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Module amenability for Banach modules

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ABSTRACT

We study the module amenability of Banach modules. This is a natural generalization of Johnson's amenability of Banach algebras. As an example we show that for a discrete abelian group G, $\ell^{\mathsf{p}}(\mathsf{G})$ is amenable as an $\ell^1(\mathsf{G})$-module if and only if G is amenable, where $\ell^1(\mathsf{G})$ is a Banach algebra with pointwise multiplication.

RESUMEN

Se estudia el módulo de receptividad de los módulos de Banach. Esta es una generalización natural de la receptividad de Johnson de las álgebras de Banach. Como ejemplo se muestra que para un grupo abeliano discreto G $\ell^{\mathsf{p}}(\mathsf{G})$ es receptivo como un G $\ell^{\mathsf{p}}(\mathsf{G})$-módulo, si y sólo si G es receptivo, donde $\ell^1(\mathsf{G})$ es un álgebra de Banach con producto punto.

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1. Introduction

The concept of amenability for Banach algebras was introduced by B.E. Johnson in [J]. The main example in [J] asserts that the group algebra $L^1(G)$ of a locally compact group G is amenable if and only if G is amenable. This is far from true for semigroups. If S is a discrete inverse semigroup with the set of idempotents E_S, $\ell^1(S)$ is amenable if and only if E_S is finite and all the maximal subgroups of S are amenable [DN]. For an arbitrary discrete semigroup S, $\ell^1(S)$ is amenable if and only if the minimum ideal of S exists and is an amenable group and S has a principal series whose corresponding quotients are regular Rees matrix semigroups of special form [DLS, 10.12].This failure is partly due to the fact that $\ell^1(S)$ is equipped with two (related) algebraic structures. It is a Banach algebra and a Banach module over $\ell^1(E_S)$. This consideration was the motivation of the second named author to study the concept of *module amenability* for Banach algebras which have an extra Banach module structure (with compatible actions) in [A]. In particular it is shown in [A] that for an inverse semigroup S, $\ell^1(S)$ is module amenable as a Banach module over $\ell^1(E_S)$ if and only if S is amenable. The authors introduced the concept of *weak module amenability* in [AE] and showed that for a commutative inverse semigroup S, $\ell^1(S)$ is always weak module amenable as a Banach module over $\ell^1(E_S)$.

The present paper investigates module amenability from a different angle. There are many examples of Banach modules which do not have any natural algebra structure. One example is $L^p(G)$ which is a left Banach $L^1(G)$-module, for a locally compact group G [D, 3.3.19]. As another example of this sort, one may consider a Banach algebraic bundle over a locally compact group G [FD]. Then the fibers on elements of G are Banach modules over the fiber on the identity. Crossed products of Banach algebra by groups are special cases of Banach algebraic bundles. The theory of module amenability developed in [A] does not cover these examples. There is one thing in common in these examples and that is the existence of a module homomorphism from the Banach module to the underlying Banach algebra. For instance in the case of crossed products, X is a Banach algebra, G is a topological group, and $X_g = X \times \{g\}$, for $g \in G$, and $\{X_g\}$ is a Banach algebraic bundle over G. In this case we have a module homomorphism $\Delta_g : X_g \to X_e$ which sends (x, g) to (x, e), where e is the identity of G. Also if G is a compact group and $f \in L^q(G)$, then one has the module homomorphism $\Delta_f : L^p(G) \to L^1(G)$ which sends g to $f * g$.

In this paper, the concept of module amenability (more precisely Δ-amenability) is defined for a Banach module E over a Banach algebra A with a given module homomorphism $\Delta : E \to A$. The next section gives the basic properties of module amenability and in particular establishes the equivalence of this concept with the existence of module virtual (approximate) diagonals in an appropriate sense. Section 3 covers the weak Δ-amenability. A few examples are discussed in the last section.

