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On subsets of ideal topological spaces

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ABSTRACT

We define some new collection of sets in ideal topological spaces and characterize them in terms of sets already defined. Also, we give a decomposition theorem for $\alpha-\mathcal{I}-$open sets and open sets. At the end, we discuss the property of some collection of subsets in $\star-$extremally disconnected spaces.

RESUMEN

Definimos una nueva colección de conjuntos en espacios topológicos ideales y caracterizamos estos en términos de conjuntos ya definidos. También damos un teorema de descomposición para $\alpha-\mathcal{I}-$ abiertos y conjuntos abiertos. Finalmente discutimos la probabilidad de algunas colecciones de subconjuntos en espacios disconexos $\star-$ extremos.

Key words and phrases: *$\star-$extremally disconnected spaces, $t-\mathcal{I}-$set, $\alpha-\mathcal{I}-$open set, pre$-\mathcal{I}-$open set, semi–$\mathcal{I}-$open set, semi$^\star-\mathcal{I}-$open, semipre$^\star-\mathcal{I}-$open, $\mathcal{C}_\mathcal{I}-$set, $\mathcal{B}_\mathcal{I}-$set, $\mathcal{B}_{1\mathcal{I}}-$set, $\mathcal{B}_{2\mathcal{I}}-$set, $\mathcal{B}_{3\mathcal{I}}-$set, $\delta-\mathcal{I}-$open, $R\mathcal{I}-$open, $\mathcal{I}-$locally closed set, weakly $\mathcal{I}-$locally closed set, $\mathcal{A}_{\mathcal{I}R}-$set, $\mathcal{D}_\mathcal{I}-$set.*

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1 Introduction

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of A in (X, τ). An *ideal* $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a $local\ function$ [14] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\mathcal{I}, \tau)$=$\{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts concerning the local function [11, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star()$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $\star - topology$, finer than τ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$ [16]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$ and $\tau^\star$ or $\tau^\star(\mathcal{I})$ for $\tau^\star(\mathcal{I}, \tau)$. $int^\star(A)$ will denote the interior of A in $(X, \tau^\star)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is $\tau^\star - closed$ or $\star - closed$ [11](resp.$\star - perfect$[10]) if $A^\star \subset A$(resp.$A = A^\star$). A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be a $t - \mathcal{I} - set$[8] if $int(A) = int(cl^\star(A))$. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\delta - \mathcal{I} - open$[2](resp. $\alpha - \mathcal{I} - open$ [8],$pre - \mathcal{I} - open$ [6],$semi - \mathcal{I} - open$ [8], $strong\, \beta - \mathcal{I} - open$[9]) if $int(cl^\star(A)) \subset cl^\star(int(A))$(resp. $A \subset int(cl^\star(int(A)))$, $A \subset int(cl^\star(A))$, $A \subset cl^\star(int(A)$, $A \subset cl^\star(int(cl^\star(A)))$. We will denote the family of all $\delta - \mathcal{I}-$open (resp. $\alpha - \mathcal{I}-$open, pre$-\mathcal{I}-$open, semi$-\mathcal{I}-$open, strong $\beta-\mathcal{I}-$open) sets by $\delta\mathcal{I}O(X)$(resp.$\alpha\mathcal{I}O(X), P\mathcal{I}O(X), S\mathcal{I}O(X), s\beta\mathcal{I}O(X)$). The largest pre$\mathcal{I}-$open set contained in A is called the $pre - \mathcal{I} - interior$ of A and is denoted by $p\mathcal{I}int(A)$. For any subset A *of an ideal space* $(X, \tau, \mathcal{I})$, $p\mathcal{I}int(A) = A \cap int(cl^\star(A))$ *[15, Lemma 1.5].*

2 Subsets of Ideal Topological Spaces

Let $(X, \tau, \mathcal{I})$ *be an ideal space. A subset* A *of* X *is said to be a* $semi^\star - \mathcal{I} - open$ *set [7] if* $A \subset cl(int^\star(A))$. *A subset* A *of* X *is said to be a* $semi^\star - \mathcal{I}-$*closed set [7] if its complement is a* $semi^\star - \mathcal{I} - open$ *set. Clearly,* A *is* $semi^\star - \mathcal{I}-$*closed if and only if* $int(cl^\star(A)) \subset A$ *if and only if* $int(cl^\star(A)) = int(A)$ *and so* $semi^\star - \mathcal{I}-$*closed sets are nothing but* $t - \mathcal{I}-$*sets. A is said to be a* $semipre^\star - \mathcal{I}-$*closed set if* $int(cl^\star(int(A))) \subset A$. *Clearly,* A *is said to be a* $semipre^\star - \mathcal{I}-$*closed if and only if* $int(cl^\star(int(A))) = int(A)$ *if and only if* A *is* $\alpha^\star - \mathcal{I}-$*set [8]. Clearly,* X *is both* $semi^\star - \mathcal{I}-$*closed and* $semipre^\star - \mathcal{I}-$*closed. The smallest* $semi^\star - \mathcal{I}-$*closed (resp.* $semipre^\star - \mathcal{I}-$*closed) set containing is called the* $semi^\star - \mathcal{I} - closure$ *(resp.*$semipre^\star - \mathcal{I} - closure$*) of* A *and is denoted by* $S\mathcal{I}cl(A)$*(resp.*$sp\mathcal{I}cl(A)$*). A subset* A *of an ideal space* $(X, \tau, \mathcal{I})$ *is said to be a* $\mathcal{B}_\mathcal{I}-$*set[8] if* $A = U \cap V$ *where* U *is open and* V *is a* $t - \mathcal{I}-$*set. The easy proof of the following Theorem 2.1 is omitted which says that the arbitrary intersection of* $semi^\star - \mathcal{I}-$*closed (resp.* $semipre^\star - \mathcal{I}-$*closed) set is a* $semi^\star - \mathcal{I}-$*closed (resp.* $semipre^\star - \mathcal{I}-$*closed) set.*

Theorem 2.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space and* $A \subset X$. *If* $\{A_\alpha \mid \alpha \in \triangle\}$ *is a family of* $semi^\star - \mathcal{I}-$*closed (resp.* $semipre^\star - \mathcal{I}-$*closed) sets, then* $\cap A_\alpha$ *is a* $semi^\star - \mathcal{I}-$*closed (resp.* $semipre^\star - \mathcal{I}-$*closed) set.*

Theorem 2.2. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space and* $A \subset X$. *Then the following hold.*
(a)$S\mathcal{I}cl(A) = A \cup int(cl^\star(A))$.

