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Fixed point theory for compact absorbing contractions in extension type spaces

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ABSTRACT

Several new fixed point results for self maps in extension type spaces are presented in this paper. In particular we discuss compact absorbing contractions.

RESUMEN

Son presentados en este artículo varios resultados nuevos de punto fijo para autoaplicaciones en espacios de tipo extensión. En particular discutimos contracciones compactas absorbentes.

Key words and phrases: *Extension spaces, fixed point theory, compact absorbing contractions.*

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1 Introduction

In Sections 2, 3 and 4 we present new results on fixed point theory in extension type spaces. Section 2 discusses compact self-maps on NES, $ANES$ and $SANES$ spaces whereas Section 3 discusses compact absorbing contractions. In Section 4 we provide an alternative approach using projective limits. These results improve those in the literature; see [1-3, 5, 8-11, 14-15] and the references therein. Our results were motivated in part from ideas in [1, 2, 9, 12, 15].

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Suppose X and Y are topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \ Fix\, F \neq \emptyset \ \text{ for all } \ F \in \mathcal{X}(Z,Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathcal{A}$ of maps is defined by the following properties:

(i). $\mathcal{A}$ contains the class $\mathcal{C}$ of single valued continuous functions;

(ii). each $F \in \mathcal{A}_c$ is upper semicontinuous and closed valued; and

(iii). $B^n \in \mathcal{F}(\mathcal{A}_c)$ for all $n \in \{1,2,....\}$; here $B^n = \{x \in \mathbf{R}^n : \ \|x\| \leq 1\}$.

Remark 1.1. The class $\mathcal{A}$ is essentially due to Ben-El-Mechaiekh and Deguire [6]. $\mathcal{A}$ includes the class of maps $\mathcal{U}$ of Park ($\mathcal{U}$ is the class of maps defined by (i), (iii) and (iv). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued). Thus if each $F \in \mathcal{A}_c$ is compact valued the class $\mathcal{A}$ and $\mathcal{U}$ coincide and this is what occurs in Section 2 since our maps will be compact.

The following result can be found in [6, Proposition 2.2] (see also [9 pp. 286] for a special case).

Theorem 1.1. *The Hilbert cube I^∞ (subset of l^2 consisting of points $(x_1, x_2, ...)$ with $|x_i| \leq \frac{1}{2^i}$ for all i) and the Tychonoff cube T (cartesian product of copies of the unit interval) are in $\mathcal{F}(\mathcal{A}_c)$.*

We next consider the class $\mathcal{U}_c^\kappa(X,Y)$ (respectively $\mathcal{A}_c^\kappa(X,Y)$) of maps $F : X \to 2^Y$ such that for each F and each nonempty compact subset K of X there exists a map $G \in \mathcal{U}_c(K,Y)$ (respectively $G \in \mathcal{A}_c(K,Y)$) such that $G(x) \subseteq F(x)$ for all $x \in K$.

Theorem 1.2. *I^∞ and T are in $\mathcal{F}(\mathcal{A}_c^\kappa)$ (respectively $\mathcal{F}(\mathcal{U}_c^\kappa)$).*

Proof: Let $F \in \mathcal{A}_c^\kappa(I^\infty, I^\infty)$ and we must show $Fix\, F \neq \emptyset$. Now by definition there exists $G \in \mathcal{A}_c(I^\infty, I^\infty)$ with $G(x) \subseteq F(x)$ for all $x \in I^\infty$, so Theorem 1.1 guarantees that there exists $x \in I^\infty$ with $x \in G\,x$. In particular $x \in F\,x$ so $Fix\, F \neq \emptyset$. Thus $I^\infty \in \mathcal{F}(\mathcal{A}_c^\kappa)$. $\square$

Notice [14] that $\mathcal{U}_c^\kappa$ is closed under compositions. The class $\mathcal{U}_c^\kappa$ include (see [3]) the Kakutani maps, the acyclic maps, the O'Neill maps, the approximable maps and the maps admissible with respect to Gorniewicz.

For a subset K of a topological space X, we denote by $Cov_X\,(K)$ the set of all coverings of K by open sets of X (usually we write $Cov\,(K) = Cov_X\,(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov\,(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$. Given two maps single valued $f, g : X \to Y$ and $\alpha \in Cov\,(Y)$, f and g are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$.

We say f and g are α-homotopic if there is a homotopy $h_h : X \to Y$ $(0 \le t \le 1)$ joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0,1]$.

The following results can be found in [4, Lemma 1.2 and 4.7].

Theorem 1.3. *Let X be a regular topological space and $F : X \to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X\left(\overline{F(X)}\right)$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1.2. From Theorem 1.3 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [5 pp. 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [13 pp. 199] so such refinements form a cofinal family of open covers). Note also uniform spaces are regular (in fact completely regular) [7 pp. 431] (see also [7 pp. 434]). Note in Theorem 1.3 if F is compact valued then the assumption that X is regular can be removed. For convenience in this paper we will apply Theorem 1.3 only when the space is uniform.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic

(ii). p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\,\tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q\, Tr\,(f_q).$$

A linear map $f : E \to E$ of a vector space E into itself is called weakly nilpotent provided for every $x \in E$ there exists n_x such that $f^{n_x}(x) = 0$.

Assume that $E = \{E_q\}$ is a graded vector space and $f = \{f_q\} : E \to E$ is an endomorphism. We say that f is weakly nilpotent iff f_q is weakly nilpotent for every q.

It is well known [9, pp 53] that any weakly nilpotent endomorphism $f : E \to E$ is a Leray endomorphism and $\Lambda(f) = 0$.

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$.

With Čech homology functor extended to a category of morphisms (see [10 pp. 364]) we have the following well known result (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

we define the induced map

$$H\,(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$).

Recall the following result [8 pp. 227].

Theorem 1.4. *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then*

$$(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star.$$

Two morphisms $\phi,\, \psi \in M(X,Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0,1], Y)$ such that $\chi(x,0) = \phi(x)$, $\chi(x,1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0,\, i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x,0)$, $i_1(x) = (x,1)$). Recall the following result [9, pp. 231]: If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p,q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p,q) \subset \phi$) if the following two conditions hold:

(i). p is a Vietoris map

and

(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Definition 1.1. A upper semicontinuous map $\phi : X \to Y$ is said to be strongly admissible [9, 10] (and we write $\phi \in Ads(X,Y)$) provided there exists a selected pair (p,q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$.