2. Module Amenability

Let A be a Banach algebra and E be a Banach space with a left A-module structure such that, for some $M > 0$,

$$\|a.x\| \leq M\|a\| \, \|x\| \quad (a \in A, x \in E),$$

then E is called a left Banach A-module. Right and two-sided Banach A-modules are defined similarly. Throughout this section E is a Banach A-bimodule and $\Delta : E \to A$ is a bounded Banach A-bimodule homomorphism.

Definition 2.1. Let X be a Banach A-bimodule. A bounded linear map $D : A \to X$ is called a module derivation (or more specifically a Δ-derivation) if

$$D(\Delta(a.x)) = a.D(\Delta(x)) + D(a).\Delta(x), \;\; D(\Delta(x.a)) = D(\Delta(x)).a + \Delta(x).D(a),$$

For each $a \in A$ and $x \in E$. Also D is called inner (or Δ-inner) if there is $f \in X$ such that

$$D(\Delta(x)) = f.\Delta(x) - \Delta(x).f =: D_f(\Delta(x)) \quad (x \in E).$$

When Δ has a dense range, D_f extends uniquely to a Δ-derivation from A to X.

Definition 2.2. A bimodule E is called *module amenable* (or more specifically Δ-amenable as a A-bimodule) if for each Banach A-bimodule X, all Δ-derivations from A to X^* are Δ-inner.

It is clear that A is A-module amenable (with $\Delta = id$) if and only if it is amenable as a Banach algebra. A right *bounded approximate identity* of E is a bounded net $\{a_\alpha\}$ in A such that for each $x \in E$, $\|\Delta(x).a_\alpha - \Delta(x)\| \to 0$, as $\alpha \to \infty$. The left and two-sided approximate identities are defined similarly.

Proposition 2.3. If E is module amenable, then E has a bounded approximate identity.

Proof Consider the double conjugate space A^{**} as a Banach A-module with

$$\langle F.a, f\rangle = \langle F, a.f\rangle, \langle a.f, b\rangle = f(ba), a.F = 0 \quad (a, b \in A, f \in A^*, F \in A^{**}).$$

Then the canonical embedding $D : A \to A^{**}$ is a module derivation, hence $D = D_F$ on $\Delta(E)$, for some $F \in A^{**}$. Choose a net $\{a_\alpha\}$ in A which is w^*-convergent to F in A^{**}. Clearly $\{a_\alpha\}$ is a left bounded approximate identity of E. Right and two sided approximate identities now could be constructed similar to the classical case [D]. $\square$

Definition 2.4. A Banach A-module X is called right Δ-essential if for each $x \in X$ there is $a \in \Delta(E)$ and $y \in X$ such that $x = y.a$. The left Δ-essential and (two sided) Δ-essential modules are defined similarly.

The following two results are proved as in the classical case [J]. We just include the proof of Lemma 2.5(i), as it involves a variation of the Cohen factorization theorem.

Lemma 2.5. (i) If Δ has a dense range and E has a (right) bounded approximate identity, then E is module amenable iff for each (right) Δ-essential Banach A-bimodule X, all Δ-derivations from A to X^* are Δ-inner.

(ii) If E and $E^{'}$ are Banach A-modules with module homomorphisms Δ and $\Delta^{'}$ and $\theta : E \to E^{'}$ is a bounded module map with dense range such that $\Delta^{'} \circ \theta = \Delta$, then Δ-amenability of E implies $\Delta^{'}$-amenability of $E^{'}$.

(iii) If J is a closed submodule of E and J_Δ is the closed ideal of A generated by $\Delta(J)$, and $q : A \to A/J_\Delta$ and $\tilde{q} : E \to E/J$ are the corresponding quotient maps, then E is Δ amenable whenever J is $\Delta_{|J}$-amenable and E/J is $\tilde{\Delta}$-amenable, where $\tilde{\Delta} : E/J \to A/J_\Delta$ is the unique A/J_Δ-module map with $\tilde{\Delta} \circ \tilde{q} = q \circ \Delta$.