(b)$sp\mathcal{I}cl(A) = A \cup int(cl^\star(int(A)))$.

Proof. *The proof follows from Theorem 1.3 and Theorem 3.1 of [5].*

Every $semi^\star - \mathcal{I}-$*closed set is a* $semipre^\star - \mathcal{I}-$*closed set but not the converse as shown by the following Example 2.3. Theorem 2.4 below shows that the reverse direction is true if the set is semi–*$\mathcal{I}-$*open. Theorem 2.5 gives a characterization of* $t - \mathcal{I}-$*sets.*

Example 2.3. *Consider the ideal space* $(X, \tau, \mathcal{I})$ *where* $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. *If* $A = \{a\}$, *then* $int(cl^\star(int(A))) = int(cl^\star(\emptyset)) = \emptyset \subset A$ *and so* A *is* $semipre^\star - \mathcal{I}-$*closed. Since* $int(cl^\star(A)) = int(cl^\star(\{a\})) = int(\{a, b, c\}) = \{a, c\} \nsubseteq \{a\}, A$ *is not* $semi^\star - \mathcal{I}-$*closed.*

Theorem 2.4. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space and* $A \subset X$ *be* $semipre^\star - \mathcal{I}-$*closed. If* A *is semi–*$\mathcal{I}-$*open, then* A *is* $semi^\star - \mathcal{I}-$*closed.*

Proof. If A is semi–$\mathcal{I}-$open, then $A \subset cl^\star(int(A))$ and so $cl^\star(A) \subset cl^\star(int(A))$. Now $int(cl^\star(A)) \subset int(cl^\star(int(A))) \subset A$ and so A is $semi^\star - \mathcal{I}-$closed.

Theorem 2.5. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space and* $A \subset X$. *Then the following are equivalent.*
(a) A *is a* $t - \mathcal{I}-$*set.*
(b) A *is* $semi^\star - \mathcal{I}-$*closed.*
(c) A *is a* $semipre^\star - \mathcal{I}-$*closed* $\mathcal{B}_\mathcal{I}-$*set.*

Proof. *Enough to prove that (c)⇒(a). Suppose* A *is a* $semipre^\star - \mathcal{I}-$closed $\mathcal{B}_\mathcal{I}-$set. Then $A = U \cap V$ where U is open and V is a $t - \mathcal{I}-$set. Now $int(cl^\star(A)) = int(cl^\star(U \cap V)) \subset int(cl^\star(U) \cap cl^\star(V)) = int(cl^\star(U)) \cap int(cl^\star(V)) = int(cl^\star(U)) \cap int(V) = int(cl^\star(U) \cap int(V)) \subset int(cl^\star(U \cap int(V)) = int(cl^\star(int(U \cap V))) = int(cl^\star(int(A))) \subset A$ and so $int(cl^\star(A)) \subset int(A)$. But $int(A) \subset int(cl^\star(A))$ and so $int(A) = int(cl^\star(A))$ which implies that A is a $t - \mathcal{I}-$set.

The following Example 2.6 shows that the union of two $semi^\star - \mathcal{I}$*closed (resp.* $semipre^\star - \mathcal{I}-$*closed) set is not a* $semi^\star - \mathcal{I}$*closed (resp.* $semipre^\star - \mathcal{I}-$*closed) set.*

Example 2.6. *Consider the ideal space*$(X, \tau, \mathcal{I})$ *of Example 2.3. If* $A = \{a, c\}$ *and* $B = \{d\}$, *then* $int(cl^\star(A)) = int(cl^\star(\{a, c\})) = int(\{a, b, c\}) = \{a, c\} = A$ *and so* A *is* $semi^\star - \mathcal{I}-$*closed and hence* $semipre^\star - \mathcal{I}-$*closed. Also,* $int(cl^\star(B)) = int(cl^\star(\{d\})) = int(\{d\}) = \{d\} = B$. *Therefore,* B *is* $semi^\star - I-$*closed and so* $semipre^\star - I-$*closed. But* $int(cl^\star(int(A \cup B))) = int(cl^\star(int(\{a, c, d\}))) = int(cl^\star(\{a, c, d\})) = int(X) = X \nsubseteq A \cup B$ *and so* $A \cup B$ *is not* $semipre^\star - \mathcal{I}-$*closed and hence* $A \cup B$ *is not* $semi^\star - \mathcal{I}-$*closed.*

A subset A *of an ideal space* $(X, \tau, \mathcal{I})$ *is said to be a* $\mathcal{C}_\mathcal{I}-$*set [8] if* $A = U \cap V$ *where* U *is open and* V *is a* $semipre^\star - \mathcal{I}-$*closed set. We will denote the family of all* $\mathcal{C}_\mathcal{I}-$*set by* $\mathcal{C}_\mathcal{I}(X)$. *The following Theorem 2.7 gives a characterization of* $\mathcal{B}_\mathcal{I}-$*sets and* $\mathcal{C}_\mathcal{I}-$*sets.*

Theorem 2.7. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space and* A *be a subset of* X. *Then the following hold.*
(a) A *is a* $\mathcal{B}_\mathcal{I}-$*set if and only if there exists an open set* U *such that* $A = U \cap S\mathcal{I}cl(A)$.
(b) A *is a* $\mathcal{C}_\mathcal{I}-$*set if and only if there exists an open set* U *such that* $A = U \cap sp\mathcal{I}cl(A)$.

Proof. (a) Suppose A is a $\mathcal{B}_\mathcal{I}-$set. Then $A = U \cap V$ where U is open and V is a $t - \mathcal{I}-$set. Since $t - \mathcal{I}-$sets are $semi^\star - \mathcal{I}-$closed sets, $S\mathcal{I}cl(V) = V$. Now $A = U \cap A \subset U \cap S\mathcal{I}cl(A) \subset U \cap S\mathcal{I}cl(V) = U \cap V = A$ and so $A = U \cap S\mathcal{I}cl(A)$. Conversely, suppose $A = U \cap S\mathcal{I}cl(A)$ for some

open set U. Since $S\mathcal{I}cl(A)$ is semi$^\star-\mathcal{I}-$closed, $int(cl^\star(S\mathcal{I}cl(A)))\subset S\mathcal{I}cl(A)$. Also, $int(S\mathcal{I}cl(A))\subset int(cl^\star(S\mathcal{I}cl(A)))\subset S\mathcal{I}cl(A)$ and so $int(S\mathcal{I}cl(A))=int(cl^\star(S\mathcal{I}cl(A)))$ which implies that $S\mathcal{I}cl(A)$ is a $t-\mathcal{I}-$set. Therefore, A is a $\mathcal{B}_\mathcal{I}-$set.
(b) The proof is similar to that of (a).

