bounded and thus

$$|\beta''(w)| \leq C|z^2w^{-4}|. \tag{3.7}$$

Summing over $Q \in \mathcal{Q}_{1,1}$, we get

$$\sum_{Q\in\mathcal{Q}_{1,1}} \iint\limits_Q \beta''(w)|d\mu(w) \leq C\iint\limits_{\Gamma_1} |z^2w^{-4}|d\mu(w) \leq C. \tag{3.8}$$

For $Q \in \mathcal{Q}_{1,2}$, $w \in Q$, we note that $5|z| \geq |z - w|$, $|z + w| \geq \frac{|z|}{2}$, therefore the quantity $w^2(3w^2 - z^2)/(w^2 - z^2)^2$ is bounded and we again arrive at (3.7). Thus

$$\sum_{Q\in\mathcal{Q}_{1,2}} \iint\limits_Q |\beta''(w)|d\mu(w) \leq C\iint\limits_{\Gamma_2} |z^2w^{-4}|d\mu(w) \leq C\epsilon|z^2|. \tag{3.9}$$

The region Γ_3 requires a harder work. In any $Q \in \mathcal{Q}_{1,3}$, we write

$$\begin{aligned}
&\mathrm{Re}\left(\sigma^2 \log\left(1-\frac{z^2}{a_Q^2}\right)+\frac{z^2}{a_Q^2}\right) - \mathrm{Re}\iint\limits_Q \left(\log\left(1-\frac{z^2}{w^2}\right)+\frac{z^2}{w^2}\right)d\mu(w)\\
&= \iint\limits_Q \mathrm{Re}\left(\log(1-\frac{z^2}{a_Q^2}) - \log\left(1-\frac{z^2}{w^2}\right)\right)d\mu(w) + \\
&\iint\limits_Q \mathrm{Re}\left(\frac{z^2}{a_Q^2}-\frac{z^2}{w^2}\right)d\mu(w).
\end{aligned} \tag{3.10}$$

Consider the second term in (3.10). We have $\frac{z^2}{a_Q^2}-\frac{z^2}{w^2} = z^2(z+w)w^{-2}a_Q^{-2}\,(z-w)$. The quantities $|z|, |w|, |a_Q|$ are of the same order, while $|z-w| \le 2\sigma\epsilon^{-1/2}$. (Of course, $|z-w|\le \sigma\sqrt{2}$ if Q is a unit square, but if Q is a triangle or a trapezium, only the bound by $2\sigma\epsilon^{-1/2}$ is guaranteed.) Therefore, the second term in (3.10) is majorized by $C\sigma\epsilon^{-\frac{1}{2}}|z^{-1}|$, and since the quantity of domains in $\mathcal{Q}_{1,3}$ is of order $\sigma^{-2}\epsilon|z|^2$, we obtain the estimate

$$\sum_{Q\in\mathcal{Q}_{1,3}} \iint\limits_Q \mathrm{Re}\left(\frac{z^2}{a_Q^2}-\frac{z^2}{w^2}\right)d\mu(w) \le C\sigma^{-1}\sqrt{\epsilon}|z|. \tag{3.11}$$

Next we estimate the first term in (3.10). We transform the integrand as

$$\log\left(1-\frac{z^2}{a_Q^2}\right) - \log\left(1-\frac{z^2}{w^2}\right) = \log\frac{a_Q-z}{w-z} + \log\frac{a_Q+z}{w+z} + 2\log\frac{w}{a_Q}. \tag{3.12}$$

In the second term in (3.12) we write

$$\left|\log\frac{a_Q+z}{w+z}\right| = \left|\log\left(1+\frac{a_Q-w}{z+w}\right)\right| \le \frac{C}{\sqrt{\epsilon}|z|}. \tag{3.13}$$

Similarly, the third term in (3.12) is estimated as

$$\left|\log\frac{w}{a_Q}\right| = \left|\log\left(1+\frac{w-a_Q}{a_Q}\right)\right| \le \frac{C}{\sqrt{\epsilon}|z|}. \tag{3.14}$$

Now we pass to the first term in (3.12). We split the sum into two: the sum over such Q that $|a_Q - z| \le 10\sigma/\sqrt{\epsilon}$ and the sum over the remaining Q. Consider the first, finite sum (recall that $|z-a_Q| > \frac{\sigma}{4}$):

$$\sum_{Q\in\mathcal{Q}_{1,3}, |a_Q-z|\le 10\sigma/\sqrt{\epsilon}} \iint\limits_Q \left|\log\left(\frac{a_Q-z}{w-z}\right)\right| d\mu(w) \le C|\log\sigma|\epsilon^{-1}. \tag{3.15}$$