Definition 1.2. A map $\phi \in Ads(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

When we talk about $\phi \in Ads$ it is assumed that we are also considering a specified selected pair (p,q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$.

Remark 1.3. In fact since we specify the pair (p,q) of ϕ it is enough to say ϕ is a Lefschetz map if $\phi_\star = q_\star\, p_\star^{-1} : H(X) \to H(X)$ is a Leray endomorphism. However for the examples of ϕ, X known in the literature [9] the more restrictive condition in Definition 1.2 works. We note [9, pp 227] that $\phi_\star$ does not depend on the choice of diagram from $[(p,q)]$, so in fact we could specify the morphism.

If $\phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [9, 10]) $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$) by

$$\mathbf{\Lambda}\,(\phi) = \Lambda(q_\star\, p_\star^{-1}).$$

If we do not wish to specify the selected pair (p,q) of ϕ then we would consider the Lefschetz set $\mathbf{\Lambda}\,(\phi) = \{\Lambda(q_\star\, p_\star^{-1}) :\ \phi = q\,(p^{-1})\}$.

Definition 1.3. A Hausdorff topological space X is said to be a Lefschetz space provided every compact $\phi \in Ads(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies ϕ has a fixed point.

Definition 1.4. A upper semicontinuous map $\phi : X \to Y$ with closed values is said to be admissible (and we write $\phi \in Ad(X,Y)$) provided there exists a selected pair (p,q) of ϕ.

Definition 1.5. A map $\phi \in Ad(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$) by

$$\mathbf{\Lambda}\,(\phi) = \left\{\Lambda(q_\star\, p_\star^{-1}) :\ (p,q) \subset \phi\right\}.$$

Definition 1.6. A Hausdorff topological space X is said to be a Lefschetz space provided every compact $\phi \in Ad(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Recall the following result [8].

Theorem 1.5. *Every open subset of the Tychonoff cube is a Lefschetz space.*

The following concepts will be needed in Section 4. Let (X,d) be a metric space and S a nonempty subset of X. For $x \in X$ let $d(x,S) = \inf_{y \in S} d(x,y)$. Also $diam\,S = \sup\{d(x,y) : x,y \in S\}$. We let $B(x,r)$ denote the open ball in X centered at x of radius r and by $B(S,r)$ we denote $\cup_{x \in S} B(x,r)$. For two nonempty subsets S_1 and S_2 of X we define the generalized Hausdorff distance H to be

$$H(S_1,S_2) = \inf\{\epsilon > 0 :\ S_1 \subseteq B(S_2,\epsilon),\ S_2 \subseteq B(S_1,\epsilon)\}.$$

Now suppose $G : S \to 2^X$. Then G is said to be hemicompact if each sequence $\{x_n\}_{n\in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha\in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha\in I} E_\alpha : \ x_\alpha = \pi_{\alpha,\beta}(x_\beta) \ \forall\, \alpha,\, \beta \in I,\, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha\in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha\in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [1, 2] $\cap_{\alpha\in I} E_\alpha$.)

2 Preliminary Fixed Point Theory

The fixed point theory presented in this section can partly be found in [9, 14]. However for the convenience of the reader we present the following elementary approach.

By a space we mean a Hausdorff topological space. Let X and Y be spaces. A space Y is an neighborhood extension space for Q (written $Y \in NES(Q)$) if $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and for any continuous function $f_0 : K \to Y$, there exists a continuous extension $f : U \to Y$ of f_0 over a neighbourhood U of K in X.

Let $X \in NES(\text{compact})$ and $F \in \mathcal{U}_c^\kappa(X, X)$ a compact map. Now let $K = \overline{F(X)}$. We know [12] that K can be embedded as a closed subset $K^\star$ of T; let $s : K \to K^\star$ be a homeomorphism. Also let $i : K \hookrightarrow X$ be an inclusion. Let U be an open neighbourhood of $K^\star$ in T and $h_U : U \to X$ be a continuous extension of $i\, s^{-1} : K^\star \to X$ on U (guaranteed since $X \in NES(\text{compact})$). Let $j_U : K^\star \hookrightarrow U$ be the natural embedding so $h_U\, j_U = i\, s^{-1}$. Finally let $G = j_U\, s\, F\, h_U$. Notice $G \in \mathcal{U}_c^\kappa(U, U)$. We now <u>assume</u>

$$G \in \mathcal{U}_c^\kappa(U, U) \ \text{ has a fixed point.} \tag{2.1}$$

Then there exists $x \in U$ with $x \in G\, x$. Let $y = h_U(x)$, so $y \in h_U\, j_U\, s\, F\,(y)$ i.e. $y = h_U\, j_U\, s\,(q)$ for some $q \in F\,(y)$. Since $h_U\, j_U\,(z) = i\, s^{-1}(z)$ for $z \in K^\star$, we have $h_U\, j_U\, s\,(q) = i\,(q)$, so $y \in F(y)$.

