CUBO A Mathematical Journal
Vol.12, N°03, (83–97). October 2010

The Semigroup and the Inverse of the Laplacian on the Heisenberg Group[1]

APARAJITA DASGUPTA
Department of Mathematics, Indian Institute of Science,
Bangalore–560012, India
email: adgupta@math.iisc.ernet.in

AND

M.W. WONG
Department of Mathematics and Statistics, York University,
4700 Keele Street, Toronto, Ontario M3J 1P3, Canada
email: mwwong@mathstat.yorku.ca

ABSTRACT

By decomposing the Laplacian on the Heisenberg group into a family of parametrized partial differential operators $\tilde{L}_\tau, \tau \in \mathbb{R} \setminus \{0\}$, and using parametrized Fourier-Wigner transforms, we give formulas and estimates for the strongly continuous one-parameter semigroup generated by $\tilde{L}_\tau$, and the inverse of $\tilde{L}_\tau$. Using these formulas and estimates, we obtain Sobolev estimates for the one-parameter semigroup and the inverse of the Laplacian.

[1]This research has been supported by the Natural Sciences and Engineering Research Council of Canada.

RESUMEN

Mediante descomposición del Laplaceano sobre el grupo de Heisenberg en una familia de operadores diferenciales parciales parametrizados $\tilde{L}_\tau, \tau \in \mathbb{R}\setminus\{0\}$, y usando transformada de Fourier-Wigner parametrizada, damos fórmulas y estimativas para la continuidad fuerte del semigrupo generado por $\tilde{L}_\tau$, y la inversa de $\tilde{L}_\tau$. Usando esas fórmulas y estimativas obtenemos estimativas de Sobolev para el semigrupo a un parámetro y la inversa del Laplaceano.

Key words and phrases: *Heisenberg group, Laplacian, parametrized partial differential operators, Hermite functions, Fourier-Wigner transforms, heat equation, one parameter semigroup, inverse of Laplacian, Sobolev spaces.*

Math. Subj. Class.: *47F05, 47G30, 35J70.*

1 The Laplacian on the Heisenberg Group

If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ via

$$\mathbb{R}^2 \ni (x,y) \leftrightarrow z = x+iy \in \mathbb{C}$$

and let

$$\mathbb{H} = \mathbb{C}\times\mathbb{R},$$

then $\mathbb{H}$ becomes a non-commutative group when equipped with the multiplication $\cdot$ given by

$$(z,t)\cdot(w,s) = \left(z+w, t+s+\frac{1}{4}[z,w]\right), \quad (z,t),(w,s)\in\mathbb{H},$$

where $[z,w]$ is the symplectic form of z and w defined by

$$[z,w] = 2\,\mathrm{Im}(z\overline{w}).$$

In fact, $\mathbb{H}$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dz\,dt$.

Let $\mathfrak{h}$ be the Lie algebra of left-invariant vector fields on $\mathbb{H}$. A basis for $\mathfrak{h}$ is then given by X, Y and T, where

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t},$$

$$Y = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial t},$$

and

$$T = \frac{\partial}{\partial t}.$$

The Laplacian $\Delta_{\mathbb{H}}$ on $\mathbb{H}$ is defined by

$$\Delta_{\mathbb{H}} = -(X^2 + Y^2 + T^2).$$

A simple computation gives

$$\Delta_{\mathbb{H}} = -\Delta - \frac{1}{4}(x^2+y^2)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2},$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Let g be the Riemannian metric on $\mathbb{R}^3$ given by

$$g(x,y,t) = \begin{bmatrix} 1 & 0 & y/2 \\ 0 & 1 & -x/2 \\ y/2 & -x/2 & \frac{1}{4}(x^2+y^2) \end{bmatrix}$$

for all $(x,y,t) \in \mathbb{R}^3$. Then $\Delta_{\mathbb{H}}$ is also given by

$$-\Delta_{\mathbb{H}} = \frac{1}{\sqrt{\det g}} \sum_{1 \le j,k \le 3} \partial_j(\sqrt{\det g}\, g_{j,k} \partial_k),$$

where $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$, $\partial_3 = \partial/\partial t$. Since the symbol $\sigma(\Delta_{\mathbb{H}})$ of $\Delta_{\mathbb{H}}$ is given by

$$\sigma(\Delta_{\mathbb{H}})(x,y,t;\xi,\eta,\tau) = \left(\xi + \frac{1}{2}y\tau\right)^2 + \left(\eta - \frac{1}{2}x\tau\right)^2 + \tau^2$$

for all (x,y,t) and (ξ,η,τ) in $\mathbb{R}^3$, it is easy to see that $\Delta_{\mathbb{H}}$ is an elliptic partial differential operator on $\mathbb{R}^3$ but not globally elliptic in the sense of Shubin [11]. Let us recall that $\Delta_{\mathbb{H}}$ is globally elliptic if there exist positive constants C and R such that

$$|\sigma(\Delta_{\mathbb{H}})(x,y,t;\xi,\eta,\tau)| \ge C\big(1 + |x| + |y| + |t| + |\xi| + |\eta| + |\tau|\big)^2$$