Proof We prove part (i) as promised. We just need to check the necessity. Let $\{a_\alpha\} \subseteq A$ be a right bounded approximate identity for E. Let X be a Banach A-bimodule. Consider $T_\alpha : X^* \to X^*$ defined by $T(f) = a_\alpha.f$, for $f \in X^*$, where $a.f(x) = f(x.a)$, for $a \in A, x \in X$. Since $\{a_\alpha\}$ is bounded in A, $\{T_\alpha\}$ is bounded in $\mathcal{B}(X^*)$. Hence it has a w^*-cluster point T. We may assume that $T_\alpha \to T$ in w^*-topology.

For each $e \in E, x \in X, f \in X^*$, we have

$$\begin{aligned}\langle x, \Delta(e), Tf\rangle &= \lim_\alpha \langle x, \Delta(e), T_\alpha f\rangle = \lim_\alpha \langle x, \Delta(e), a_\alpha.f\rangle \\ &= \lim_\alpha \langle x, \Delta(e)a_\alpha, f\rangle = \langle x, \Delta(e), f\rangle.\end{aligned}$$

Hence $T - I : X^* \to (X.\Delta(E))^\perp$ is a bounded projection and we have the admissible short exact sequence

$$0 \to (X.\Delta(E))^\perp \to X^* \to (X.\Delta(E))^* \to 0$$

of Banach A-bimodules. But $\Delta(E).(X/\overline{(X.\Delta(E))}) = 0$ and Δ has a dense range, hence each bounded Δ-derivation $D_1 : A \to (X.\Delta(E))^\perp = (X/\overline{(X.\Delta(E))})^*$ is zero. On the other hand, each bounded Δ-derivation $D_2 : A \to (X.\Delta(E))^*$ is Δ-inner, by assumption. Therefore each bounded Δ-derivation $D : A \to (X.\Delta(E))^*$ is Δ-inner, and we are done. $\square$

Lemma 2.6. Assume that A and B are Banach algebras, J is a closed ideal of A, E is a Banach A-module, and $\Delta : E \to A$ is an A-module homomorphism.

(i) If F is a Banach A-module and $\Phi : E \to F$ is an A-module homomorphism with dense range, then Δ-amenability of E implies $\Delta \circ \Phi$-amenability of F.

(ii) If $\Psi : A \to B$ is a Banach algebra epimorphism with

$$E.\mathrm{Ker}(\Psi) = \mathrm{Ker}(\Psi).E = \{0\},$$

and E is considered as a B-module via

$$b.x := a.x,\ x.b := x.a \quad (b \in B, x \in E),$$

where $a \in A$ on the right hand side is any element with $b = \Psi(a)$. Then Δ-amenability of E, as an A-module, implies $\Psi \circ \Delta$-amenability of E as a B-module.

(iii) In (ii), if $B = A/J$, $\Psi : A \to A/J$ is the quotient map, and $E.J = J.E = \{0\}$, then Δ-amenability of E, as an A-module, implies $\Psi \circ \Delta$-amenability of E as a A/J-module.

(iv) If I is a closed ideal of A, $E^{'}$ is the closed submodule of E generated by IE, and $\Delta^{'} : E^{'} \to I$ is the restriction of $\Delta : E \to A$, then $E^{'}$ is $\Delta^{'}$-amenable whenever E is Δ-amenable and $E^{'}$ has a bounded approximate identity.

Proposition 2.7. If I is a closed ideal of A which contains a bounded approximate identity (of itself), E is a Banach A-bimodule with module homomorphism $\Delta : E \to A$, and X is an essential Banach I-module, then X is (canonically) a Banach A-module and each $\Delta_{|I}$-derivation $D : I \to X^*$ uniquely extends to a Δ-derivation $\tilde{D} : A \to X^*$ which is continuous with respect to the strict topology of A (induced by I) and w^*-topology of X^*.