Theorem 2.1. *Let $X \in NES(\text{compact})$ and $F \in \mathcal{U}_c^\kappa(X, X)$ a compact map. Also assume* (2.1) *holds with K, $K^\star$, U, s, i, j_U and h_U as described above. Then F has a fixed point.*

We discuss Theorem 2.1 for the class $Ad(X, X)$. Let $X \in NES(\text{compact})$ and $F \in Ad(X, X)$ a compact map. Also let K, $K^\star$, U, s, i, j_U and h_U as described above. Let (p, q) be a selected pair for F. Now since $F\, h_U \in Ad(U, X)$ then [9, Section 40] guarantees that there exists a selected pair (p', q') of $F\, h_U$ with $(q')_\star\, (p')_\star^{-1} = q_\star\, p_\star^{-1}\, (h_U)_\star$. Notice

$$(q')_\star\, (p')_\star^{-1}\, (j_U)_\star\, s_\star = q_\star\, p_\star^{-1}\, (h_U)_\star\, (j_U)_\star\, s_\star = q_\star\, p_\star^{-1}$$

since $h_U\, j_U\, s = i\, s^{-1}\, s$. Next note $G = j_U\, s\, F\, h_U \in Ad(U, U)$ has a selected pair $(p', j_U\, s\, q')$ (since $j_U\, s\, q'\,(p')^{-1}(x) \subseteq j_U\, s\, F\, h_U(x) = G(x)$ for $x \in U$) and from Theorem 1.5 we know U is a Lefschetz space so $(j_U\, s\, q')_\star\, (p')_\star^{-1}$ is a Leray endomorphism. Notice $(j_U)_\star\, s_\star\, (q')_\star\, (p')_\star^{-1} = (j_U\, s\, q')_\star\, (p')_\star^{-1}$

and from above $(q')_\star\,(p')_\star^{-1}\,(j_U)_\star\,s_\star = q_\star\,p_\star^{-1}$ so [8, page 314, see (1.3)] (here $E' = U'$, $E'' = U''$, $u = (q')_\star\,(p')_\star^{-1}$, $v = (j_U)_\star\,s_\star$, $f' = (j_U\,s\,q')_\star\,(p')_\star^{-1}$ and $f'' = q_\star\,p_\star^{-1}$) guarantees that $q_\star\,p_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,((j_U\,s\,q')_\star\,(p')_\star^{-1})$. Thus $\mathbf{\Lambda}\,(F)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F) \neq \{0\}$. Then there exists a selected pair (p,q) as described above with $\Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Let p' and q' be as described above with $\Lambda\,((j_U\,s\,q')_\star\,(p')_\star^{-1}) = \Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Now since U is a Lefschetz space there exists $x \in U$ with $x \in j_U\,s\,q'\,(p')^{-1}(x)$ i.e. $x \in G(x)$ so (2.1) is satisfied. Combining with Theorem 2.1 we have the following result.

Theorem 2.2. *Let $X \in NES(compact)$ and $F \in Ad(X,X)$ a compact map. Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.*

Remark 2.1. Theorem 2.2 says that $NES(\text{compact})$ spaces are Lefschetz spaces (for the class Ad).

Remark 2.2. Essentially the same reasoning as in Theorem 2.2 establishes: Let $X \in NES(\text{compact})$ and $F \in Ads(X,X)$ a compact map. Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq 0$ then F has a fixed point i.e. $NES(\text{compact})$ spaces are Lefschetz spaces (for the class Ads).

A space Y is a approximate neighborhood extension space for Q (written $Y \in ANES(Q)$) if $\forall\,\alpha \in Cov\,(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$, there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α-close.

Let $X \in ANES(\text{compact})$ be a uniform space and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact upper semicontinuous map with closed values. Also let $\alpha \in Cov_X\,(K)$ where $K = \overline{F(X)}$. To show F has a fixed point it suffices (Theorem 1.3 with Remark 1.2) to show F has an α–fixed point. Let $\alpha' = \alpha \cup \{X \setminus K\}$ and let $K^\star$, s and i be as above. Since $X \in ANES(\text{compact})$ there exists an open neighborhood U_α of $K^\star$ in T and $f_\alpha : U_\alpha \to X$ a continuous function such that $f_\alpha|_{K^\star}$ and s^{-1} are α'–close and as a result $f_\alpha\,j_{U_\alpha}\,s : K \to X$ and $i : K \to X$ are α–close; here $j_{U_\alpha} : K^\star \hookrightarrow U_\alpha$ is the natural imbedding. Finally let $G_\alpha = j_{U_\alpha}\,s\,F\,f_\alpha$. Notice $G_\alpha \in \mathcal{U}_c^\kappa(U_\alpha,U_\alpha)$ is a compact upper semicontinuous map with closed values. We now <u>assume</u>

$$G_\alpha \in \mathcal{U}_c^\kappa(U_\alpha,U_\alpha) \text{ has a fixed point for each } \alpha \in Cov_X\,(\overline{F(X)}). \tag{2.2}$$

We still have $\alpha \in Cov_X\,(K)$ fixed and we let x be a fixed point of G_α. Now let $y = f_\alpha(x)$ so $y \in f_\alpha\,j_{U_\alpha}\,s\,F(y)$ i.e. $y = f_\alpha\,j_{U_\alpha}\,s(q)$ for some $q \in F(y)$. Now since $f_\alpha\,j_{U_\alpha}\,s$ and i are α–close there exists $U \in \alpha$ with $f_\alpha\,j_{U_\alpha}\,s(q) \in U$ and $i(q) \in U$ i.e. $q \in U$ and $y = f_\alpha\,j_{U_\alpha}\,s(q) \in U$. Thus $q \in U$ and $y \in U$ so

$$y \in U \quad \text{and} \quad F(y) \cap U \neq \emptyset \quad \text{since} \quad q \in F\,(y).$$

Theorem 2.3. *Let $X \in ANES(compact)$ be a uniform space and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact upper semicontinuous map with closed values. Also assume* (2.2) *holds with K, $K^\star$, U_α, s, j_{U_α}, i and f_α as described above. Then F has a fixed point.*

We discuss Theorem 2.3 for the class $Ad(X,X)$. First however we need the following definition.

A space Y is a strongly approximate neighborhood extension space for Q (written $Y \in SANES(Q)$) if $\forall\,\alpha \in Cov\,(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$, there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α close and α-homotopic.