whenever

$$|x| + |y| + |t| + |\xi| + |\eta| + |\tau| \ge R.$$

The aim of this paper is to give new estimates for the strongly continuous one-parameter semigroup $e^{-u\Delta_{\mathbb{H}}}$, $u > 0$, generated by $\Delta_{\mathbb{H}}$ and the inverse $\Delta_{\mathbb{H}}^{-1}$ of $\Delta_{\mathbb{H}}$. More precisely, we use the Sobolev spaces $L^2_s(\mathbb{H})$, $s \in \mathbb{R}$, as in [1, 2] to estimate $\|e^{-u\Delta_{\mathbb{H}}} f\|_{L^2_s(\mathbb{H})}$, $u > 0$, in terms of $\|f\|_{L^2(\mathbb{H})}$ for all f in $L^2(\mathbb{H})$, and to give an estimate for $\|e^{-u\Delta_{\mathbb{H}}} f\|_{L^2(\mathbb{H})}$ in terms of $\|f\|_{L^2_s(\mathbb{H})}$. These Sobolev spaces are also used to estimate $||\Delta_{\mathbb{H}}^{-1} f||_{L^2_{s+2}(\mathbb{H})}$ in terms of $||f||_{L^2_s(\mathbb{H})}$ for all f in $L^2_s(\mathbb{H})$.

The function F on $\mathbb{H}\times(0,\infty)$ given by

$$F(z,t,u)=(e^{-u\Delta_{\mathbb{H}}}f)(z,t),\quad (z,t)\in\mathbb{H},\, u>0,$$

is in fact the solution of the initial value problem

$$\begin{cases} \frac{\partial F}{\partial u}(z,t,u)=-(\Delta_{\mathbb{H}}F)(z,t,u), & (z,t)\in\mathbb{H},\, u>0,\\ F(z,t,0)=f(z,t), & (z,t)\in\mathbb{H}, \end{cases}$$

for the Laplacian $\Delta_{\mathbb{H}}$.

Using the same techniques as in [1], we get for all $f\in L^2(\mathbb{H})$ and $u>0$,

$$(e^{-u\Delta_{\mathbb{H}}}f)(z,t)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-it\tau}(e^{-u\tilde{L}_\tau}f^\tau)(z)\,d\tau,\quad (z,t)\in\mathbb{H}, \tag{1.1}$$

where $\tilde{L}_\tau$, $\tau\in\mathbb{R}\setminus\{0\}$, is given by

$$\tilde{L}_\tau=-\Delta+\frac{1}{4}(x^2+y^2)\tau^2-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\tau+\tau^2$$

and f^τ is the function on $\mathbb{C}$ given by

$$f^\tau(z)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{it\tau}f(z,t)\,dt,\quad z\in\mathbb{C},$$

provided that the integral exists. In fact, $f^\tau(z)$ is the inverse Fourier transform of $f(z,t)$ with respect to t evaluated at τ. In this paper, the nonzero parameter τ can be looked at as Planck's constant.

To obtain the estimates in this paper, we use formulas for $e^{-u\tilde{L}_\tau}$ and $\tilde{L}_\tau^{-1}$ in terms of the τ-Weyl transforms and the τ-Fourier–Wigner transforms of Hermite functions, $\tau\in\mathbb{R}\setminus\{0\}$, which we recall in, respectively, Section 2 and Section 3. The L^2-boundedness and the Hilbert–Schimdt property of τ-Weyl transforms are instrumental in obtaining the estimates.

Basic information on the classical Fourier–Wigner transforms, Wigner transforms and Weyl transforms can be found in [13] among others.

In Section 2, we introduce the τ-Weyl transforms and prove results on the L^2-boundedness and the Hilbert–Schmidt property of the τ-Weyl transforms. The τ-Fourier–Wigner transforms of Hermite functions are recalled in Section 3. A formula for $e^{-u\tilde{L}_\tau}f$, $u>0$, for every function f in $L^2(\mathbb{C})$ and an estimate for $\|e^{-u\tilde{L}_\tau}f\|_{L^2(\mathbb{C})}$, $u>0$, in terms of $\|f\|_{L^p(\mathbb{C})}$, $1\le p\le 2$, are given in Section 4. This formula gives a formula for $e^{-u\Delta_{\mathbb{H}}}$, $u>0$, immediately using the inverse Fourier transform as indicated by (1.1). In Section 5, we use the family $L^2_s(\mathbb{H})$, $s\in\mathbb{R}$, of Sobolev spaces with respect to the center of the Heisenberg group as in [1, 2] to obtain Sobolev estimates for $e^{-u\Delta_{\mathbb{H}}}f$, $u>0$, in terms of $\|f\|_{L^2(\mathbb{H})}$, and Sobolev estimates for

$\|e^{-u\Delta_{\mathbb{H}}}f\|_{L^2(\mathbb{H})}$, $u > 0$, in terms of the Sobolev norms $\|f\|_{L^2_s(\mathbb{H})}$ of f in $L^2_s(\mathbb{H})$. In Section 6, we obtain a formula for $\tilde{L}_\tau^{-1}$ and estimates for $\tilde{L}_\tau^{-1}$ which are then used to estimate $\Delta_{\mathbb{H}}^{-1}$. In Section 7, estimates for $\|\Delta_{\mathbb{H}}^{-1}f\|_{L^2_{s+2}(\mathbb{H})}$ in terms of $\|f\|_{L^2_s(\mathbb{H})}$ for all f in $L^2_s(\mathbb{H})$ are given.