For the second sum we have $|w-z| \ge \frac{5\sigma}{\sqrt{\epsilon}}$ therefore for $w\in Q$

$$\left|\log\left(\frac{a_Q-z}{w-z}\right)\right| = \left|\log\left(1+\frac{a_Q-w}{w-z}\right)\right| \le C|w-z|^{-1},$$

and thus

$$\sum_{Q\in\mathcal{Q}_{1,3},|a_Q-z|\geq\frac{10\sigma}{\sqrt{\epsilon}}}\iint_Q\left|\log\left|\frac{a_Q-z}{w-z}\right|\right|d\mu(w)\leq C\iint_{|w|\in(|z|/2,2|z|),|w-z|\geq\frac{5\sigma}{\sqrt{\epsilon}}}\frac{d\mu(w)}{|w-z|}\leq C\sigma|z|. \tag{3.16}$$

Summing the estimates (3.5), (3.6), (3.8), (3.9), (3.11), (3.13), (3.14), (3.15), (3.16), we obtain the following inequality.

Proposition 3.2. *For a given ϵ and functions $V_\epsilon(z)$ and $W_\epsilon(z)$ defined as in* (3.1) *and* (3.3),

$$|V_\epsilon(z)-\text{Re } W_\epsilon(z)|\leq C\epsilon|z|^2+C'|\log\sigma|\epsilon^{-1}+C\sigma|z|,$$

as $|z|\to\infty$ and z avoids $\frac{\sigma}{4}$-neighborhoods of the points a_Q. In particular, using the asymptotics (A.3) *for W_ϵ, we have*

$$V_\epsilon(z)=\frac{1}{2}|z|^2\log|z|\sin(2\epsilon)\cos(2\psi)+\epsilon O(|z|^2)+O(\log|\sigma|)+C\sigma|z|, \tag{3.17}$$

if z tends to infinity along the line $z=|z|e^{i\psi}$.

It remains to estimate the difference in question for z in $\sigma/4$ neighborhood of the point a_{Q_0} for some $Q_0\in\mathcal{Q}$. Note that by our construction, there can be only one such point a_{Q_0}. Here we can simply separate the term corresponding to $Q=Q_0$ in the sum (3.4). For the sum of remaining terms the inequality we just obtained holds. This gives us the following estimate.

Proposition 3.3. *For a given ϵ and functions $V_e(z)$ and $W_\epsilon(z)$ defined as in* (3.1) *and* (3.3),

$$\left|V_\epsilon(z)-\text{Re } W_\epsilon(z)-\sigma^2\log\left(1-\frac{z}{a_{Q_0}}\right)\right|\leq C\epsilon|z|^2+C'|\log\sigma|\epsilon^{-1}+C\sigma|z|, \tag{3.18}$$

as $|z|\to\infty$ and z lies in the σ-neighborhood of the point a_{Q_0}.

The estimates (3.17), (3.18) lead to the following inequalities for the exponent of $V_\epsilon(z)$.

Proposition 3.4. *For a positive κ, we have*

$$|\exp(\kappa V_\epsilon(z))|\leq\exp(\kappa\frac{1}{2}|z|^2\log|z|\sin(2\epsilon)\cos(2\psi)+\epsilon O(\kappa|z|^2)+O(\log|\sigma|)), \tag{3.19}$$

as z tends to infinity along the line $z=|z|e^{i\psi}$.

Proof. If z tends to infinity along the line $z=|z|e^{i\psi}$ avoiding the $\sigma/4$-neighborhoods of the points a_Q, (3.19) follows from (3.17). For z in these neighborhoods, we apply (3.18) and use that $|\exp(\log\left(1-\frac{z}{a_{Q_0}}\right))|$ is bounded. ∎

3.2 Construction of entire functions

We return to our initial problem. We recall that $b_2 = 1$ and that $|b_1|$ is sufficiently small (how small will be determined later). We set $\kappa = \frac{c_0 \sin(\alpha)}{\pi \sin(2\epsilon)}$; $c_0 = 1 - b_1$.

We chose $\sigma = \kappa^{-\frac{1}{2}}$ and define the function $\Phi(z)$ as the Weierstrass product:

$$\Phi(z) = \prod_{Q \in \mathcal{Q}_+} \left[\left(1 - \frac{z^2}{a_Q^2}\right) \exp\left(\frac{z^2}{a_Q^2}\right) \right].$$

We also denote

$$\Phi_\alpha(z) = \Phi\left(ze^{-i\frac{\alpha+\pi}{2}}\right).$$

By our choice of σ, this function is related to the function V_ϵ considered in Section 3.1:

$$\log|\Phi_\alpha(z)|^2 = \sigma^{-2} V_\epsilon\left(ze^{-i\frac{\alpha+\pi}{2}}\right) = \kappa V_\epsilon\left(ze^{-i\frac{\alpha+\pi}{2}}\right).$$