Let $X \in SANES(\text{compact})$ be a uniform space and $F \in Ad(X, X)$ a compact map. Also let K, $K^\star$, U_α, s, j_{U_α}, i and f_α as described above. Let (p, q) be a selected pair for F. Now since $F\, f_\alpha \in Ad(U_\alpha, X)$ then [9, Section 40] guarantees that there exists a selected pair (p'_α, q'_α) of $F\, f_\alpha$ with $(q'_\alpha)_\star\, (p'_\alpha)_\star^{-1} = q_\star\, p_\star^{-1}\, (f_\alpha)_\star$. As a result

$$(q'_\alpha)_\star\, (p'_\alpha)_\star^{-1}\, (j_{U_\alpha})_\star\, s_\star = q_\star\, p_\star^{-1}\, (f_\alpha)_\star\, (j_{U_\alpha})_\star\, s_\star = q_\star\, p_\star^{-1}$$

since $f_\alpha\, j_{U_\alpha}\, s$ is α homotopic to i (note $f_\alpha|_{K^\star}$ and s^{-1} are α'-homotopic by definition). Next note $G_\alpha = j_{U_\alpha}\, s\, F\, f_\alpha \in Ad(U_\alpha, U_\alpha)$ has a selected pair $(p'_\alpha, j_{U_\alpha}\, s\, q'_\alpha)$ and from Theorem 1.5 we have that $(j_{U_\alpha}\, s\, q'_\alpha)_\star\, (p'_\alpha)_\star^{-1}$ is a Leray endomorphism. Now since $(j_{U_\alpha})_\star\, s_\star\, (q'_\alpha)_\star\, (p'_\alpha)_\star^{-1} = (j_{U_\alpha}\, s\, q'_\alpha)_\star\, (p'_\alpha)_\star^{-1}$ and from above $(q'_\alpha)_\star\, (p'_\alpha)_\star^{-1}\, (j_{U_\alpha})_\star\, s_\star = q_\star\, p_\star^{-1}$ then [8, page 314, see (1.3)] guarantees that $q_\star\, p_\star^{-1}$ is a Leray endomorphism and we have $\Lambda\,(q_\star\, p_\star^{-1}) = \Lambda\,((j_{U_\alpha}\, s\, q'_\alpha)_\star\, (p'_\alpha)_\star^{-1})$. Thus $\boldsymbol{\Lambda}\,(F)$ is well defined.

Next suppose $\boldsymbol{\Lambda}\,(F) \neq \{0\}$. Then there exists a selected pair (p, q) as described above with $\Lambda\,(q_\star\, p_\star^{-1}) \neq 0$. Let p'_α and q'_α be as described above with $\Lambda\,((j_{U_\alpha}\, s\, q'_\alpha)_\star\, (p'_\alpha)_\star^{-1}) = \Lambda\,(q_\star\, p_\star^{-1}) \neq 0$. Now since U_α is a Lefschetz space there exists $x \in U_\alpha$ with $x \in j_{U_\alpha}\, s\, q'_\alpha\, (p'_\alpha)^{-1}(x)$ i.e. $x \in G_\alpha(x)$ so (2.2) is satisfied. Combining with Theorem 2.3 we have the following result.

Theorem 2.4. *Let $X \in SANES(compact)$ be a uniform space and $F \in Ad(X, X)$ a compact map. Then $\boldsymbol{\Lambda}\,(F)$ is well defined and if $\boldsymbol{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.*

Remark 2.3. Theorem 2.4 says that $SANES(\text{compact})$ uniform spaces are Lefschetz spaces (for the class Ad).

Remark 2.4. Essentially the same reasoning as in Theorem 2.4 establishes: Let $X \in SANES(\text{compact})$ be a uniform space and $F \in Ads(X, X)$ a compact map. Then $\boldsymbol{\Lambda}\,(F)$ is well defined and if $\boldsymbol{\Lambda}\,(F) \neq 0$ then F has a fixed point i.e. $SANES(\text{compact})$ uniform spaces are Lefschetz spaces (for the class Ads).

One could in fact generalize Theorem 2.2 and Theorem 2.4 by using some results in [1]. Let X be a subset of a Hausdorff topological space and let X be a uniform space. Then X is said to be Schauder admissible if for every compact subset K of X and every open covering $\alpha \in Cov_X(K)$ there exists a continuous function $\pi_\alpha : K \to E$ such that

(i). π_α and $i : K \hookrightarrow X$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset C of X with C a Lefschetz space;

and

(iii). π_α and $i : K \hookrightarrow X$ are homotopic.

Remark 2.5. For example we could take $C \in NES(\text{compact})$ or $C \in SANES(\text{compact})$ in (ii) above (for both Ad and Ads maps).

The following result can be found in [1].

Theorem 2.5. *Let X be a subset of a Hausdorff topological space and let X be a uniform space. Also suppose X is Schauder admissible. If $F \in Ad(X,X)$ is a compact map then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point (i.e. Schauder admissible uniform spaces are Lefschetz spaces (for the class Ad)).*

Let X be a Hausdorff topological space and let $\alpha \in Cov(X)$. X is said to be Schauder admissible α-dominated if there exists a Schauder admissible space X_α and two continuous functions $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and $i : X \to X$ are α–close and also that $r_\alpha\, s_\alpha \sim Id_X$. X is said to be almost Schauder admissible dominated if X is Schauder admissible α-dominated for every $\alpha \in Cov(X)$.

The following result can be found in [1].

Theorem 2.6. *Let X be a subset of a Hausdorff topological space and let X be a uniform space. Also suppose X is almost Schauder admissible dominated. If $F \in Ad(X,X)$ is a compact map then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point (i.e. almost Schauder admissible dominated uniform spaces are Lefschetz spaces (for the class Ad)).*

Remark 2.6. A similar result holds if $F \in Ad(X,X)$ is replaced by $F \in Ads(X,X)$ in Theorem 2.5 and 2.6.

3 Asymptotic Fixed Point Theory

Let X be a Hausdorff topological space. A map $F \in Ad(X,X)$ is said to be a compact absorbing contraction (written $F \in CAC(X,X)$ or $F \in CAC(X)$) if there exists $Y \subseteq X$ such that

(i). $F(Y) \subseteq Y$;

(ii). $F|_Y \in Ad(Y,Y)$ (automatically satisfied) is a compact map with Y a Lefschetz space;

(iii). for every compact $K \subseteq X$ there is an integer $n = n(K)$ such that $F^n(K) \subseteq Y$.

Remark 3.1. Examples of Lefschetz spaces Y can be found in Section 2. For example Y could be NES(compact) or a $SANES$(compact) uniform space.