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be a $\mathcal{A}_{1\mathcal{I}}-$set (resp. $\mathcal{B}_{1\mathcal{I}}-$set [4]($\alpha_\mathcal{I}M_1-$set [1])) if $A=U\cap V$ where U is open (resp.$\alpha-\mathcal{I}-$open) and $cl^\star(int(V))=X$. We will denote the family of all $\mathcal{B}_{1\mathcal{I}}-$sets (resp. $\mathcal{A}_{1\mathcal{I}}-$sets) by $\mathcal{B}_{1\mathcal{I}}(X)$ (resp.$\mathcal{A}_{1\mathcal{I}}(X)$). Clearly, $\mathcal{A}_{1\mathcal{I}}(X)\subset\mathcal{B}_{1\mathcal{I}}(X)$. The following Theorem 2.8 shows that $\mathcal{B}_{1\mathcal{I}}-$sets and $\mathcal{A}_{1\mathcal{I}}-$sets are nothing but $\alpha-\mathcal{I}-$open sets.

Theorem 2.8. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then $\mathcal{B}_{1\mathcal{I}}(X)=\alpha\mathcal{I}O(X)=\mathcal{A}_{1\mathcal{I}}(X)$.*

Proof. *Suppose $A\in\mathcal{B}_{1\mathcal{I}}(X)$. Then $A=U\cap V$ where U is $\alpha-\mathcal{I}-$open and $cl^\star(int(V))=X$. Since $V\subset X=int(cl^\star(int(V)))$, $V\in\alpha\mathcal{I}O(X)$. Since $\alpha\mathcal{I}O(X)$ is a topology on X, $A\in\alpha\mathcal{I}O(X)$ and so $\mathcal{B}_{1\mathcal{I}}(X)\subset\alpha\mathcal{I}O(X)$.*
Suppose $A\in\alpha\mathcal{I}O(X)$. Then $A\subset int(cl^\star(int(A)))$ and so $A=int(cl^\star(int(A)))\cap(X-(int(cl^\star(int(A)))-A))=int(cl^\star(int(A)))\cap((X-int(cl^\star(int(A))))\cup A)$. Also, $cl^\star(int((X-int(cl^\star(int(A))))\cup A))\supset cl^\star(int(X-int(cl^\star(int(A))))\cup int(A))=cl^\star(int(X-int(cl^\star(int(A)))))\cup cl^\star(int(A))\supset cl^\star(int(X-cl^\star(int(A))))\cup cl^\star(int(A))\supset int(X-cl^\star(int(A)))\cup cl^\star(int(A))\supset int((X-cl^\star(int(A)))\cup cl^\star(int(A)))=int(X)=X$. Therefore, $A\in\mathcal{A}_{1\mathcal{I}}(X)$ which implies that $\alpha\mathcal{I}O(X)\subset\mathcal{A}_{1\mathcal{I}}(X)$.
Clearly, $\mathcal{A}_{1\mathcal{I}}(X)\subset\mathcal{B}_{1\mathcal{I}}(X)$. This completes the proof.

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be an $R\mathcal{I}-$open set [17] if $A=int(cl^\star(A))$. We will denote the family of all $R\mathcal{I}-$open sets by $R\mathcal{I}O(X)$. In [17], it is established that $R\mathcal{I}O(X)$ is a base for a topology $\tau_\mathcal{I}$ and $\tau_s\subset\tau_\mathcal{I}\subset\tau$ where τ_s is the semiregularization of τ. The following Theorem 2.9 gives characterizations of pre$-\mathcal{I}-$open sets in terms of $R\mathcal{I}-$open sets.

Theorem 2.9. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subset X$. Then the following are equivalent.*
(a) A is pre$-\mathcal{I}-$open.
(b) There exists an $R\mathcal{I}-$open set G such that $A\subset G$ and $cl^\star(G)=cl^\star(A)$.
(c)$A=G\cap D$ where G is $R\mathcal{I}-$open and D is $\tau^\star-$dense.
(d)$A=G\cap D$ where G is open and D is $\tau^\star-$dense.

Proof. *(a)$\Rightarrow$(b). Suppose A is pre$-\mathcal{I}-$open. If $G=int(cl^\star(A))$, then $A\subset G$ and $int(cl^\star(G))=int(cl^\star(int(cl^\star(A))))=int(cl^\star(A))=G$ which implies that G is an $R\mathcal{I}-$open set containing A. Also, $cl^\star(A)\subset cl^\star(G)=cl^\star(int(cl^\star(A)))\subset cl^\star(A)$ which implies that $cl^\star(A)=cl^\star(G)$. This proves (b).*
(b)$\Rightarrow$(c). Suppose G is an $R\mathcal{I}-$open set such that $A\subset G$ and $cl^\star(G)=cl^\star(A)$. If $D=A\cup(X-G)$, then $A=G\cap D$ and D is $\tau^\star-$dense. This proves (c).
(c)$\Rightarrow$(d) is clear.
(d)$\Rightarrow$(a) follows from Lemma 4.3 of [3].

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be a $\mathcal{A}_{2\mathcal{I}}-$set (resp. $\mathcal{B}_{2\mathcal{I}}-$set [4]($\alpha_\mathcal{I}M_2-$set [1])) if $A=U\cap V$ where U is open (resp.$\alpha-\mathcal{I}-$open) and $cl^\star(V)=X$. We will denote the family of all $\mathcal{A}_{2\mathcal{I}}-$sets (resp. $\mathcal{B}_{2\mathcal{I}}-$sets) by $\mathcal{A}_{2\mathcal{I}}(X)$ (resp.$\mathcal{B}_{2\mathcal{I}}(X)$). Clearly, $\mathcal{A}_{2\mathcal{I}}(X)\subset\mathcal{B}_{2\mathcal{I}}(X)$. The following Theorem 2.10 shows that $\mathcal{A}_{2\mathcal{I}}-$sets and $\mathcal{B}_{2\mathcal{I}}-$sets are nothing but pre$-\mathcal{I}-$open sets. Also, it shows that the converse of Proposition 2.6 of [4] is true.