Therefore, by (3.17), with $z = re^{i\psi}$

$$\begin{aligned} \log|\Phi_\alpha(z)|^2 &\le \frac{1}{2}|z|^2 \log|z| \sin(2\epsilon)\kappa \cos(2(\psi - \frac{\alpha+\pi}{2})) \\ +\kappa(\epsilon+1)O(|z|^2) &= -\frac{c \sin\alpha}{2\pi}|z|^2 \log|z| \cos(2(\psi - \alpha/2)) \\ &\quad + \sin\alpha(\epsilon+1)/\sin(2\epsilon) O(|z|^2). \end{aligned} \tag{3.20}$$

With ϵ chosen as $\alpha^{1/2}$ (thus $\sigma \sim \alpha^{-\frac{1}{2}}$), we have

$$\sin\alpha(\epsilon+1)/\sin(2\epsilon) \le C\alpha^{1/2}.$$

For a function $G(z)$, to be specified later, we consider the integral

$$I(G) = \iint_{\mathbb{C}} e^{2F(z)}|\Phi_\epsilon(z)|^2|G(z)|^2 d\mu(z) = \iint_{\mathbb{C}} e^{2F(z)+2\log|\Phi_\epsilon(z)|}|G(z)|^2 d\mu(z). \tag{3.21}$$

From the estimates for F in (2.5) and for $\log\Phi$ in (3.20), we see that the terms with $|z|^2 \log|z|$ in the exponent in (3.21) cancel and therefore

$$\begin{aligned} I(G) &\le C \iint_{\Omega_1} e^{|b_1|x_2^2 + c_0\alpha^{1/2}|z|^2}|G(z)|^2 d\mu(z) \\ +C &\iint_{\Omega_2} e^{-x_2^2 + c_0\alpha^{1/2}|z|^2}|G(z)|^2 d\mu(z) = I_1(G) + I_2(G). \end{aligned} \tag{3.22}$$

Now we choose the function $G(z)$. We take it in the form

$$G(z) = \exp(-1/4 \left(ze^{-i\frac{\alpha}{2}}\right)^2)P(z),$$

where $P(z)$ is an arbitrary polynomial. Then (3.22) implies

$$I_1(G) \le C \iint_{\Omega_1} e^{|b_1|\sin^2\alpha|z|^2 + c_0\alpha^{1/2}|z^2| - \frac{1}{2}|z|^2\cos 2(\psi-\alpha/2)} d\mu(z)$$
$$\le \iint_{\Omega_1} e^{(|b_1|\alpha^2 + c_0\alpha^{1/2})|z^2| - \frac{1}{2}|z^2|\cos(\alpha)} |P(z)|^2 d\mu(z) < \infty,$$

as soon as $\alpha, |b_1|\alpha$ are sufficiently small. Further on, we split the integral $I_2(G)$ as $I_2(G)' + I_2(G)''$, so that $I_2(G)'$ involves integration over the region in Ω_2 where $|\arg z| < \alpha^{1/5}$ or $|\arg z - \pi| < \alpha^{1/10}$, and $I_2(G)''$ involves the integration over the rest of Ω_2, i.e., the region where $|\arg z| > \alpha^{1/5}$ and $|\arg z - \pi| > \alpha^{1/10}$. This gives us

$$\begin{gathered} I_2(G) = I_2(G)' + I_2(G)'' \le \\ \iint_{\Omega_2} e^{-\frac{1}{2}\cos(\frac{3}{2}\alpha^{1/5})|z|^2 + c_0\alpha^{1/2}|z|^2} |P(z)|^2 d\mu(z) + \\ \iint_{\Omega_2} e^{-\sin^2\alpha^{1/5}|z|^2 + c_0\alpha^{1/2}|z|^2} |P(z)|^2 d\mu(z). \end{gathered} \tag{3.23}$$

Both integrals in (3.23) converge, again, as soon as α is small enough.

Thus any entire analytical function $u(z)$ of the form

$$u(z) = \Phi_\alpha(z)\exp(-1/4\left(ze^{-i\frac{\alpha}{2}}\right)^2)P(z)$$

belongs to $L_2(\mathbb{C})$ with weight $\exp(2F(z))$ and therefore the dimension of the corresponding subspace is infinite.

4 Nonexistence of Zero Modes

In this Section we prove the following theorem about the non-existence of zero modes.

Theorem 4.1. *Suppose that the angle α is sufficiently close to π and $|b_1| < \frac{1}{2}$. Then the space of analytical functions u satisfying $u\exp(\pm F) \in L_2$ consists only of the zero function.*

4.1 A half-plane

We consider the case of $\alpha = \pi$ first. So, let us have $B(x) = 2b_1 < 0$ in the half-plane $\mathbb{C}^+ = x_2 > 0$ and $B(x) = 2b_2 = 2$ in the half-plane $\mathbb{C}^- = x_2 < 0$. The potential, the solution of the equation $\Delta F = B$ can be taken in the form

$$F(z) = b_1 x_2^2, x_2 > 0; F(z) = x_2^2, x_2 < 0.$$

We will show that no nontrivial entire analytical function $u(z)$ can belong to L_2 with weight $e^{F(z)}$ or $e^{-F(z)}$. Actually, a more general statement is correct.