Remark 3.2. If $Y = U$ is an open subset of X then (iii) could be changed to

(iii)'. for every $x \in X$ there exists an integer $n = n(x)$ such that $F^{n(x)}(x) \subseteq Y = U$.

To see this we show (iii)' implies (iii). For each $x \in X$ there exists $n(x)$ such that $F^{n(x)}(x) \subseteq Y = U$ so by upper semicontinuity there exists an open neighborhood U_x of x in X such that $F^{n(x)}(y) \subseteq Y = U$ for $y \in U_x$. Let K be a compact subset of X. Then there exists an open covering $\{U_{x_1}, ..., U_{x_n}\}$ of K. Let $n = \max\{n(x_1), ..., n(x_n)\}$ and so for $x \in K$ we have $F^n(x) \subseteq U = Y$, so (iii) is true.

Remark 3.3. For a discussion on compact absorbing contractions see [9, Section 42] and [12, Section 15.5].

Theorem 3.1. *Let X be a Hausdorff topological space and $F \in CAC(X,X)$. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point.*

Proof: Let Y be as described above. Let (p,q) be a selected pair for F so in particular $q\,p^{-1}(Y) \subseteq F(Y)$. Consider $F|_Y$ and let $q',\,p' : p^{-1}(Y) \to Y$ be given by $p'(u) = p(u)$ and $q'(u) = q(u)$. Notice (p',q') is a selected pair for $F|_Y$. Now since Y is a Lefschetz space then $q'_\star\,(p')_\star^{-1}$ is a Leray endomorphism. Now [9, Proposition 42.2, pp 208] guarantees (see (iii)) that the homeomorphism

$$q''_\star\,(p'')_\star^{-1} : H(X,Y) \to H(X,Y)$$

is weakly nilpotent (here p'', q'' : $(\Gamma, p^{-1}(Y)) \to (X,Y)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$). Then [9, pp 53] guarantees that $q''_\star\,(p'')_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q''_\star\,(p'')_\star^{-1}) = 0$. Also [9, Property 11.5, pp 52] guarantees that $q_\star\,p_\star^{-1}$ is a Leray endomorphism (with $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1})$) so $\mathbf{\Lambda}(F)$ is well defined.

Next suppose $\mathbf{\Lambda}(F) \neq \{0\}$. Then there exists a selected pair (p,q) of F with $\Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Let (p',q') be as described above with $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1})$. Then $\Lambda\,(q'_\star\,(p')_\star^{-1}) \neq 0$ so since Y is a Lefschetz space then there exists $x \in Y$ with $x \in F|_Y\,(x)$ i.e. $x \in F\,x$. $\square$

Remark 3.4. A map $F \in Ads(X,X)$ is said to be a compact absorbing contraction (written $F \in CACs(X,X)$ or $F \in CACs(X)$) if there exists $Y \subseteq X$ such that

(i). $F(Y) \subseteq Y$;

(ii). $F|_Y \in Ads(Y,Y)$ (automatically satisfied) is a compact map with Y a Lefschetz space;

(iii). for every compact $K \subseteq X$ there is an integer $n = n(K)$ such that $F^n(K) \subseteq Y$.

Essentially the same reasoning as in Theorem 3.1 establishes the following: Let X be a Hausdorff topological space and $F \in CACs(X,X)$. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq 0$ then F has a fixed point.

4 Fixed point theory in Fréchet spaces

We now present another approach based on projective limits. Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1,2,....\}$. We assume that the family of seminorms satisfies

$$(4.1) \qquad |x|_1 \leq |x|_2 \leq |x|_3 \leq .... \quad \text{for every } x \in E.$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x,r) = \{y \in E : |x-y|_n \leq r\, \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$(4.2) \qquad x \sim_n y \quad \text{iff} \quad |x-y|_n = 0.$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (4.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (4.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m / \sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now $\mu_{n,m}\,\mu_{m,k} = \mu_{n,k}$ if $n \leq m \leq k$ and $\mu_n = \mu_{n,m}\,\mu_m$ if $n \leq m$. We now assume the following condition holds:

$$(4.3) \qquad \begin{cases} \text{for each } n \in N, \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases}$$

Remark 4.1. (i). For convenience the norm on E_n is denoted by $|\cdot|_n$.

(ii). In many applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii). Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally we assume

$$(4.4) \qquad \begin{cases} E_1 \supseteq E_2 \supseteq \ldots\ldots\ldots \text{ and for each } n \in N, \\ |j_n\,\mu_{n,n+1}\,j_{n+1}^{-1}\,x|_n \leq |x|_{n+1}\ \forall\, x \in E_{n+1} \end{cases}$$

(here we use the notation from [1, 2] i.e. decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [1, 2]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\,\mu_{n,m}\,j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\,\mu_n(X)$, and we let $\overline{X_n}$, $int\,X_n$ and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. For $r > 0$ and $x \in E_n$ we denote $B_n(x,r) = \{y \in E_n : |x-y|_n \leq r\}$.

Let $M \subseteq E$ and consider the map $F : M \to 2^E$. Assume for each $n \in N$ and $x \in M$ that $j_n\,\mu_n\,F(x)$ is closed. Let $n \in N$ and $M_n = j_n\,\mu_n(M)$. Since we first consider Volterra type operators we assume (note this assumption is only needed in Theorems 4.1 and 4.2)

$$(4.5) \qquad \text{if } x,\, y \in E \text{ with } |x-y|_n = 0 \text{ then } H_n(F\,x, F\,y) = 0;$$

here H_n denotes the appropriate generalized Hausdorff distance (alternatively we could assume $\forall n \in N, \forall x,\, y \in M$ if $j_n\,\mu_n\,x = j_n\,\mu_n\,y$ then $j_n\,\mu_n\,F\,x = j_n\,\mu_n\,F\,y$ and of course here we do not need to assume that $j_n\,\mu_n\,F(x)$ is closed for each $n \in N$ and $x \in M$). Now (4.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For $y \in M_n$ there exists a $x \in M$ with $y = j_n\,\mu_n(x)$ and we let

$$F_n\,y = j_n\,\mu_n\,F\,x$$

(we could of course call it $F\,y$ since it is clear in the situation we use it); note $F_n : M_n \to C(E_n)$ and note if there exists a $z \in M$ with $y = j_n\,\mu_n(z)$ then $j_n\,\mu_n\,F\,x = j_n\,\mu_n\,F\,z$ from (4.5) (here $C(E_n)$

denotes the family of nonempty closed subsets of E_n). In this paper we assume F_n will be defined on $\overline{M_n}$ i.e. we assume the F_n described above admits an extension (again we call it F_n) $F_n : \overline{M_n} \to 2^{E_n}$ (we will assume certain properties on the extension).