Theorem 2.10. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then $\mathcal{A}_{2\mathcal{I}}(X)=P\mathcal{I}O(X)=\mathcal{B}_{2\mathcal{I}}(X)$.*

Proof. *By Theorem 2.9(d), $\mathcal{A}_{2\mathcal{I}}(X) = P\mathcal{I}O(X)$. Since $\mathcal{A}_{2\mathcal{I}}(X) \subset \mathcal{B}_{2\mathcal{I}}(X)$, it is enough to prove that $\mathcal{B}_{2\mathcal{I}}(X) \subset \mathcal{A}_{2\mathcal{I}}(X)$. Suppose $A \in \mathcal{B}_{2\mathcal{I}}(X)$. Then $A = U \cap V$ where U is $\alpha - \mathcal{I}-$open and $cl^\star(V) = X$. Now $A \subset U \subset int(cl^\star(int(U))) = int(cl^\star(int(U \cap X))) = int(cl^\star(int(U \cap cl^\star(V)))) \subset int(cl^\star(int(cl^\star(U \cap V)))) = int(cl^\star(U \cap V)) = int(cl^\star(A))$ and so $A \in P\mathcal{I}O(X)$. This completes the proof.*

Clearly, $A_{1\mathcal{I}}(X) \subset A_{2\mathcal{I}}(X)$. The following Example 2.11 shows that an $A_{2\mathcal{I}}-$set need not be an $A_{1\mathcal{I}}-$set.

Example 2.11. *Consider the ideal space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. If $A = \{a, c\}$, then A is an $A_{2\mathcal{I}}-$set. But $cl^\star(int(A)) = int(A) \cup (int(A))^\star = \{c\} \neq X$. Hence A is not an $A_{1\mathcal{I}}-$set.*

A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be an $\alpha_\mathcal{I} N_5-$set [1] if $A = U \cap V$ where U is $\alpha - \mathcal{I}-$open and V is $\star-$closed. We will denote the family of all $\alpha_\mathcal{I} N_5-$sets of an ideal space $(X, \tau, \mathcal{I})$ by $\alpha_\mathcal{I} N_5(X)$. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}-$locally closed [6] (resp. weakly $\mathcal{I}-$locally closed [13]) set if $A = U \cap V$ where U is open and V is a $\star-$perfect (resp. $\star-$closed) set. By Theorem 2.9 of [15], A is weakly $\mathcal{I}-$locally closed if and only if $A = U \cap cl^\star(A)$ for some open set U. The family of all weakly $\mathcal{I}-$locally closed sets is denoted by $W\mathcal{I}LC(X)$. Clearly, every weakly $\mathcal{I}-$locally closed set is an $\alpha_\mathcal{I} N_5-$set but not the converse as shown by the following Example 2.12. Theorem 2.13 below gives a characterization of $\alpha_\mathcal{I} N_5-$sets.

Example 2.12. *[4, Example 2.2]Consider the ideal space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. If $A = \{a, b\}$, then $int(cl^\star(int(A))) = int(cl^\star(int(\{a, b\}))) = int(cl^\star(\{a\})) = int(\{a, b, c\}) = X \supset A$ and so A is $\alpha - \mathcal{I}-$open and hence an $\alpha_\mathcal{I} N_5-$set. But there is no open set U such that $A = U \cap cl^\star(A)$ where $cl^\star(A) = X$. Hence A is not a weakly $\mathcal{I}-$locally closed set.*

Theorem 2.13. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. Then A is $\alpha_\mathcal{I} N_5-$set if and only if $A = U \cap cl^\star(A)$ for some $U \in \alpha\mathcal{I}O(X)$.*

Proof. *If A is an $\alpha_\mathcal{I} N_5-$set, then $A = U \cap V$ where U is $\alpha - \mathcal{I}-$open and V is $\star-$closed. Since $A \subset V$, $cl^\star(A) \subset cl^\star(V) = V$ and so $U \cap cl^\star(A) \subset U \cap V = A \subset U \cap cl^\star(A)$ which implies that $A = U \cap cl^\star(A)$. Conversely, suppose $A = U \cap cl^\star(A)$ for some $U \in \alpha\mathcal{I}O(X)$. Since $cl^\star(A)$ is $\star-$closed, A is an $\alpha_\mathcal{I} N_5-$set.*

A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}R-$closed set [1] if $A = cl^\star(int(A))$. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be an $\alpha\mathcal{A}_I-$set [4]($\alpha_\mathcal{I} N_2-$set [1]) (resp.$\mathcal{A}_{\mathcal{I}R}-$set [1]) if $A = U \cap V$ where U is an $\alpha-\mathcal{I}-$open (resp. open) set and V is an $\mathcal{I}R-$closed set. $\mathcal{A}_{\mathcal{I}R}-$sets are called as $\mathcal{A}_\mathcal{I}-$sets in [4]. We will denote the family of all $\alpha\mathcal{A}_I-$sets (resp.$\mathcal{A}_{\mathcal{I}R}-$sets) by $\alpha\mathcal{A}_\mathcal{I}(X)$(resp.$\mathcal{A}_{\mathcal{I}R}(X)$). Clearly, every $\mathcal{A}_{\mathcal{I}R}-$set is an $\alpha\mathcal{A}_I-$set but the converse is not true [4, Example 2.2]. Theorem 2.14 below shows that $\alpha\mathcal{A}_I-$sets are nothing but semi$-\mathcal{I}-$open sets which shows that the reverse direction of Proposition 2.4 of [4] is true and each such set is both a strong $\beta - \mathcal{I}-$open set and an $\alpha_\mathcal{I} N_5-$set.