Proposition 4.2. *Let $h(s), s \geq 0$ be a positive function, $h(s) \to \infty$ as $s \to \infty$. Then the set of functions $u(z)$, analytical in the half-plane $x_2 > 0$ and continuous up to the boundary such that*

$$\iint_{\mathbb{C}^+} e^{x_2 h(x_2)} |u(z)|^2 d\lambda(z) < \infty \tag{4.1}$$

consists only of a zero function.

It is clear that the absence of nontrivial entire functions in the case we started with follows from Proposition 4.2 applied separately to half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$. Moreover, Proposition 4.2 will be the destination point for other configurations of B to be considered: having obtained a lower estimate for the function F in some domain, we make a conformal mapping of this domain onto the upper half-plane, where the Proposition can be applied.

Proof. It follows from the condition (4.1) that for any fixed $y_0 > 0$, the function $u_{y_0}(x_1) = u(x_1 + iy_0)$ belongs to $L_2(\mathbb{R}^1)$ as a function of x_1, moreover, $u(x_1 + ix_2)$ tends to 0 as $x_2 \to \infty$, uniformly in $x_1 \in \mathbb{R}^1$. Thus $u(x_1 + ix_2)$ is a bounded harmonic function in a half-plane $\mathbb{C}_+^{y_0} = x_2 \geq y_0$ with boundary values in $L_2(\mathbb{R}^1)$, Such function can be expressed by means of the Fourier transform:

$$u(x_1 + ix_2) = \mathcal{F}^{-1}_{\xi \to x_1} e^{-(x_2 - y_0)|\xi|^2} \widehat{u_{y_0}}(|\xi|).$$

Therefore,

$$\iint_{x_2 > y_0} e^{x_2 h(x_2)} |u(x_1 + ix_2)|^2 d\mu(z) = \iint_{x_2 > y_0} e^{x_2 h(x_2) - 2(x_2 - y_0)|\xi|} |\widehat{u_{y_0}}(\xi)|^2 d\xi dx_2,$$

and since $h(s) \to \infty$ as $s \to \infty$, the integral diverges unless $u_{y_0} \equiv 0$ for any $y_0 > 0$. $\square$

The case of a sector configuration of B will be reduced to Proposition 4.2 by means of a conformal mapping. The following subsection will be devoted to the proof that such special mapping is, in fact univalent.

4.2 Univalentness property

Proposition 4.3. *Denote for $R > 0$ by $\mathbb{C}_R^+$ the upper half-plane $\mathbb{C}^+$ with the disk $|z| \leq R$ removed. Then for any $A \neq 0$ there exists a number $R_A > 1$ such that the function*

$$\zeta(\omega) = \omega(\log \omega - \frac{\pi}{2} i)^A$$

is analytical and univalent in $\mathbb{C}_{R_A}^+$ and maps it onto a set $\Omega_A \subset \mathbb{C}$. The boundaries of Ω_A are described by

$$\begin{aligned} \varkappa &= -\frac{\pi A}{2 \log \rho} + O(\frac{\log \log \rho}{\log^2 \rho}),\ \zeta = \rho e^{i\varkappa}, \varkappa \text{ near } 0; \\ \varkappa &= \pi + \frac{\pi A}{2 \log \rho} + O(\frac{\log \log \rho}{\log^2 \rho}),\ \zeta = \rho e^{i\varkappa} \varkappa \text{ near } \pi. \end{aligned} \tag{4.2}$$

In other words, the function $\zeta(\omega)$ maps conformally the upper half-plane, with a disk cut away, onto a slightly, logarithmically, deformed half-plane, with a compact set cut away.

Proof. The fact that the function ξ is analytical in the domain $\mathbb{C}^+_{R_A}$ and the asymptotic expressions (4.2) for the mapping of the boundaries follow directly from the definition of the function. What, actually, requires being checked is that the function is univalent for $R = R_A$ sufficiently large. We start with an intermediate mapping onto a strip. Set

$$z = z(\omega) = \log\omega - \frac{\pi}{2}i. \tag{4.3}$$

The mapping (4.3) transforms $\mathbb{C}^+$ onto the strip $\{z = x+iy : |y| < \frac{\pi}{2}\}$ and the domain $\mathbb{C}_R$ onto the half-strip $\Pi_\varsigma = \{z = x+iy : |y| < \frac{\pi}{2}, x > \varsigma\}$, $\varsigma = \log R$. Since $\omega = ie^z$, it is sufficient to check that the function $\zeta(z) = e^z z^A$ is univalent in Π_ς for ς large enough.