We end this section by putting in perspectives the results in this paper. While the semigroup and the inverse can be studied in the framework of functional analysis as explained in [3, 4, 5, 8, 9, 16], the results and methods in this paper are based on explicit formulas in *hard* analysis and are related to the works in [1, 2, 6, 7, 10, 12, 14, 15].

2 τ-Weyl Transforms

Let f and g be functions in $L^2(\mathbb{R})$. Then for τ in $\mathbb{R}\setminus\{0\}$, the τ-Fourier–Wigner transform $V_\tau(f,g)$ is defined by

$$V_\tau(f,g)(q,p) = (2\pi)^{-1/2}|\tau|^{1/2}\int_{-\infty}^{\infty} e^{i\tau qy} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}\,dy$$

for all q and p in $\mathbb{R}$. In fact,

$$V_\tau(f,g)(q,p) = |\tau|^{1/2}V(f,g)(\tau q,p), \quad q,p\in\mathbb{R},$$

where $V(f,g)$ is the classical Fourier–Wigner transform of f and g. A proof can be found in [1].

It can be proved that $V_\tau(f,g)$ is a function in $L^2(\mathbb{C})$ and we have the Moyal identity stating that

$$\|V_\tau(f,g)\|_{L^2(\mathbb{C})} = \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}, \quad \tau\in\mathbb{R}\setminus\{0\}. \tag{2.1}$$

We define the τ-Wigner transform $W_\tau(f,g)$ of f and g by

$$W_\tau(f,g) = V_\tau(f,g)^\wedge. \tag{2.2}$$

Then we have the following connection of the τ-Wigner transform with the usual Wigner transform.

Theorem 2.1. *Let $\tau\in\mathbb{R}\setminus\{0\}$. Then for all functions f and g in $L^2(\mathbb{R})$,*

$$W_\tau(f,g)(x,\xi) = |\tau|^{-1/2}W(f,g)(x/\tau,\xi), \quad x,\xi\in\mathbb{R},$$

where $W(f,g)$ is the classical Wigner transform of f and g.

It is obvious that

$$W_\tau(f,g) = \overline{W_\tau(g,f)}, \quad f,g\in L^2(\mathbb{R}). \tag{2.3}$$

Let $\sigma \in L^p(\mathbb{C})$, $1 \le p \le \infty$. Then for all τ in $\mathbb{R}\setminus\{0\}$ and all functions f in the Schwartz space $\mathscr{S}(\mathbb{R})$ on $\mathbb{R}$, we define $W_\sigma^\tau f$ to be the tempered distribution on $\mathbb{R}$ by

$$(W_\sigma^\tau f, g) = (2\pi)^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(x,\xi) W_\tau(f,g)(x,\xi)\,dx\,d\xi \tag{2.4}$$

for all g in $\mathscr{S}(\mathbb{R})$, where (F,G) is defined by

$$(F,G) = \int_{\mathbb{R}^n} F(z)\overline{G(z)}\,dz$$

for all measurable functions F and G on $\mathbb{R}^n$, provided that the integral exists. We call W_σ^τ the τ-Weyl transform associated to the symbol σ. It is easy to see that if σ is a symbol in the Schwartz space $\mathscr{S}(\mathbb{C})$ on $\mathbb{C}$, then $W_\sigma^\tau f$ is a function in $\mathscr{S}(\mathbb{R})$ for all f in $\mathscr{S}(\mathbb{R})$.

We have the following estimate for the norm of the Weyl transform $W_{\hat{\sigma}}^\tau$ in terms of the L^p norm of the symbol σ when $\sigma \in L^p(\mathbb{C})$, $1 \le p \le 2$.

Theorem 2.2. *Let $\sigma \in L^p(\mathbb{C})$, $1 \le p \le 2$. Then $W_{\hat{\sigma}}^\tau : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and*

$$\|W_{\hat{\sigma}}^\tau\|_* \le (2\pi)^{-1/p} |\tau|^{-(1/2)+(1/p)} \|\sigma\|_{L^p(\mathbb{C})},$$

where $\|W_{\hat{\sigma}}^\tau\|_*$ is the operator norm of $W_{\hat{\sigma}}^\tau : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Proof Let f and g be functions in $\mathscr{S}(\mathbb{R})$. Then

$$\begin{aligned}
(W_{\hat{\sigma}}^\tau f, g) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\sigma}(x,\xi) W_\tau(f,g)(x,\xi)\,dx\,d\xi \\
&= (2\pi)^{-1} |\tau|^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\sigma}(x,\xi) W(f,g)(x/\tau,\xi)\,dx\,d\xi \\
&= (2\pi)^{-1} |\tau|^{1/2} \int_{-\infty}^{\infty}\int_{\infty}^{\infty} \hat{\sigma}(\tau x,\xi) W(f,g)(x,\xi)\,dx\,d\xi.
\end{aligned}$$