Theorem 2.14. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\alpha\mathcal{A}_\mathcal{I}(X) = s\beta\mathcal{I}O(X) \cap \alpha_\mathcal{I} N_5(X) = S\mathcal{I}O(X)$.*

Proof. *Suppose $A \in \alpha\mathcal{A}_\mathcal{I}(X)$. Then $A = U \cap V$ where $U \in \alpha\mathcal{I}O(X)$ and V is an $\mathcal{I}R-$closed set. Now $A = U \cap V \subset int(cl^\star(int(U))) \cap cl^\star(int(V)) \subset cl^\star(int(cl^\star(int(U))) \cap int(V)) = cl^\star(int(cl^\star(int(U)) \cap int(V))) \subset cl^\star(int(cl^\star(int(U) \cap int(V)))) = cl^\star(int(cl^\star(int(U \cap V)))) = cl^\star(int(U \cap V)) = cl^\star(int(A)) \subset$*

$cl^\star(int(cl^\star(A)))$ and so $A \in s\beta\mathcal{I}O(X)$. Since V is $\star-$closed, $A \in \alpha_{\mathcal{I}}N_5(X)$ and so $\alpha\mathcal{A}_{\mathcal{I}}(X) \subset s\beta\mathcal{I}O(X) \cap \alpha_{\mathcal{I}}N_5(X)$. Conversely, suppose $A \in s\beta\mathcal{I}O(X) \cap \alpha_{\mathcal{I}}N_5(X)$. $A \in s\beta\mathcal{I}O(X)$ implies that $A \subset cl^\star(int(cl^\star(A)))$ and $A \in \alpha_{\mathcal{I}}N_5(X)$ implies that $A = U \cap cl^\star(A)$ where $U \in \alpha\mathcal{I}O(X)$. Since $A \subset U$, $A \subset U \cap cl^\star(int(cl^\star(A))) \subset U \cap cl^\star(A) = A$ and so $A = U \cap cl^\star(int(cl^\star(A)))$. Since $cl^\star(int(cl^\star(A)))$ is $\mathcal{I}R-$closed, $A \in \alpha\mathcal{A}_{\mathcal{I}}(X)$ and so $s\beta\mathcal{I}O(X) \cap \alpha_{\mathcal{I}}N_5(X) \subset \alpha\mathcal{A}_{\mathcal{I}}(X)$. Therefore, $\alpha\mathcal{A}_{\mathcal{I}}(X) = s\beta\mathcal{I}O(X) \cap \alpha_{\mathcal{I}}N_5(X)$.

If $A \in S\mathcal{I}O(X)$, then $A \in s\beta\mathcal{I}O(X)$ by Proposition 1(d) of [9]. Moreover, if $V = A \cup (X - cl^\star(int(A)))$, then $A = V \cap cl^\star(int(A))$. Also, $int(cl^\star(int(V))) = int(cl^\star(int(A \cup (X - cl^\star(int(A)))))) \supset int(cl^\star(int(A) \cup int(X - cl^\star(int(A))))) = int(cl^\star(int(A)) \cup cl^\star(int(X - cl^\star(int(A))))) \supset int(cl^\star(int(A)) \cup int(X - cl^\star(int(A)))) \supset int(int(cl^\star(int(A)) \cup (X - cl^\star(int(A))))) = int(X) = X \supset V$ and so V is $\alpha - \mathcal{I}-$open. Therefore, $A \in \alpha_{\mathcal{I}}N_5(X)$ and hence $S\mathcal{I}O(X) \subset s\beta\mathcal{I}O(X) \cap \alpha_{\mathcal{I}}N_5(X)$. Conversely, suppose $A \in s\beta\mathcal{I}O(X) \cap \alpha_{\mathcal{I}}N_5(X)$. $A \in \alpha_{\mathcal{I}}N_5(X)$ implies that $A = U \cap V$ where U is $\alpha - \mathcal{I}-$open and V is $\star-$closed. Since $A \in s\beta\mathcal{I}O(X)$, $A \subset cl^\star(int(cl^\star(A))) = cl^\star(int(cl^\star(U \cap V))) \subset cl^\star(int(cl^\star(int(cl^\star(int(U))) \cap V))) \subset$
$cl^\star(int(cl^\star(int(cl^\star(int(U)))) \cap V)) = cl^\star(int(cl^\star(int(U)) \cap V)) = cl^\star(int(cl^\star(int(U))) \cap int(V)) \subset cl^\star(int(cl^\star(int(U) \cap int(V)))) = cl^\star(int(cl^\star(int(U \cap V)))) = cl^\star(int(U \cap V)) = cl^\star(int(A))$. Therefore, $A \in S\mathcal{I}O(X)$ which implies that $s\beta\mathcal{I}O(X) \cap \alpha_{\mathcal{I}}N_5(X) \subset S\mathcal{I}O(X)$. Hence $s\beta\mathcal{I}O(X) \cap \alpha_{\mathcal{I}}N_5(X) = S\mathcal{I}O(X)$.

Corollary 2.15. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. Then the following are equivalent.*
(a) A is $\alpha - \mathcal{I}-$open.
(b) A is pre–$\mathcal{I}-$open and semi–$\mathcal{I}-$open [4, Proposition 1.1].
(c) A is a $\mathcal{B}_{2\mathcal{I}}-$set and $\alpha\mathcal{A}_{\mathcal{I}}-$set[4, Theorem 2.3].

Proof. (a) and (b) are equivalent by Proposition 1.1 of [4]. (b) and (c) are equivalent by Theorem 2.10 and Theorem 2.14.

Corollary 2.16. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. Then the following are equivalent.*
(a) A is open.
(b) A is $\alpha - \mathcal{I}-$open and $\mathcal{A}_{\mathcal{I}R} -$ set.
(c) A is pre–$\mathcal{I}-$open and $\mathcal{A}_{\mathcal{I}R}-$set.
(d) A is $\alpha - \mathcal{I}-$open and weakly $\mathcal{I}-$locally closed.
(e) A is $\alpha - \mathcal{I}-$open and $\mathcal{B}_{\mathcal{I}}-$set.
(f) A is $\alpha - \mathcal{I}-$open and $\mathcal{C}_{\mathcal{I}}-$set.

Proof. (a) and (b) are equivalent by Theorem 2.1 of [4].
That (b) implies (c) is clear.
(c) and (d) are equivalent by Proposition 2.2 of [4].
(d) implies (e) and (e) implies (f) are clear.
(f) implies (a) follows from Proposition 3.3 of [8].