Now we present some Lefschetz type theorems in Fréchet spaces. Our first two results are motivated by Fredholm type operators.

Theorem 4.1. *Let E and E_n be as described above, $C \subseteq E$ and $F : C \to 2^E$. Also assume for each $n \in N$ and $x \in C$ that $j_n\,\mu_n\,F(x)$ is closed and also for each $n \in N$ that $F_n : \overline{C_n} \to 2^{E_n}$ as described above is a closed map with $x \notin F_n(x)$ in E_n for $x \in \partial\, C_n$. Suppose the following conditions are satisfied:*

$$\begin{cases} \text{for each } \ n \in N, \ F_n \in CAC(C_n, C_n) \ \text{ and} \\ F_n : \overline{C_n} \to 2^{E_n} \text{ is hemicompact,} \end{cases} \tag{4.6}$$

$$\text{for each } \ n \in N, \ \mathbf{\Lambda}_{C_n}(F_n) \neq \{0\} \tag{4.7}$$

and

$$\begin{cases} \text{for each } \ n \in \{2,3,....\} \ \text{ if } \ y \in C_n \ \text{ solves } \ y \in F_n\,y \ \text{ in } \ E_n \\ \text{then } \ j_k\,\mu_{k,n}\,j_n^{-1}(y) \in C_k \ \text{ for } \ k \in \{1,...,n-1\}. \end{cases} \tag{4.8}$$

Then F has a fixed point in E.

Proof: For each $n \in N$ there exists $y_n \in C_n$ with $y_n \in F_n\,y_n$ in E_n. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in C_1$ and $j_1\,\mu_{1,k}\,j_k^{-1}(y_k) \in C_1$ for $k \in N\backslash\{1\}$ from (4.8). Note $j_1\,\mu_{1,n}\,j_n^{-1}(y_n) \in F_1\,(j_1\,\mu_{1,n}\,j_n^{-1}(y_n))$ in E_1; to see note for $n \in N$ fixed there exists a $x \in E$ with $y_n = j_n\,\mu_n\,(x)$ so $j_n\,\mu_n\,(x) \in F_n\,(y_n) = j_n\,\mu_n\,F(x)$ on E_n so on E_1 we have

$$\begin{aligned} j_1\,\mu_{1,n}\,j_n^{-1}(y_n) &= j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,(x) \in j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,F(x) \\ &= j_1\,\mu_{1,n}\,\mu_n\,F(x) = j_1\,\mu_1\,F(x) = F_1(j_1\,\mu_1\,(x)) \\ &= F_1(j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,(x)) = F_1\,(j_1\,\mu_{1,n}\,j_n^{-1}(y_n)). \end{aligned}$$

Now (4.6) guarantees that there exists is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{C_1}$ with $j_1\,\mu_{1,n}\,j_n^{-1}(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$ and $z_1 \in F_1\,z_1$ since F_1 is a closed map. Note $z_1 \in C_1$ since $x \notin F_1(x)$ in E_1 for $x \in \partial\, C_1$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $j_2\,\mu_{2,n}\,j_n^{-1}(y_n) \in C_2$ for $n \in N_1$ together with (4.6) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{C_2}$ with $j_2\,\mu_{2,n}\,j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$ and $z_2 \in F_2\,z_2$. Also $z_2 \in C_2$. Note from (4.4) and the uniqueness of limits that $j_1\,\mu_{1,2}\,j_2^{-1}\,z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$ (note $j_1\,\mu_{1,n}\,j_n^{-1}(y_n) = j_1\,\mu_{1,2}\,j_2^{-1}\,j_2\,\mu_{2,n}\,j_n^{-1}(y_n)$ for $n \in N_2^\star$). Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ......, \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in \overline{C_k}$ with $j_k\,\mu_{k,n}\,j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$ and $z_k \in F_k\,z_k$. Also $z_k \in C_k$. Note $j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = z_k$ in E_k for $k \in \{1,2,...\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Now $z_k \in F_k\,z_k$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,j_{k+1}\,\mu_{k+1,k+2}\,j_{k+2}^{-1}\,z_{k+2} \\ &= j_k\,\mu_{k,k+2}\,j_{k+2}^{-1}\,z_{k+2} = ... = j_k\,\mu_{k,m}\,j_m^{-1}\,z_m = \pi_{k,m}\,z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $z_k \in C_k$ for each $k \in N$. Thus for each $k \in N$ we have

$$j_k\, \mu_k\,(y) = z_k \in F_k\, z_k = j_k\, \mu_k\, F\, y \quad \text{in} \quad E_k$$

so $y \in F\, y$ in E. $\square$

Remark 4.2. Of course one could remove $x \notin F_n(x)$ in E_n for $x \in \partial\, C_n$ for each $n \in N$ if C is a closed subset of E. The proof follows as in Theorem 4.1 except in this case $z_k \in \overline{C_k}$ (but not necessarily in C_k). Also from $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}\,(y_m) \to z_k$ in E_k as $m \to \infty$ we can conclude that $y \in \overline{C} = C$ (note $q \in \overline{C}$ iff for every $k \in N$ there exists $(x_{k,m}) \in C$, $x_{k,m} = \pi_{k,n}\,(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k\, \mu_k\,(q)$ in E_k as $m \to \infty$). Thus $z_k = j_k\, \mu_k\,(y) \in C_k$ and so $j_k\, \mu_k\,(y) \in j_k\, \mu_k\, F\,(y)$ in E_k as before. Note in fact we can remove the assumption that C is a closed subset of E if we assume $F : Y \to 2^E$ with $C \subseteq Y$ and $\overline{C_n} \subseteq Y_n$ for each $n \in N$.