But

$$\hat{\sigma}(\tau x,\xi) = |\tau|^{-1}\widehat{\sigma_{1/\tau}}(x,\xi), \quad x,\xi \in \mathbb{R},$$

where $\sigma_{1/\tau}$ is the dilation of σ with respect to the first variable by the amount $1/\tau$. More precisely,

$$\sigma_{1/\tau}(q,p) = \sigma(q/\tau, p), \quad q,p \in \mathbb{R}.$$

So,

$$\begin{aligned}
(W_{\hat{\sigma}}^\tau f, g) &= (2\pi)^{-1/2} |\tau|^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{\sigma_{1/\tau}}(x,\xi) W(f,g)(x,\xi)\,dx\,d\xi \\
&= |\tau|^{-1/2} (W_{\widehat{\sigma_{1/\tau}}} f, g),
\end{aligned}$$

where $W_{\widehat{\sigma_{1/\tau}}}$ is the classical Weyl transform with symbol $\widehat{\sigma_{1/\tau}}$. Thus, it follows from Theorem 21.1 in [14] that $W_{\hat{\sigma}}^{\tau} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and

$$\|W_{\hat{\sigma}}^{\tau}\|_* \le |\tau|^{-1/2}(2\pi)^{-1/p}\|\sigma_{1/\tau}\|_{L^p(\mathbb{C})} = (2\pi)^{-1/p}|\tau|^{-(1/2)+(1/p)}\|\sigma\|_{L^p(\mathbb{C})}.$$

□

We have the following result for the Hilbert–Schmidt norm of the Weyl transform $W_{\hat{\sigma}}^{\tau}$ in terms of the L^2 norm of the symbol σ when $\sigma \in L^2(\mathbb{C})$.

Theorem 2.3. *Let $\sigma \in L^2(\mathbb{C})$. Then $W_{\hat{\sigma}}^{\tau} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Hilbert–Schmidt operator and*

$$\|W_{\hat{\sigma}}^{\tau}\|_{HS} = (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{C})},$$

where $\|W_{\hat{\sigma}}^{\tau}\|_{HS}$ is the Hilbert–Schmidt norm of $W_{\hat{\sigma}}^{\tau} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Proof Let f and g be functions in $\mathscr{S}(\mathbb{R})$. Then

$$\begin{aligned}
(W_{\hat{\sigma}}^{\tau} f, g) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\sigma}(x,\xi)W_{\tau}(f,g)(x,\xi)\,dx\,d\xi \\
&= (2\pi)^{-1/2}|\tau|^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\sigma}(x,\xi)W(f,g)(x/\tau,\xi)\,dx\,d\xi \\
&= (2\pi)^{-1/2}|\tau|^{1/2}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\hat{\sigma}(\tau x,\xi)W(f,g)(x,\xi)\,dx\,d\xi.
\end{aligned}$$

But

$$\hat{\sigma}(\tau x,\xi) = |\tau|^{-1/2}\widehat{\sigma_{1/\tau}}(x,\xi), \quad x,\xi \in \mathbb{R},$$

where $\sigma_{1/\tau}$ is the dilation of σ with respect to the first variable by the amount $1/\tau$, *i.e.*,

$$\sigma_{1/\tau}(q,p) = \sigma(q/\tau,p), \quad q,p \in \mathbb{R}.$$

So,

$$\begin{aligned}
(W_{\hat{\sigma}}^{\tau} f, g) &= (2\pi)^{-1}|\tau|^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{\sigma_{1/\tau}}(x,\xi)W(f,g)(x,\xi)\,dx\,d\xi \\
&= |\tau|^{-1/2}(W_{\widehat{\sigma_{1/\tau}}} f, g),
\end{aligned}$$

where $W_{\widehat{\sigma_{1/\tau}}}$ is the classical Weyl transform with symbol $\widehat{\sigma_{1/\tau}}$. Thus, it follows from Theorem 7.5 in [13] that $W_{\hat{\sigma}}^{\tau} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Hilbert–Schmidt operator and

$$\begin{aligned}
\|W_{\hat{\sigma}}^{\tau}\|_{HS} &= |\tau|^{-1/2}\|W_{\widehat{\sigma_{1/\tau}}}\|_{HS} \\
&= (2\pi)^{-1/2}|\tau|^{-1/2}\|\sigma_{1/\tau}\|_{L^2(\mathbb{C})} \\
&= (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{C})}.
\end{aligned}$$

□

3 Fourier–Wigner Transforms of Hermite Functions

For $\tau \in \mathbb{R}\setminus\{0\}$ and for $k = 0,1,2,\dots$, we define e_k^τ to be the function on $\mathbb{R}$ by

$$e_k^\tau(x) = |\tau|^{1/4} e_k(\sqrt{|\tau|}x), \quad x \in \mathbb{R}.$$

Here, e_k is the Hermite function of order k defined by

$$e_k(x) = \frac{1}{(2^k k!\sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R},$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2/2} \left(\frac{d}{dx}\right)^k (e^{-x^2}), \quad x \in \mathbb{R}.$$

For $j,k = 0,1,2,\dots$, we define $e_{j,k}^\tau$ on $\mathbb{R}^2$ by

$$e_{j,k}^\tau = V_\tau(e_j^\tau, e_k^\tau).$$

The following theorem gives the connection of $\{e_{j,k}^\tau : j,k = 0,1,2,\dots\}$ with $\{e_{j,k} : j,k = 0,1,2,\dots\}$, where

$$e_{j,k} = V(e_j, e_k), \quad j,k = 0,1,2,\dots.$$

A proof can be found in [1].