A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be a $\mathcal{A}_{3\mathcal{I}}-$set (resp.$\mathcal{B}_{3\mathcal{I}}-$set [4]($\alpha_{\mathcal{I}}N_1-$set [1])) if $A = U \cap V$ where U is open (resp.$\alpha - \mathcal{I}-$open) and $cl^\star(int(V)) \subset V$. We will denote the family of all $\mathcal{A}_{3\mathcal{I}}-$sets (resp. $\mathcal{B}_{3\mathcal{I}}-$sets) by $\mathcal{A}_{3\mathcal{I}}(X)$ (resp.$\mathcal{B}_{3\mathcal{I}}(X)$). Clearly, $\mathcal{A}_{3\mathcal{I}}(X) \subset \mathcal{B}_{3\mathcal{I}}(X)$. The following Example 2.17 shows that the reverse direction is not true. Example 2.18 below shows that $A_{2\mathcal{I}}-$sets and $A_{3\mathcal{I}}-$sets are independent concepts. Theorem 2.19 below gives a characterization of $\mathcal{A}_{\mathcal{I}R}-$sets in terms of $\mathcal{A}_{3\mathcal{I}}-$sets.

Example 2.17. *Consider the ideal space $(X,\tau,\mathcal{I})$ of Example 2.12. If $A=\{a,b\}$, then $int(cl^\star(int(A)))$ $= int(cl^\star(int(\{a,b\}))) = int(cl^\star(\{a\})) = int(X) = X \supset A$ and so A is an $\alpha-\mathcal{I}-$open set and hence A is a $\mathcal{B}_{3\mathcal{I}}-$set. Since $cl^\star(int(A)) \nsubseteq A$ and X is the only open set containing A, A is not an $A_{3\mathcal{I}}-$set.*

Example 2.18. *(a) Consider the ideal space $(X,\tau,\mathcal{I})$ of Example 2.12. If $A=\{a,b\}$, then A is not an $A_{3\mathcal{I}}-$set. Since $cl^\star(A) = A\cup A^\star = \{a,b\}\cup X = X$, and so A is an $A_{2\mathcal{I}}-$set.*

(b) Consider the ideal space $(X,\tau,\mathcal{I})$ where $X=\{a,b,c,d\}$, $\tau=\{\emptyset,\{d\},\{a,c\},$ $\{a,c,d\},X\}$ and $\mathcal{I}=\{\emptyset,\{c\},\{d\},\{c,d\}\}$. If $A=\{a,b,c\}$, then $cl^\star(int(A)) = int(A)\cup(int(A))^\star =$ $\{a,c\}\cup\{a,b,c\} = \{a,b,c\} = A$ and so A is an $A_{3\mathcal{I}}-$set. Since $cl^\star(A) = A\cup A^\star = \{a,b,c\} \neq X$, A is not an $A_{2\mathcal{I}}-$set.

Theorem 2.19. *Let $(X,\tau,\mathcal{I})$ be an ideal space . Then $\mathcal{A}_{\mathcal{I}R}(X) = S\mathcal{I}O(X)\cap\mathcal{A}_{3\mathcal{I}}(X)$.*

Proof. *Suppose $A\in\mathcal{A}_{\mathcal{I}R}(X)$. Clearly, $A\in\mathcal{A}_{3\mathcal{I}}(X)$. By Theorem 3.3 of [1], $A\in S\mathcal{I}O(X)$. Therefore, $\mathcal{A}_{\mathcal{I}R}(X)\subset S\mathcal{I}O(X)\cap\mathcal{A}_{3\mathcal{I}}(X)$. Conversely, suppose $A\in S\mathcal{I}O(X)\cap\mathcal{A}_{3\mathcal{I}}(X)$. $A\in\mathcal{A}_{3\mathcal{I}}(X)$ implies that $A=U\cap V$ where U is open and $cl^\star(int(V))\subset V$. $A\in S\mathcal{I}O(X)$ implies that $A\subset cl^\star(int(A))$ and so $A = A\cap cl^\star(int(A)) = (U\cap V)\cap cl^\star(int(U\cap V)) \subset U\cap cl^\star(int(U\cap V)) = U\cap cl^\star(U\cap int(V)) \subset$ $U\cap cl^\star(U)\cap cl^\star(int(V)) \subset U\cap V = A$ and so $A = U\cap cl^\star(int(U\cap V)) = U\cap cl^\star(int(A))$. Since $cl^\star(int(A))$ is $\mathcal{I}R-$closed, $A\in\mathcal{A}_{\mathcal{I}R}(X)$. Therefore, $\mathcal{A}_{\mathcal{I}R}(X) = S\mathcal{I}O(X)\cap\mathcal{A}_{3\mathcal{I}}(X)$.*

Corollary 2.20. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subset X$. Then the following are equivalent.*
(a)$A\in\mathcal{A}_{\mathcal{I}R}(X)$.
(b)$A\in S\mathcal{I}O(X)\cap\mathcal{A}_{3\mathcal{I}}(X)$.
(c) $A\in\alpha\mathcal{A}_{\mathcal{I}}(X)\cap\mathcal{A}_{3\mathcal{I}}(X)$.
(d)$A\in s\beta\mathcal{I}O(X)\cap\alpha_{\mathcal{I}}N_5(X)\cap\mathcal{A}_{3\mathcal{I}}(X)$.
(e)$A\in s\beta\mathcal{I}O(X)\cap W\mathcal{I}LC(X)$.

Proof. *(a), (b), (c) and (d) are equivalent by Theorem 2.14 and Theorem 2.19.*
(a) and (e) are equivalent by Theorem 2.10 of [15].