Remark 4.3. If we replace $F_n : \overline{C_n} \to 2^{E_n}$ is hemicompact in (4.6) with $F_n : C_n \to 2^{E_n}$ is hemicompact then one can remove $x \notin F_n(x)$ in E_n for $x \in \partial\, C_n$ and $F_n : \overline{C_n} \to 2^{E_n}$ is a closed map for each $n \in N$ in the statement of Theorem 4.1 since if for each $n \in N$, $F_n : C_n \to 2^{E_n}$ is hemicompact then we automatically have that $z_k \in C_k$.

Essentially the same reasoning as in Theorem 4.1 (with Remark 4.2) yields the following result.

Theorem 4.2. *Let E and E_n be as described above, $C \subseteq E$ and $F : C \to 2^E$. Also assume C is a closed subset of E, for each $n \in N$ and $x \in C$ that $j_n\, \mu_n\, F\,(x)$ is closed and also for each $n \in N$ that $F_n : \overline{C_n} \to 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:*

(4.9) $$\text{for each } n \in N,\ F_n \in CAC(\overline{C_n}, \overline{C_n}) \text{ is hemicompact,}$$

(4.10) $$\text{for each } n \in N,\ \mathbf{\Lambda}_{\overline{C_n}}\,(F_n) \neq \{0\}$$

and

(4.11) $$\begin{cases} \text{for each } n \in \{2, 3, ...\} \text{ if } y \in \overline{C_n} \text{ solves } y \in F_n\, y \text{ in } E_n \\ \text{then } j_k\, \mu_{k,n}\, j_n^{-1}\,(y) \in \overline{C_k} \text{ for } k \in \{1, ..., n-1\}. \end{cases}$$

Then F has a fixed point in E.

Remark 4.4. Note we can remove the assumption in Theorem 4.2 that C is a closed subset of E if we assume $F : Y \to 2^E$ with $C \subseteq Y$ and $\overline{C_n} \subseteq Y_n$ for each $n \in N$.

Our result two results are motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (4.16).

Theorem 4.3. *Let E and E_n be as described above, $C \subseteq E$ and $F : C \to 2^E$. Also for each $n \in N$ assume there exists $F_n : C_n \to 2^{E_n}$ and suppose the following conditions are satisfied:*

(4.12) $$\text{for each } n \in N,\ F_n \in CAC(C_n, C_n)$$

(4.13) $$\text{for each } n \in N,\ \mathbf{\Lambda}_{C_n}\,(F_n) \neq \{0\}$$

$$(4.14)\qquad \begin{cases} \textit{for each } n \in \{2,3,....\} \textit{ if } y \in C_n \textit{ solves } y \in F_n\, y \textit{ in } E_n \\ \textit{then } j_k\, \mu_{k,n}\, j_n^{-1}(y) \in C_k \textit{ for } k \in \{1,...,n-1\} \end{cases}$$

$$(4.15)\qquad \begin{cases} \textit{for any sequence } \{y_n\}_{n\in N} \textit{ with } y_n \in C_n \\ \textit{and } y_n \in F_n\, y_n \textit{ in } E_n \textit{ for } n \in N \textit{ and} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, ...\},\ N_k \subseteq N_{k-1} \textit{ for} \\ k \in \{1,2,....\},\ N_0 = N, \textit{ and a } z_k \in C_k \textit{ with} \\ j_k\, \mu_{k,n}\, j_n^{-1}(y_n) \to z_k \textit{ in } E_k \textit{ as } n \to \infty \textit{ in } N_k \end{cases}$$

and

$$(4.16)\qquad \begin{cases} \textit{if there exists a } w \in C \textit{ and a sequence } \{y_n\}_{n\in N} \\ \textit{with } y_n \in C_n \textit{ and } y_n \in F_n\, y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \textit{ of } N \textit{ with } j_k\, \mu_{k,n}\, j_n^{-1}(y_n) \to w \\ \textit{in } E_k \textit{ as } n \to \infty \textit{ in } S, \textit{ then } w \in F\, w \textit{ in } E. \end{cases}$$

Then F has a fixed point in E.

Remark 4.5. Notice to check (4.15) we need to show that for each $k \in N$ the sequence $\{j_k\, \mu_{k,n}\, j_n^{-1}(y_n)\}_{n \in N_{k-1}} \subseteq C_k$ is sequentially compact.

Proof: Fix $n \in N$. Now there exists $y_n \in C_n$ with $y_n \in F_n\, y_n$ in E_n. Lets look at $\{y_n\}_{n\in N}$. Notice $y_1 \in C_1$ and $j_1\, \mu_{1,k}\, j_k^{-1}(y_k) \in C_1$ for $k \in \{2,3,...\}$. Now (4.15) with $k=1$ guarantees that there exists a subsequence $N_1 \subseteq \{2,3,....\}$ and a $z_1 \in C_1$ with $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) \to z_1$ in E_1 as $n \to \infty$ in N_1. Look at $\{y_n\}_{n\in N_1}$. Now $j_2\, \mu_{2,n}\, j_n^{-1}(y_n) \in C_2$ for $k \in N_1$. Now (4.15) with $k=2$ guarantees that there exists a subsequence $N_2 \subseteq \{3,4,...\}$ of N_1 and a $z_2 \in C_2$ with $j_2\, \mu_{2,n}\, j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in N_2. Note from (4.4) and the uniqueness of limits that $j_1\, \mu_{1,2}\, j_2^{-1}\, z_2 = z_1$ in E_1 since $N_2 \subseteq N_1$ (note $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) = j_1\, \mu_{1,2}\, j_2^{-1}\, j_2\, \mu_{2,n}\, j_n^{-1}(y_n)$ for $n \in N_2$). Proceed inductively to obtain subsequences of integers

$$N_1 \supseteq N_2 \supseteq ......, \quad N_k \subseteq \{k+1, k+2, ...\}$$

and $z_k \in C_k$ with $j_k\, \mu_{k,n}\, j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in N_k. Note $j_k\, \mu_{k,k+1}\, j_{k+1}^{-1}\, z_{k+1} = z_k$ in E_k for $k \in \{1,2,...\}$.