Theorem 3.1. *For $\tau \in \mathbb{R}\setminus\{0\}$ and for $j,k = 0,1,2,\dots$,*

$$e_{j,k}^\tau(q,p) = |\tau|^{1/2} e_{j,k}\left(\frac{\tau}{\sqrt{|\tau|}}q, \sqrt{|\tau|}p\right), \quad q,p \in \mathbb{R}.$$

Theorem 3.2. *$\{e_{j,k}^\tau : j,k = 0,1,2,\dots\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$.*

Theorem 3.2 follows from Theorem 3.1 and Theorem 21.2 in [13] to the effect that $\{e_{j,k} : j,k = 0,1,2,\dots\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

Theorem 3.3. *For $j,k = 0,1,2,\dots$,*

$$\tilde{L}_\tau e_{j,k}^\tau = (2k+1+|\tau|)|\tau| e_{j,k}^\tau.$$

Theorem 3.3 can be proved using Theorem 3.1, Theorem 3.3 in [2] and Theorem 22.2 in [13] telling us that for $j,k = 0,1,2,\dots$, $e_{j,k}$ is an eigenfunction of L_1 corresponding to the eigenvalue $2k+1$ and the fact that, $\tilde{L}_\tau = L_\tau + \tau^2$.

4 A Formula and an Estimate for $e^{-u\tilde{L}_\tau}$, $u > 0$

Let $\tau \in \mathbb{R}\setminus\{0\}$. Then a formula for $e^{-u\tilde{L}_\tau}$, $u > 0$, is given by the following theorem.

Theorem 4.1. *Let $f \in L^2(\mathbb{C})$. Then for $u > 0$,*

$$e^{-u\tilde{L}_\tau} f = (2\pi)^{1/2} \sum_{k=0}^{\infty} e^{-(2k+1+|\tau|)|\tau|u} V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau),$$

where the convergence of the series is understood to be in $L^2(\mathbb{C})$.

Proof Let $f \in L^2(\mathbb{C})$. Then from Theorem 3.3 we have for $u > 0$

$$e^{-u\tilde{L}_\tau} f = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} e^{-(2k+1+|\tau|)|\tau|u}(f, e_{j,k}^\tau)e_{j,k}^\tau = e^{-|\tau|^2 u} e^{-uL_\tau} f, \tag{4.1}$$

where the series is convergent in $L^2(\mathbb{C})$. Now, using the formula for $e^{-uL_\tau} f$ in [2] and (4.1), we get

$$e^{-u\tilde{L}_\tau} f = (2\pi)^{1/2} \sum_{k=0}^{\infty} e^{-(2k+1+|\tau|)|\tau|u} V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau)$$

for all f in $L^2(\mathbb{C})$ and $u > 0$. □

For all τ in $\mathbb{R}\setminus\{0\}$, we have the following estimate for the L^2 norm of $e^{-u\tilde{L}_\tau} f$, $u > 0$, in terms of the L^p norm of f.

Theorem 4.2. *Let $\tau \in \mathbb{R}\setminus\{0\}$. Then for all functions f in $L^p(\mathbb{C})$, $1 \le p \le 2$,*

$$\|e^{-u\tilde{L}_\tau} f\|_{L^2(\mathbb{C})} \le (2\pi)^{-(1/p)+(1/2)}|\tau|^{-(1/2)+(1/p)} e^{-\tau^2 u} \frac{1}{2\sinh(|\tau|u)} \|f\|_{L^p(\mathbb{C})}.$$

Proof By Theorem 4.1, the Moyal identity (2.1) and the fact that

$$\|e_k^\tau\|_{L^2(\mathbb{R})} = 1, \quad k = 0, 1, 2, \ldots,$$

we get

$$\|e^{-u\tilde{L}_\tau} f\|_{L^2(\mathbb{C})} \le (2\pi)^{1/2} e^{-(|\tau|+|\tau|^2)u} \sum_{k=0}^{\infty} e^{-2k|\tau|u} \|W_{\hat{f}}^\tau e_k^\tau\|_{L^2(\mathbb{R})}, \quad u > 0. \tag{4.2}$$

Applying Theorem 2.2 to (4.2), we get

$$\begin{aligned} &\|e^{-u\tilde{L}_\tau} f\|_{L^2(\mathbb{C})} \\ \le\ & (2\pi)^{-(1/p)+(1/2)}|\tau|^{-(1/2)+(1/p)} e^{-(|\tau|+|\tau|^2)u} \left(\sum_{k=0}^{\infty} e^{-2k|\tau|u}\right) \|f\|_{L^p(\mathbb{C})} \\ =\ & (2\pi)^{-(1/p)+(1/2)}|\tau|^{-(1/2)+(1/p)} e^{-|\tau|^2 u} \frac{1}{2\sinh(|\tau|u)} \|f\|_{L^p(\mathbb{C})}, \end{aligned}$$

as asserted.