By Remark 3.3 of [8], every $\mathcal{B}_{\mathcal{I}}-$set is a $\mathcal{C}_{\mathcal{I}}-$set but the reverse direction is not true. The following Theorem 2.22 gives characterizations of $\mathcal{B}_{\mathcal{I}}-$sets in terms of $\mathcal{C}_{\mathcal{I}}-$sets. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be an $\alpha\mathcal{B}_{\mathcal{I}}-$set ($\alpha_{\mathcal{I}}N_3-$set [1]) if $A=U\cap V$ where $U\in\alpha\mathcal{I}O(X)$ and V is a $t-\mathcal{I}-$set. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be an $\alpha\mathcal{C}_{\mathcal{I}}-$set [4]($\alpha_{\mathcal{I}}N_4-$set [1]) if $A=U\cap V$ where $U\in\alpha\mathcal{I}O(X)$ and V is a $\alpha^\star-\mathcal{I}-$set. Clearly every $\alpha\mathcal{B}_{\mathcal{I}}-$set is an $\alpha\mathcal{C}_{\mathcal{I}}-$set [1, Proposition 3.2(c)] but not the converse [1, Example 3.4]. We will denote the family of all $\alpha\mathcal{B}_{\mathcal{I}}-$sets (resp.$\alpha\mathcal{C}_{\mathcal{I}}-$sets) in $(X,\tau,\mathcal{I})$ by $\alpha\mathcal{B}_{\mathcal{I}}(X)$ (resp.$\alpha\mathcal{C}_{\mathcal{I}}(X)$). We define $\mathcal{D}_{\mathcal{I}}(X) = \{A\subset X \mid int(A) = p\mathcal{I}int(A)\}$ and if $A\in\mathcal{D}_{\mathcal{I}}$, then A is called a $\mathcal{D}_{\mathcal{I}}-$set. The following Lemma 2.21 characterizes $\alpha\mathcal{B}_{\mathcal{I}}-$sets and $\alpha\mathcal{C}_{\mathcal{I}}-$sets, the proof, which is similar to the proof of Theorem 2.7, is omitted. Corollary 2.23 follows from Theorem 2.22.

Lemma 2.21. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A be a subset of X. Then the following hold.*
(a) A is a $\alpha\mathcal{B}_{\mathcal{I}}-$set if and only if there exists an $\alpha-\mathcal{I}-$open set U such that $A=U\cap S\mathcal{I}cl(A)$.
(b) A is an $\alpha\mathcal{C}_{\mathcal{I}}-$set if and only if there exists an $\alpha-\mathcal{I}-$open set U such that $A=U\cap sp\mathcal{I}cl(A)$.

Theorem 2.22. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subset X$. Then the following are equivalent.*
(a) A is a $\mathcal{D}_{\mathcal{I}}-$set and a $\mathcal{C}_{\mathcal{I}}-$set.
(b) A is a $\delta-\mathcal{I}-$open set and a $\mathcal{C}_{\mathcal{I}}-$set.

(c) A is a $\mathcal{B}_{\mathcal{I}}-$set.
(d) A is an $\alpha\mathcal{B}_{\mathcal{I}}-$set and a $\mathcal{C}_{\mathcal{I}}-$set.

Proof. *(a)⇒(b). Suppose $A \in \mathcal{D}_{\mathcal{I}}(X) \cap \mathcal{C}_{\mathcal{I}}(X)$. If $A \in \mathcal{D}_{\mathcal{I}}(X)$, then $int(A) = p\mathcal{I}int(A)$. Now $int(cl^\star(A)) = cl^\star(A) \cap int(cl^\star(A)) \subset cl^\star(A \cap int(cl^\star(A))) = cl^\star$ $(p\mathcal{I}int(A)) = cl^\star(int(A))$ and so A is a $\delta - \mathcal{I}-$open set. This proves (b).*
(b)⇒(c). Suppose A is a $\delta - \mathcal{I}-$open set and a $\mathcal{C}_{\mathcal{I}}-$set. Then, by Theorem 2.4 of [12], $int(cl^\star(A)) = int(cl^\star(int(A)))$ and so $A \cup int(cl^\star(A)) = A \cup int(cl^\star(int(A)))$ which implies that $S\mathcal{I}cl(A) = sp\mathcal{I}cl(A)$. If A is a $\mathcal{C}_{\mathcal{I}}-$set, then Theorem 2.7, $A = U \cap sp\mathcal{I}cl(A)$ for some open set U and so $A = U \cap S\mathcal{I}cl(A)$ for some open set U which implies that A is a $\mathcal{B}_{\mathcal{I}}-$set.
(c)⇒(a). Clearly, every $\mathcal{B}_{\mathcal{I}}-$set is a $\mathcal{C}_{\mathcal{I}}-$set. If A is a $\mathcal{B}_{\mathcal{I}}-$set, then $A = U \cap V$ where U is open and $int(cl^\star(V)) = int(V)$. Now $p\mathcal{I}int(A) = A \cap int(cl^\star(A)) = A \cap int(cl^\star(U \cap V)) \subset A \cap int(cl^\star(U) \cap cl^\star(V)) = A \cap int(cl^\star(U)) \cap int(cl^\star(V)) = (U \cap V) \cap int(cl^\star(U)) \cap int(V) = U \cap int(V) = int(U \cap V) = int(A)$. But always, $int(A) \subset p\mathcal{I}int(A)$ and so $int(A) = p\mathcal{I}int(A)$ which implies that A is a $\mathcal{D}_{\mathcal{I}}-$set. This proves (a).
(c)⇒(d) is clear.
(d)⇒(c). If A is an $\alpha\mathcal{B}_{\mathcal{I}}-$set, then $A = U \cap V$ where U is $\alpha - \mathcal{I}-$open and $int(cl^\star(V)) = int(V)$. Now $A \subset U$ implies that $A \subset int(cl^\star(int(U)))$ and so $int(cl^\star(A)) \subset int(cl^\star(int(cl^\star(int(U))))) = int(cl^\star(int(U))) \subset int(cl^\star(U))$. Again,$A \subset V$ implies that $int(cl^\star(A)) \subset int(cl^\star(V)) = int(V)$. Therefore,
$int(cl^\star(A)) \subset int(cl^\star(U)) \cap int(V) \subset cl^\star(int(U) \cap int(V)) \subset cl^\star(int(U \cap V)) = cl^\star(int(A))$ and so $int(cl^\star(A)) = int(cl^\star(int(A)))$ which implies that $A \cup int(cl^\star(A))$ $= A \cup int(cl^\star(int(A)))$. Hence $S\mathcal{I}cl(A) = sp\mathcal{I}cl(A)$. Since A is a $\mathcal{C}_{\mathcal{I}}-$set, by Theorem 2.7, $A = G \cap sp\mathcal{I}cl(A)$ for some open set G and so $A = G \cap S\mathcal{I}cl(A)$. Therefore, A is a $\mathcal{B}_{\mathcal{I}}-$set.

Corollary 2.23. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following hold.*
(a) Every $\mathcal{B}_{\mathcal{I}}-$set is a $\mathcal{D}_{\mathcal{I}}-$set.
(b) Every $\mathcal{B}_{\mathcal{I}}-$set is a $\alpha\mathcal{B}_{\mathcal{I}}-$set.
(c) Every $\mathcal{D}_{\mathcal{I}}-$set is a $\delta - \mathcal{I}-$open set (Proof follows from (a)⇒(b) of Theorem 2.22).