We choose ς so that $\varsigma > 2\pi$ and moreover

$$|\arg(z^A)| < \frac{\pi}{40},\ |\arg(1 + Az^{-1})| < \frac{\pi}{40},\ z \in \Pi_\varsigma. \tag{4.4}$$

We show first that the function $\zeta(z)$ is univalent in any substrip

$$D = \{z = x+iy :\ x > \varsigma, |y - y_D| < 0.4\pi\},$$

such that ς satisfies (4.4) and $D \subset \Pi_\varsigma$. Let $z_D = \varsigma + 1 + iy_D$,

$$\nu_D = \arg(\zeta'(z_D)) = y_D + \arg(z_D^A) + \arg(1 + A/z_D). \tag{4.5}$$

By (4.4), for any $z = x+iy \in D$, we have

$$\begin{gathered} |\arg\zeta'(z) - \arg\zeta'(z_D)| \\ \le |y - y_D| + |\arg z^A| + |\arg(1+A/z)| + |\arg z_D^A| + |\arg(1+A/z_D)| \le \\ 0.4\pi + 4\cdot\frac{\pi}{40} = \frac{\pi}{2}. \end{gathered} \tag{4.6}$$

Now let z_1, z_2 be some points in D, $\frac{z_2-z_1}{|z_2-z_1|} = e^{i\chi}$. Then we have

$$\begin{aligned} \zeta(z_2) - \zeta(z_1) &= \int\limits_{[z_1,z_2]} \zeta'(t)dt = e^{i\chi} \int\limits_0^{|z_2-z_1|} \zeta'(z_1 + \tau e^{i\chi})d\tau \\ &= e^{i(\chi+\nu_D)} \int\limits_0^{|z_2-z_1|} e^{-i\nu_D}\zeta'(z_1 + \tau e^{i\chi})d\tau, \end{aligned}$$

and therefore, by (4.5), (4.6)

$$|\zeta(z_1)-\zeta(z_2)| = \left|\int_0^{|z_2-z_1|} e^{-i\nu_D}\zeta'(z_1+\tau e^{i\chi})d\tau\right|$$
$$\geq \left|\int_0^{|z_2-z_1|} \mathrm{Re}\,\left(e^{-i\nu_D}\zeta'(z_1+\tau e^{i\chi})\right)d\tau\right| > 0.$$

Now we consider the whole half-strip Π_ς. Let us take arbitrary points $z_1, z_2 \in \Pi_\varsigma$. If $|\,\mathrm{Im}\,(z_1-z_2)| < 0.8\pi$ then z_1, z_2 lie in some half-strip D of the type just considered and thus $\zeta(z_1) = \zeta(z_2)$ is impossible. On the other hand, if $\mathrm{Im}\,(z_2-z_1) \geq 0.8\pi$, we have

$$\arg\zeta(z_2) - \arg\zeta(z_1) = \mathrm{Im}\,(z_2-z_1) + (\arg(z_2^A) - \arg(z_2^A)) > 0.8\pi - \frac{2\pi}{40} > 0,$$

and again $\zeta(z_1) = \zeta(z_2)$ is impossible. □

4.3 Estimates for a large angle

We consider the case when in the setting of Section 2 the angle α is close to π. We are going to show that if $\theta = \pi - \alpha$ is small enough then there are no nontrivial entire functions $u(z)$ such that $u\exp(F)$ or $u\exp(-F)$ belong to $L_2(\mathbb{C})$. To do this, we consider two adjoining sectors $\Xi_\pm$ with slightly curved boundaries, where the functions $\pm F$ are positive. By performing the conformal mapping of $\Xi_\pm$ onto the upper half-plane, we arrive at the situation described in Subsection 4.1. Recall that in polar coordinates r, ψ the function $F(z)$ has the form

$$F(re^{i\psi}) = \phi(z) - \frac{c_0\sin\theta}{2\pi}r^2\log r\cos(2(\psi-\alpha/2)) + \frac{c_0\sin\theta}{2\pi}\psi\sin(2(\psi-\frac{\alpha}{2}))r^2.$$

Recall that the polar angle ψ lies in $[\alpha, \alpha+2\pi)$ in this representation, $c_0 = 1 + |b_1|$.

We suppose that α is close to π and consider two sectors in the complex plane, $S = \{z : \psi \in (\frac{\alpha}{2} + \frac{7}{4}\pi, \frac{\alpha}{2} + \frac{9}{4}\pi)\}$ and $T = \{z : \psi \in (\frac{\alpha}{2} + \frac{9}{4}\pi, \alpha + 2\pi) \cap [\alpha, \frac{\alpha}{2} + \frac{3}{4}\pi)\} = T_1 \cup T_2$. In these sectors the log-quadratic part of F is negative, resp. positive.