□

5 Sobolev Estimates for $e^{-\Delta_{\mathbb{H}}}$, $u > 0$

Let $s \in \mathbb{R}$. Then we define $L^2_s(\mathbb{H})$ to be the set of all tempered distributions f in $\mathscr{S}'(\mathbb{H})$ such that $f^\tau(z)$ is a measurable function and

$$\int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} |f^\tau(z)|^2 d\tau\, dz < \infty.$$

For every f in $L^2_s(\mathbb{H})$, we define the norm $\|f\|_{L^2_s(\mathbb{H})}$ by

$$\|f\|^2_{L^2_s(\mathbb{H})} = \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} |f^\tau(z)|^2 d\tau\, dz.$$

Then it can be shown easily that $L^2_s(\mathbb{H})$ is an inner product space in which the inner product $(\,,\,)_{L^2_s(\mathbb{H})}$ is given by

$$(f,g)_{L^2_s(\mathbb{H})} = \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} f^\tau(z) \overline{g^\tau(z)}\, d\tau\, dz$$

for all f and g in $L^2_s(\mathbb{H})$.

Theorem 5.1. *Let $s \ge 1$. Then for $u > 0$, $e^{-u\Delta_{\mathbb{H}}} : L^2(\mathbb{H}) \to L^2_s(\mathbb{H})$ is a bounded linear operator and*

$$\|e^{-u\Delta_{\mathbb{H}}} f\|_{L^2_s(\mathbb{H})} \le \frac{c_s}{2u^s} \|f\|_{L^2(\mathbb{H})}, \quad f \in L^2(\mathbb{H}),$$

where

$$c_s = \sup_{\tau \in \mathbb{R}\setminus\{0\}} (|\tau|^s / \sinh|\tau|).$$

Proof Let $u > 0$ and $f \in L^2(\mathbb{H})$. Then by (1.1), Fubini's theorem, Plancherel's theorem and Theorem 4.2 with $p = 2$,

$$\begin{aligned}
\|e^{-u\Delta_{\mathbb{H}}} f\|^2_{L^2_s(\mathbb{H})} &= \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} |(e^{-u\Delta_{\mathbb{H}}} f)^\tau(z)|^2 d\tau\, dz \\
&= \int_{-\infty}^{\infty} |\tau|^{2s} \left(\int_{\mathbb{C}} |(e^{-u\Delta_{\mathbb{H}}} f)^\tau(z)|^2 dz \right) d\tau \\
&= \int_{-\infty}^{\infty} |\tau|^{2s} \left(\int_{\mathbb{C}} |(e^{-u\tilde{L}_\tau} f^\tau)(z)|^2 dz \right) d\tau \\
&= \int_{-\infty}^{\infty} |\tau|^{2s} \|e^{-u\tilde{L}_\tau} f^\tau\|^2_{L^2(\mathbb{C})} d\tau \\
&\le \frac{1}{4} \left(\int_{-\infty}^{\infty} \frac{e^{-2\tau^2 u} |\tau|^{2s}}{\sinh^2(|\tau| u)} \|f^\tau\|^2_{L^2(\mathbb{C})} d\tau \right) \\
&\le \frac{1}{4} \int_{-\infty}^{\infty} \frac{|\tau|^{2s}}{\sinh^2(|\tau| u)} \left(\int_{\mathbb{C}} |f^\tau(z)|^2 dz \right) d\tau
\end{aligned}$$

$$= \frac{1}{4u^{2s+1}} \int_{-\infty}^{\infty} \frac{|\tau|^{2s}}{\sinh^2(|\tau|u)} \left(\int_{\mathbb{C}} |\check{f}(z,\tau/u)|^2 dz \right) d\tau,$$

where $\check{f}$ is the inverse Fourier transform of f with respect to t. So, using a simple change of variable and letting

$$C_s = \sup_{\tau \in \mathbb{R}\setminus\{0\}} (|\tau|^{2s}/\sinh^2|\tau|),$$

we get

$$\|e^{-u\Delta_{\mathbb{H}}} f\|^2_{L^2_s(\mathbb{H})} \leq \frac{C_s}{4u^{2s}} \int_{-\infty}^{\infty} \left(\int_{\mathbb{C}} |\check{f}(z,\tau)|^2 dz \right) d\tau = \frac{C_s}{4u^{2s}} \|f\|^2_{L^2(\mathbb{H})}$$

and this completes the proof. □

The following result complements Theorem 5.1.

Theorem 5.2. *Let $s \leq -1$. Then for $u > 0$, $e^{-u\Delta_{\mathbb{H}}} : L^2_s(\mathbb{H}) \to L^2(\mathbb{H})$ is a bounded linear operator and*

$$\|e^{-u\Delta_{\mathbb{H}}} f\|_{L^2(\mathbb{H})} \leq \frac{c_{-s}}{2u^{-s}} \|f\|_{L^2_s(\mathbb{H})}, \quad f \in L^2_s(\mathbb{H}),$$

where

$$c_{-s} = \sup_{\tau \in \{0\}} (|\tau|^{-s} \sinh|\tau|).$$

The proof of Theorem 5.2 is very similar to that of Theorem 5.1 and is hence omitted.

6 Two Formulas and an Estimate for $\tilde{L}_\tau^{-1}$

Let $\tau \in \mathbb{R} \setminus \{0\}$. Then a formula for L_τ^{-1} is given by the following theorem.

Theorem 6.1. *Let $f \in L^2(\mathbb{C})$. Then*

$$\tilde{L}_\tau^{-1} f = (2\pi)^{1/2} \sum_{k=0}^{\infty} \frac{1}{(2k+1+|\tau|)|\tau|} V_\tau(W^\tau_{\hat{f}} e^\tau_k, e^\tau_k),$$

where the convergence of the series is understood to be in $L^2(\mathbb{C})$.