Fix $k \in N$. Note

$$\begin{aligned} z_k &= j_k\, \mu_{k,k+1}\, j_{k+1}^{-1}\, z_{k+1} = j_k\, \mu_{k,k+1}\, j_{k+1}^{-1}\, j_{k+1}\, \mu_{k+1,k+2}\, j_{k+2}^{-1}\, z_{k+2} \\ &= j_k\, \mu_{k,k+2}\, j_{k+2}^{-1}\, z_{k+2} = ... = j_k\, \mu_{k,m}\, j_m^{-1}\, z_m = \pi_{k,m}\, z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $y \in C$ since $z_k \in C_k$ for each $k \in N$. Also since $y_n \in F_n\, y_n$ in E_n for $n \in N_k$ and $j_k\, \mu_{k,n}\, j_n^{-1}(y_n) \to z_k = y$ in E_k as $n \to \infty$ in N_k we have from (4.16) that $y \in F\, y$ in E. $\square$

Remark 4.6. From the proof we see that condition (4.14) can be removed from the statement of Theorem 4.3. We include it only to explain condition (4.15) (see Remark 4.5).

Remark 4.7. Suppose in Theorem 4.3 we have

$$(4.15)^\star \qquad \begin{cases} \text{for any sequence } \{y_n\}_{n\in N} \text{ with } y_n \in C_n \\ \text{and } y_n \in F_n\, y_n \text{ in } E_n \text{ for } n \in N \text{ and} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, ...\},\ N_k \subseteq N_{k-1} \text{ for} \\ k \in \{1, 2, ...\},\ N_0 = N, \text{ and a } z_k \in \overline{C_k} \text{ with} \\ j_k\, \mu_{k,n}\, j_n^{-1}(y_n) \to z_k \text{ in } E_k \text{ as } n \to \infty \text{ in } N_k \end{cases}$$

instead of (4.15) and $F : C \to 2^E$ is replaced by $F : Y \to 2^E$ with $C \subseteq Y$ and $\overline{C_n} \subseteq Y_n$ for each $n \in N$ and suppose (4.16) is true with $w \in C$ replaced by $w \in Y$. Then the result in Theorem 4.3 is again true.

The proof follows the reasoning in Theorem 4.3 except in this case $z_k \in \overline{C_k}$ (but not necessarily in C_k) and $y \in Y$.

In fact we could replace $\overline{C_n} \subseteq Y_n$ above with $\overline{C_n}$ a subset of the closure of Y_n in E_n if Y is a closed subset of E (so in this case we can take $Y = C$ if C is a closed subset of E). To see this note $z_k \in \overline{C_k}$, $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}(y_m) \to z_k$ in E_k as $m \to \infty$ and we can conclude that $y \in \overline{Y} = Y$.

In fact in this remark we could replace (in fact we can remove it as mentioned in Remark 4.6) (4.14) with

$$(4.14)^\star \qquad \begin{cases} \text{for each } n \in \{2, 3, ...\} \text{ if } y \in C_n \text{ solves } y \in F_n\, y \text{ in } E_n \\ \text{then } j_k\, \mu_{k,n}\, j_n^{-1}(y) \in \overline{C_k} \text{ for } k \in \{1, ..., n-1\} \end{cases}$$

and the result above is again true.

Essentially the same reasoning as in Theorem 4.3 (with Remark 4.7) yields the following result.

Theorem 4.4. *Let E and E_n be as described above, $C \subseteq E$ and $F : C \to 2^E$. Also assume C is a closed subset of E and for each $n \in N$ that $F_n : \overline{C_n} \to 2^{E_n}$ and suppose the following conditions are satisfied:*

$$(4.17) \qquad \begin{cases} \textit{for each } n \in \{2, 3, ...\} \textit{ if } y \in \overline{C_n} \textit{ solves } y \in F_n\, y \textit{ in } E_n \\ \textit{then } j_k\, \mu_{k,n}\, j_n^{-1}(y) \in \overline{C_k} \textit{ for } k \in \{1, ..., n-1\} \end{cases}$$

$$(4.18) \qquad \textit{for each } n \in N,\ F_n \in CAC(\overline{C_n}, \overline{C_n})$$

$$(4.19) \qquad \textit{for each } n \in N,\ \mathbf{\Lambda}_{\overline{C_n}}(F_n) \neq \{0\}$$

$$(4.20)\quad \begin{cases} \textit{for any sequence } \{y_n\}_{n\in N} \textit{ with } y_n \in \overline{C_n} \\ \textit{and } y_n \in F_n\, y_n \textit{ in } E_n \textit{ for } n \in N \textit{ and} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, ...\},\ N_k \subseteq N_{k-1} \textit{ for} \\ k \in \{1, 2, ...\},\ N_0 = N, \textit{ and a } z_k \in \overline{C_k} \textit{ with} \\ j_k\, \mu_{k,n}\, j_n^{-1}\, (y_n) \to z_k \textit{ in } E_k \textit{ as } n \to \infty \textit{ in } N_k \end{cases}$$

and

$$(4.21)\quad \begin{cases} \textit{if there exists a } w \in C \textit{ and a sequence } \{y_n\}_{n\in N} \\ \textit{with } y_n \in \overline{C_n} \textit{ and } y_n \in F_n\, y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \textit{ of } N \textit{ with } j_k\, \mu_{k,n}\, j_n^{-1}\, (y_n) \to w \\ \textit{in } E_k \textit{ as } n \to \infty \textit{ in } S, \textit{ then } w \in F\, w \textit{ in } E. \end{cases}$$

Then F has a fixed point in E.

Remark 4.8. Condition (4.17) can be removed from the statement of Theorem 4.4.

Remark 4.9. Note we can remove the assumption in Theorem 4.4 that C is a closed subset of E if we assume $F : Y \to 2^E$ with $C \subseteq Y$ and $\overline{C_n} \subseteq Y_n$ (or $\overline{C_n}$ a subset of the closure of Y_n in E_n if Y is a closed subset of E) for each $n \in N$ with of course $w \in C$ replaced by $w \in Y$ in (4.21).

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