Consider the sector S. If $z = re^{i\psi} \in S$, we have $z \in \Omega_1$, $\phi(z) = b_1 r^2\sin^2\psi$, and therefore

$$-F(z) = \left(|b_1|\sin^2\psi - \frac{c_0\sin\theta}{2\pi}\psi\sin(2(\psi-\frac{\alpha}{2}))\right)r^2 \qquad (4.7)$$
$$+\frac{c_0\sin\theta}{2\pi}\cos(2(\psi-\frac{\alpha}{2}))r^2\log r.$$

It follows from (4.7)that $-F \geq Cr^2\log r$ for sufficiently large r in S with arbitrarily small sectors near the boundary of S removed. To estimate F near the boundary of S, we chose θ so small that

$$\frac{c_0\sin\theta}{2\pi}(\frac{9}{4}\pi + \frac{\alpha}{2}) \leq \frac{1}{2}\sin^2(\frac{9}{4}\pi + \frac{\alpha}{2})|b_1|. \qquad (4.8)$$

Then, by (4.8), for $z = re^{i\psi}$, ψ close to $\frac{9}{4}\pi + \frac{\alpha}{2}$, we have

$$\left(|b_1|\sin^2\psi - \frac{c_0 \sin\theta}{2\pi}\psi \sin(2(\psi - \frac{\alpha}{2}))\right) r^2 \geq \frac{|b_1|}{2} r^2 \geq Cr^2. \tag{4.9}$$

for some positive constant C. For ψ close to $\frac{7}{4}\pi + \frac{\alpha}{2}$ we note that $\sin(2(\psi - \frac{\alpha}{2}))$ is negative, therefore

$$\left(|b_1|\sin^2\psi - \frac{c_0 \sin\theta}{2\pi}\psi \sin(2(\psi - \frac{\alpha}{2}))\right) r^2 \geq Cr^2 \tag{4.10}$$

Now, it follows from (4.9), (4.10) that for some constant $A > 0$, small enough, the quadratic term in (4.7) majorates the log-quadratic term in the domains $S_1(A) = \{z = re^{i\psi}, |\psi - (\frac{7}{4}\pi + \frac{\alpha}{2})| \leq \frac{A}{\log r}\}$ and $S_2(A) = \{z = re^{i\psi}, |\psi - (\frac{9}{4}\pi + \frac{\alpha}{2})| \leq \frac{A}{\log r}\}$:

$$\left|\frac{c_0 \sin\theta}{2\pi} \cos(2(\psi - \frac{\alpha}{2})) r^2 \log r\right| \leq \frac{C}{2} r^2, \;\; z \in S_1(A) \cup S_2(A).$$

Therefore, in the domain $S' = S \cup S_1(A) \cup S_2(A)$, S' slightly larger than the quarter-plane, we have

$$-F(z) \geq \frac{C}{2}|z|^2.$$

Now suppose that for some nontrivial analytical function $u(z)$ we have

$$\iint_{S'} |u(z)|^2 e^{-2F(z)} d\mu(z) < \infty \tag{4.11}$$

In order to obtain a contradiction, we make a conformal mapping of the domain S' onto a domain covering the upper halve-plane. We will do it in two steps.

Let $\zeta = \rho e^{i\varkappa} = (ze^{i\frac{7}{4}\pi})^2$. Under this mapping, the domain S' is transformed conformally onto

$$\tilde{S} = \{\zeta : \;\; \rho > r_0^2, \;\; -\delta(\rho) < \arg\varkappa < \pi + \delta(\rho),$$

where

$$\delta(\rho) \sim \frac{2A}{\log \rho}.$$

Next we make a change of variables in the integral in (4.11) by setting $v(\zeta) = \frac{u(\sqrt{\zeta})}{\sqrt{\zeta}}$. We obtain

$$\iint_{\tilde{S}} |v(\zeta)|^2 e^{|\zeta|} d\mu(\zeta) < \infty. \tag{4.12}$$

As it follows from our construction, the domain $\tilde{S}$ is slightly, logarithmically, larger than the upper half-plane with a disk removed. Now we map conformally this set onto the upper half-plane with a compact set removed. It is more convenient to do it by considering the inverse mapping.