Proof Let $f \in L^2(\mathbb{C})$. Then

$$\tilde{L}_\tau^{-1} f = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(2k+1+|\tau|)|\tau|} (f, e^\tau_{j,k}) e^\tau_{j,k}, \tag{6.1}$$

where the series is convergent in $L^2(\mathbb{C})$. Now, by Plancherel's theorem and (2.2)–(2.4),

$$(f, e^\tau_{j,k}) = \int_{\mathbb{C}} f(z) \overline{V_\tau(e^\tau_j, e^\tau_k)(z)} dz = \int_{\mathbb{C}} \hat{f}(\zeta) \overline{V_\tau(e^\tau_j, e^\tau_k)^\wedge(\zeta)} d\zeta$$

$$= \int_{\mathbb{C}} \hat{f}(\zeta)\overline{W_\tau(e_j^\tau, e_k^\tau)(\zeta)}\,d\zeta = (2\pi)^{1/2}(W_{\hat{f}}e_k^\tau, e_j^\tau) \tag{6.2}$$

for $j,k = 0,1,2,\ldots$. Similarly, for $j,k = 0,1,2,\ldots$, and g in $L^2(\mathbb{C})$, we get

$$(e_{j,k}^\tau, g) = \overline{(g, e_{j,k}^\tau)} = (2\pi)^{1/2}\overline{(W_{\hat{g}}^\tau e_k^\tau, e_j^\tau)} = (2\pi)^{1/2}(e_j^\tau, W_{\hat{g}}^\tau e_k^\tau). \tag{6.3}$$

So, by (6.1)–(6.3), Fubini's theorem and Parseval's identity,

$$\begin{aligned}(\tilde{L}_\tau^{-1}f, g) &= 2\pi\sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|}\sum_{j=0}^\infty (W_{\hat{f}}^\tau e_k^\tau, e_j^\tau)(e_j^\tau, W_{\hat{g}}^\tau e_k^\tau)\\ &= 2\pi\sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|}(W_{\hat{f}}^\tau e_k^\tau, W_{\hat{g}}^\tau e_k^\tau).\end{aligned} \tag{6.4}$$

By Plancherel's theorem and (2.2)–(2.4),

$$\begin{aligned}(W_{\hat{f}}^\tau e_k^\tau, W_{\hat{g}}^\tau e_k^\tau) &= (2\pi)^{-1/2}\overline{\int_{\mathbb{C}} \hat{g}(z)W_\tau(e_k^\tau, W_{\hat{f}}^\tau e_k^\tau)(z)\,dz}\\ &= (2\pi)^{-1/2}\int_{\mathbb{C}} W_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau)(z)\overline{\hat{g}(z)}\,dz\\ &= (2\pi)^{-1/2}\int_{\mathbb{C}} V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau)(z)\overline{g(z)}\,dz\end{aligned} \tag{6.5}$$

for $k = 0,1,2,\ldots$. Thus, by (6.4), (6.5) and Fubini's theorem,

$$\begin{aligned}(\tilde{L}_\tau^{-1}f, g) &= (2\pi)^{1/2}\sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|}(V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau), g)\\ &= (2\pi)^{1/2}\left(\sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|}V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau), g\right)\end{aligned} \tag{6.6}$$

for all f and g in $L^2(\mathbb{C})$. Thus, by (6.6),

$$\tilde{L}_\tau^{-1}f = (2\pi)^{1/2}\sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|}V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau)$$

for all f in $L^2(\mathbb{C})$. □

The formula (6.4) is an important formula in its own right and we upgrade it to the status of a theorem.

Theorem 6.2. *For all $\tau \in \mathbb{R}\setminus\{0\}$, the inverse $\tilde{L}_\tau^{-1}$ of the parametrized partial differential operators $\tilde{L}_\tau$ is given by*

$$(\tilde{L}_\tau^{-1}f, g) = 2\pi\sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|}(W_{\hat{f}}^\tau e_k^\tau, W_{\hat{g}}^\tau e_k^\tau), \quad f, g \in L^2(\mathbb{C}).$$

For all τ in $\mathbb{R}\setminus\{0\}$, we have the following estimate for the L^2 norm of $\tilde{L}_\tau^{-1}f$ in terms of the L^2 norm of f.

Theorem 6.3. *Let $\tau\in\mathbb{R}\setminus\{0\}$. Then for all functions f in $L^2(\mathbb{C})$,*

$$\|\tilde{L}_\tau^{-1}f\|_{L^2(\mathbb{C})}\le|\tau|^{-2}\|f\|_{L^2(\mathbb{C})}.$$

Proof Let f and g be functions in $L^2(\mathbb{R})$. Then by Theorems 2.3 and 6.2,

$$\begin{aligned}|(\tilde{L}_\tau^{-1}f,g)| &\le 2\pi\frac{1}{|\tau|^2}\sum_{k=0}^{\infty}|(W_{\hat{f}}^\tau e_k^\tau, W_{\hat{g}}^\tau e_k^\tau)|\\ &\le 2\pi\frac{1}{|\tau|^2}\|W_{\hat{f}}^\tau\|_{HS}\|W_{\hat{g}}^\tau\|_{HS}\\ &= \frac{1}{|\tau|^2}\|f\|_{L^2(\mathbb{C})}\|g\|_{L^2(\mathbb{C})}\end{aligned}$$

and this completes the proof. □

7 Sobolev Estimates for $\Delta_{\mathbb{H}}^{-1}$

We have the following simple result giving the connection of $\Delta_{\mathbb{H}}^{-1}$ with $\tilde{L}_\tau^{-1}$, $\tau\in\mathbb{R}\setminus\{0\}$, which can be proved easily using the elementary properties of the Fourier transform and the Fourier inversion formula.