The following Theorem 2.24 characterizes $\alpha\mathcal{B}_{\mathcal{I}}-$open sets in terms of $\delta - \mathcal{I}-$open sets and $\alpha\mathcal{C}_{\mathcal{I}}-$open sets. Example 2.25 below shows that $\delta - \mathcal{I}-$openness and $\alpha\mathcal{C}_{\mathcal{I}}-$openness are independent concepts.

Theorem 2.24. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\alpha\mathcal{B}_{\mathcal{I}}(X) = \delta\mathcal{I}O(X) \cap \alpha\mathcal{C}_{\mathcal{I}}(X)$.*

Proof. *Clearly, $\alpha\mathcal{B}_{\mathcal{I}}(X) \subset \alpha\mathcal{C}_{\mathcal{I}}(X)$. If $A \in \alpha\mathcal{B}_{\mathcal{I}}(X)$, then $A = U \cap V$ where U is $\alpha - \mathcal{I}-$open and V is a $t - \mathcal{I}-$set. $A \subset U$ implies that $int(cl^\star(A)) \subset int(cl^\star(U)) \subset int(cl^\star(int(cl^\star(int(U))))) \subset int(cl^\star(int(U))) \subset cl^\star(int(U))$. Also, $A \subset V$implies that $int(cl^\star(A)) \subset int(cl^\star(V)) = int(V)$ and so $int(cl^\star(A)) \subset cl^\star(int(U)) \cap int(V) \subset cl^\star(int(U) \cap int(V)) = cl^\star(int(U \cap V)) = cl^\star(int(A)$. Therefore, $A \in \delta\mathcal{I}O(X)$. Hence $\alpha\mathcal{B}_{\mathcal{I}}(X) \subset \delta\mathcal{I}O(X) \cap \alpha\mathcal{C}_{\mathcal{I}}(X)$. Conversely, suppose $A \in \delta\mathcal{I}O(X) \cap \alpha\mathcal{C}_{\mathcal{I}}(X)$. $A \in \delta\mathcal{I}O(X)$ implies that $int(cl^\star(A)) = int(cl^\star(int(A)))$ and so $S\mathcal{I}cl(A) = sp\mathcal{I}cl(A)$. $A \in \alpha\mathcal{C}_{\mathcal{I}}(X)$ implies that $A = U \cap sp\mathcal{I}cl(A)$ for some $\alpha - \mathcal{I}-$open set U by Lemma 2.21 and so $A = U \cap S\mathcal{I}cl(A)$ for some $\alpha - \mathcal{I}-$open set U which implies that $A \in \alpha\mathcal{B}_{\mathcal{I}}(X)$. Therefore, $\delta\mathcal{I}O(X) \cap \alpha\mathcal{C}_{\mathcal{I}}(X) \subset \alpha\mathcal{B}_{\mathcal{I}}(X)$. This completes the proof.*

Example 2.25. *(a) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, b\}, \{a, b, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\ \}$. If $A = \{a, c\}$, then $int(cl^\star(int(A))) = int(cl^\star(int(\{a, c\}))) = int(cl^\star(\emptyset)) = \emptyset = int(A)$. Therefore, A is*

an $\alpha^\star-\mathcal{I}-$set and hence an $\alpha\mathcal{C}_\mathcal{I}-$set. But $int(cl^\star(A)) = int(\{a,b,c\}) = \{a,b\}$ and $cl^\star(int(A)) = cl^\star(\emptyset) = \emptyset$ and so A is not a $\delta-\mathcal{I}-$set.
(b) Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,\{a\},\{c\},\{a,c\},X\}$ and $\mathcal{I}=\{\emptyset,\{a\}\ \}$. If $A=\{a,b,c\}$, then A is neither open nor an $\alpha^\star-\mathcal{I}-$set and so A is not an $\alpha\mathcal{C}_\mathcal{I}-$set. But $int(cl^\star(A)) = int(\{a,b,c,d\}) = X$ and $cl^\star(int(A)) = cl^\star(\{a,c\}) = X$ and so A is a $\delta-\mathcal{I}-$set.

An ideal space $(X,\tau,\mathcal{I})$ is said to be $\star-$extremally disconnected [7] if the $\tau^\star-$closure ($\star-$closure) of every open set is open. Clearly, $\mathcal{B}_{3\mathcal{I}}(X)\subset\alpha\mathcal{C}_\mathcal{I}(X)$. By Example 3.6 of [1] the reverse direction is not true. The following Theorem 2.26 shows that for $\star-$extremally disconnected spaces, the two collection of sets coincide. Example 2.27 below shows that $\alpha C_\mathcal{I}(X)=B_{3\mathcal{I}}(X)$ does not imply that the space is $\star-$extremally disconnected.

Theorem 2.26. *Let $(X,\tau,\mathcal{I})$ be a $\star-$extremally disconnected ideal space. Then $\mathcal{B}_{3\mathcal{I}}(X)=\alpha\mathcal{C}_\mathcal{I}(X)$.*

Proof. *Enough to prove that $\alpha\mathcal{C}_\mathcal{I}(X)\subset\mathcal{B}_{3\mathcal{I}}(X)$. Suppose $A\in\alpha\mathcal{C}_\mathcal{I}(X)$. Then $A=U\cap V$ where U is $\alpha-\mathcal{I}-$open and $int(cl^\star(int(V)))=int(V)$. Since $(X,\tau,\mathcal{I})$ is $\star-$extremally disconnected, $cl^\star(int(V))$ is open and so $int(V)=int(cl^\star(int(V)))=cl^\star(int(V))$. Therefore, $A\in\mathcal{B}_{3\mathcal{I}}(X)$. This completes the proof.*

Example 2.27. *Consider the ideal space $(X,\tau,\mathcal{I})$ where $X=\{a,b,c\}$, $\tau=\{\emptyset,\{b\},\{c\},\{b,c\},X\}$ and $\mathcal{I}=\{\emptyset,\{a\}\}$. If $A=\{b\}$, A is open and $cl^\star(A)=\{b\}\cup\{a,b\}=\{a,b\}$, which is not open. Hence $(X,\tau,\mathcal{I})$ is not $\star-$extremally disconnected but $\wp(X)=\alpha C_\mathcal{I}(X)=B_{3\mathcal{I}}(X)$.*

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