We set $\zeta = \zeta(\omega) = \omega(\log(\omega - \frac{\pi i}{2}))^A$. If A is small enough and a is large enough, the image under this mapping of the upper half-plane with the disk $|\omega| < a$ removed, lies in $\tilde{S}$. By Proposition

4.3, the mapping $\zeta(\omega)$ is univalent in this set, so the inverse, $\omega = \omega(\zeta)$ exists, maps the image of $\zeta(\omega)$ onto the $\mathbb{C}_a$ and its asymptotics as $|\zeta| \to \infty$ can be easily found. We change variables in the integral in (4.12) which gives

$$\iint_{\mathbb{C}_a^+} |v(\zeta(\omega))|^2 |\zeta'(\omega)| e^{\frac{C}{4}|\omega||\log|\omega||^A} d\mu(\omega) < \infty. \tag{4.13}$$

Since $|\zeta'(\omega)|$ behaves logarithmically at infinity, we can apply Proposition 4.2 with $h(x_2) = (\log(|x_2|+1))^A - C$. Thus, (4.13) can hold only for $v \equiv 0$, or $u \equiv 0$.

Now we consider the sector T. From the expression for $F(z)$ in (2.5) we obtain for small θ

$$F(re^{i(\frac{\alpha}{2}+\frac{3}{4}\pi)}) = \left(\frac{1}{2} + O(\theta)\right) r^2 \geq \frac{1}{4} r^2. \tag{4.14}$$

We consider now an auxiliary entire analytical function $H(z) = -\frac{i}{6} e^{-\alpha} z^2$, so that Re $H(z) = -\frac{1}{6} r^2 \sin 2(\psi - \frac{\alpha}{2} - \frac{\pi}{2})$. Then, by (4.14) and (4.9) for the function $\tilde{F}(z) = F(z) - \text{Re } H(z)$ we have

$$\tilde{F}(z) \geq \frac{1}{12} r^2 \text{ for } \psi = \frac{9}{4}\pi + \frac{\alpha}{2},\ \tilde{F}(z) \geq \frac{1}{24} r^2 \text{ for } \psi = \frac{3}{4}\pi + \frac{\alpha}{2}.$$

So, the log-quadratic term in $\tilde{F}$ is positive in the sector T and the quadratic term in $\tilde{F}$ is positive on the boundaries of T. Therefore, similar to the above consideration in the sector S, the function $\tilde{F}$ admits a quadratic lower estimate in a domain slightly larger than T:

$$\tilde{F}(z) \geq C|z|^2, \psi \in [\frac{\alpha}{2}, \frac{3}{4}\pi + \frac{A}{\log r}] \cup [\frac{9}{4}\pi + \frac{\alpha}{2} - \frac{A}{\log r}, \frac{\alpha}{2} + 2\pi].$$

If an entire analytical function u satisfies $\iint_{\mathbb{C}} e^{2F} |u|^2 \delta\mu(z) < \infty$ then the function $\tilde{u} = u \exp(H)$ satisfies $\iint_{\mathbb{C}} e^{2\tilde{F}} |\tilde{u}|^2 \delta\mu(z) < \infty$. It remains to repeat the reasoning with the conformal mapping used for the sector S to show that the function $\tilde{u}$ and therefore u must necessarily be zero.

This concludes the proof of Theorem 4.1.

5 The Non-Resonance Case

As it can be observed from the calculations above, the main trouble in the study of the 'sector' configuration is created by power-logarithmic behavior of the function F. In the non-resonance case such terms are not present in F, therefore the analysis of zero modes is considerably easier.

Theorem 5.1. *Let $B(z) = B(re^{i\psi})$ be radial homogeneous of degree $s \in (-2, 0]$ and $\beta_0 = (2\pi)^{-1} \int_0^{2\pi} B(e^{i\psi}) d\psi \neq 0$. For $s = 0, -1$ we suppose that $\int_0^{2\pi} B(e^{i\psi}) e^{i(s+2)\psi} d\psi = 0$. Then, if $\int_0^{2\pi} |B(e^{i\psi}) - \beta_0|^2 d\psi$ is sufficiently small then the space of zero modes is infinite-dimensional.*

Proof. Suppose that $\beta_0 > 0$ and set $\tilde{B}(x) = B(x) - \beta_0 r^s$. Then the solution F of the Poisson equation $\Delta F = B$ can be represented as $F = \Phi + \tilde{F}$, where $\Phi(x) = \beta_0 (s+2)^{-2} |r|^{s+2}$ and

$\tilde{F} = \varphi(\psi)r^{s+2}$ with $\varphi(\psi)$ being a solution of $\varphi''(\psi) + (s+2)^2\varphi(\psi) = \tilde{B}(\psi)$. Such solution exists (for $s = 0$ or $s = -1$ we use the orthogonality condition and require also that $\varphi(\psi)$ is again orthogonal to $e^{i(s+2)\psi}$). Moreover, the solution $\varphi(\psi)$, by ellipticity, belongs to the Sobolev space H^2 on the circle $\mathbf{S}^1$ with estimate $||\varphi||_{H^2(\mathbf{S}^1)} \leq C||\tilde{B}||_{L_2(\mathbf{S}^1)}$. Thus, if the latter norm is small enough, then, by the embedding theorem, $||\varphi||_{C(\mathbf{S}^1)}$ ia also small and can be made smaller than $|\beta_0|$. In this case, it turns out that $F(x) \geq C|x|^{s+2}$ and therefore for any polynomial $p(z)$ the integral $\iint_{\mathbb{C}} |p(z)|^2 e^{-2F} d\mu(z)$ converges. □

We explain here the role of the orthogonality condition in the cases $s = 0$ and $s = -1$. If it is violated, the solution of the Poisson equation contains necessarily a log-power term, similar to the one considered in Sections 3,4, and therefore will not be sign-definite. The same complication arises in the case $s = -2$.