Theorem 7.1. *Let $f\in L^2(\mathbb{H})$. Then*

$$(\Delta_{\mathbb{H}}^{-1}f)(z,t)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-it\tau}(\tilde{L}_\tau^{-1}f^\tau)(z)\,d\tau,\quad (z,t)\in\mathbb{H}.$$

We can now give the following theorem, which can be seen as another manifestation of the ellipticity of $\Delta_{\mathbb{H}}$.

Theorem 7.2. *Let $s\in\mathbb{R}$. Then $\Delta_{\mathbb{H}}^{-1}:L_s^2(\mathbb{H})\to L_{s+2}^2(\mathbb{H})$ and*

$$\|\Delta_{\mathbb{H}}^{-1}f\|_{L_{s+2}^2(\mathbb{H})}\le\|f\|_{L_s^2(\mathbb{H})},\quad f\in L_s^2(\mathbb{H}).$$

Proof By Fubini's theorem, Plancherel's theorem, Theorems 6.3 and 7.1,

$$\begin{aligned}\|\Delta_{\mathbb{H}}^{-1}f\|_{L_{s+2}^2(\mathbb{H})}^2 &= \int_{\mathbb{C}}\int_{-\infty}^{\infty}|\tau|^{2(s+2)}|(\Delta_{\mathbb{H}}^{-1}f)^\tau(z)|^2d\tau\,dz\\ &= \int_{-\infty}^{\infty}|\tau|^{2(s+2)}\left(\int_{\mathbb{C}}|(\Delta_{\mathbb{H}}^{-1}f)^\tau(z)|^2dz\right)d\tau\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} |\tau|^{2(s+2)} \left(\int_{\mathbb{C}} |(\tilde{L}_\tau^{-1} f^\tau)(z)|^2 dz \right) d\tau \\
&= \int_{-\infty}^{\infty} |\tau|^{2(s+2)} \|\tilde{L}_\tau^{-1} f^\tau\|^2_{L^2(\mathbb{C})} d\tau \\
&\leq \int_{-\infty}^{\infty} |\tau|^{2s} \|f^\tau\|^2_{L^2(\mathbb{C})} d\tau \\
&= \int_{-\infty}^{\infty} |\tau|^{2s} \left(\int_{\mathbb{C}} |f^\tau(z)|^2 dz \right) d\tau \\
&= \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} |f^\tau(z)|^2 d\tau\, dz \\
&= \|f\|^2_{L^2_s(\mathbb{H})},
\end{aligned}
$$

as asserted. □

References

[1] DASGUPTA, A AND WONG, M.W., *Weyl transforms and the inverse of the sub-Laplacian on the Heisenberg group*, in *Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis*, Fields Institute Communications, **52**, American Mathematical Society, 2007, 27–36.

[2] DASGUPTA, A. AND WONG, M.W., *Weyl transforms and the heat equation for the sub-Laplacian on the Heisenberg group*, in *New Developments in Pseudo-Differential Operators*, Operator Theory: Advances and Applications, **189**, Birkhäuser, 2009, 33–42.

[3] DAVIES, E.B., *One-Parameter Semigroups*, Academic Press, 1980.

[4] DAVIES, E.B., *Linear Operators and their Spectra*, Cambridge University Press, 2007.

[5] FRIEDMAN, A., *Partial Differential Equations*, Holt, Reinhart and Winston, 1969.

[6] FURUTANI, K., *The heat kernel and the spectrum of a class of manifolds*, Comm. Partial Differential Equations, **21** (1996), 423–438.

[7] GAVEAU, B., *Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents*, Acta Math., **139** (1977), 95–153.

[8] HILLE, E. AND PHILLIPS, R.S., *Functional Analysis and Semi-groups*, Third Printing of Revised Version of 1957, American Mathematical Society, 1974.

[9] IANCU, G.M. AND WONG, M.W., *Global solutions of semilinear heat equations in Hilbert spaces*, Abstr. Appl. Anal., **1** (1996), 263–276.

[10] KIM, J. AND WONG, M.W., *Positive definite temperature functions on the Heisenberg group*, Appl. Anal., **85** (2006), 987–1000.

[11] SHUBIN, M.A., *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, 1987.

[12] STEIN, E.M., *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[13] WONG, M.W., *Weyl Transforms*, Springer-Verlag, 1998.

[14] WONG, M.W., *Weyl transforms, the heat kernel and Green function of a degenerate elliptic operator*, Ann. Global Anal. Geom., **28** (2005), 271–283.

[15] WONG, M.W., *The heat equation for the Hermite operator on the Heisenberg group*, Hokkaido Math. J., **34** (2005), 393–404.

[16] YOSIDA, K., *Functional Analysis*, Reprint of Sixth Edition, Springer-Verlag, 1995.