Finally, we present a construction showing that in the nonresonance case the absence of zero modes can also occur.

Example 5.2. Let $s \in (-1, 0]$. We construct the function $F(z) = F(re^{ip}) = f(\psi)r^{s+2}$ in the following way. For some $\epsilon > 0$, $\epsilon < \frac{1}{4}(1+s)$, we consider two disjoint arcs I_+, I_- in $\mathbf{S}^1$ having length $\pi(s+2-\epsilon)^{-1} < \frac{\pi}{2}$. We set $f(\psi) = \beta_+ > 0$ on I_+, $f(\psi) = \beta_- < 0$ on I_- with some constants $\beta_\pm$ and define $f(\psi)$ in an arbitrary way on the complement of $I_\pm$, to obtain a smooth function on the circle. Denote by $S_\pm$ the sectors in $\mathbb{C}$ defined by the arcs $I_\pm$. Supposing that there exists an analytical function $u(z)$ satisfying $\iint_{S_+} e^{2F}|u(z)|^2 d\mu(z) < \infty$, we make a conformal mapping $z(\zeta)$ of the upper half-plane $\mathbb{C}_+$ onto the sector S_+, $z = z_0\zeta^{(s+2-\epsilon)^{-1}}$, $|z_0| = 1$. Then the integral transforms to

$$(s+2-\epsilon)^{-1} \iint_{\mathbb{C}_+} e^{2F(z_0\zeta^{(s+2-\epsilon)^{-1}})} |u(z_0\zeta^{(s+2-\epsilon)^{-1}})|^2 |\zeta|^{(s+2-\epsilon)^{-1}-1} d\mu(\zeta). \tag{5.1}$$

For the exponent $2F(z_0\zeta^{(s+2-\epsilon)^{-1}})$ we have the lower estimate by $|z|^{\frac{s+2}{s+2-\epsilon}}$, and by Proposition 4.2 the function u should be zero. In a similar way, the integral $\iint_{S_-} e^{-2F}|u(z)|^2 d\mu(z)$ cannot be finite unless $u \equiv 0$.

A Some Integrals

We present here the calculation of the integral in (3.3). we show first that

$$v(z) = \int_1^\infty \left(t\log(1 - \frac{z^2}{t^2}) + \frac{z^2}{t^2} \right) dt = \frac{z^2 - 1}{2}\log(1 - z^2) - \frac{z^2}{2}. \tag{A.1}$$

It is clear that $v(0) = 0$. We find the derivative of $v(z)$. For $|z| < 1$ it is legal to differentiate under the integral sign, therefore

$$v'(z) = -z\int_1^\infty (\frac{1}{t-z} + \frac{1}{t+z} - \frac{2}{t})dt = z\log(1 - z^2).$$

The derivative of the right-hand side in (A.1) gives the same expression.

For $|z| \geq 1, \mathrm{Im}\, z \neq 0$, we can continue analytically the expression in (A.1) separately to the upper and lower half-planes, and the corresponding branches of the logarithm should be used.

Next we consider the integral in (3.3). We represent it, using (A.1) as

$$\begin{gathered} W_\epsilon(z) = \int_{-\epsilon}^{\epsilon} d\vartheta \int_{1}^{\infty} \left(\log(1 - \frac{z^2}{\tau^2} e^{-2i\vartheta}) + \frac{z^2}{\tau^2} e^{-2i\vartheta} \right) \tau d\tau = \\ \frac{1}{2}\int_{-\epsilon}^{\epsilon} (ze^{i\theta})^2 \log(1-(ze^{i\theta})^2) d\theta - \frac{1}{2}\int_{-\epsilon}^{\epsilon} (ze^{i\vartheta})^2 d\vartheta - \frac{1}{2}\int_{-\epsilon}^{\epsilon} \log(1-(ze^{i\vartheta})^2) d\vartheta. \end{gathered} \tag{A.2}$$

We estimate the ϑ integral for small ϵ, obtaining

$$W_\epsilon(z) = \frac{1}{2}|z|^2 \log|z| \sin(2\epsilon)\cos(2\phi) + \epsilon O(|z|^2) \tag{A.3}$$

as z tends to infinity along the line $z = |z|e^{i\phi}$.

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