Proof Each $x \in X$ decomposes (not uniquely) as $x = a.y$, for some $a \in I$ and $y \in X$. It is easy to see that X is a left Banach A-module under the action

$$b.x = ba.y \quad (a \in I, b \in A, x, y \in X, x = a.y).$$

This is well defined, as I has a bounded approximate identity. Define $\tilde{D} : A \to X^*$ by

$$\tilde{D}(b) = w^*\text{-}\lim_{\alpha}(D(be_\alpha) - b.D(e_\alpha)),$$

where $\{e_\alpha\}$ is a bounded approximate identity of I. Now $be_\alpha \to b$ strictly, for each $b \in A$. Hence, given $b \in A$ and $e \in E$, we have

$$\begin{aligned}\tilde{D}(\Delta(b.e)) = \tilde{D}(b\Delta(e)) &= w^*\text{-}\lim_{\alpha} w^*\text{-}\lim_{\beta} D(be_\alpha\Delta(e)e_\beta)\\ &= w^*\text{-}\lim_{\alpha} w^*\text{-}\lim_{\beta}[be_\alpha D(\Delta(e)e_\beta) + D(be_\alpha).\Delta(e)e_\beta]\\ &= b\tilde{D}(\Delta(e)) + \tilde{D}(b).\Delta(e).\end{aligned}$$

Hence $\tilde{D}$ is a Δ-derivation. The rest of the proof is similar to [Ru, 2.1.6]. □

Proposition 2.8. If $\Delta : E \to A$ has a dense range, then Δ-amenability of E is equivalent to amenability of A.

Proof If E is Δ-amenable, then each derivation $D : A \to X^*$, where X is a Banach A-module, is a module derivation and so inner on $\Delta(E)$. By continuity, D is inner on A. Conversely each module derivation $D : A \to X^*$ is a derivation. Indeed, given $b \in A$, there is a sequence $\{x_n\} \subseteq E$ such that $\Delta(x_n) \to b$, and so

$$D(ab) = \lim_{n} D(a\Delta(x_n)) = \lim_{n}(D(a).\Delta(x_n) + a.D(\Delta(x_n))) = D(a).b + a.D(b),$$

for each $a \in A$. Hence, if A is amenable, then E is Δ-amenable. □

Definition 2.9. Let $\pi : A\hat{\otimes}A \to A$ be the continuous lift of the multiplication map of A to the projective tensor product $A\hat{\otimes}A$. A *module approximate diagonal* of E is a bounded net $\{e_\alpha\}$ in $A\hat{\otimes}A$ such that

$$\|e_\alpha.\Delta(x) - \Delta(x).e_\alpha\| \to 0, \ \|\pi(e_\alpha).\Delta(x) - \Delta(x)\| \to 0 \quad (x \in E),$$

as $\alpha \to \infty$. A *module virtual diagonal* of E is an element M in $(A\hat{\otimes}A)^{**}$ such that

$$M.\Delta(x) - \Delta(x).M = 0, \ \pi^{**}(M).\Delta(x) - \Delta(x) = 0 \quad (x \in E).$$

It is clear that if E has a module virtual diagonal, then A contains a bounded approximate identity.

Theorem 2.10. *Consider the following assertions.*

(i) E *is module amenable,*

(ii) E *has a module virtual diagonal,*

(iii) E *has a module approximate diagonal.*

We have $(i) \to (ii) \leftrightarrow (iii)$. *If moreover* Δ *has a dense range, all the assertions are equivalent.*

Proof $(i) \to (ii)$. By Proposition 2.3, we may choose a bounded approximate identity $\{e_\alpha\}$ for E. We may assume that $\{e_\alpha \otimes e_\alpha\}$ is w^*-convergent to a point $P \in (A\hat{\otimes}A)^{**}$. Then for each $x \in E$ and $f \in A^*$,

$$\begin{aligned}\langle \pi^{**}((D_P \circ \Delta)(x), f\rangle &= w^*\text{-}\lim_\alpha \langle \pi^*(f).\Delta(x) - \Delta(x).\pi^*(f), e_\alpha \otimes e_\alpha\rangle \\ &= w^*\text{-}\lim_\alpha f(\Delta(x.e_\alpha)e_\alpha - e_\alpha\Delta(e_\alpha.x)) = 0.\end{aligned}$$

Hence $\mathrm{Im}(D_P \circ \Delta) \subseteq \mathrm{Ker}(\pi^{**})$. Now $\mathrm{Ker}(\pi^{**})$ is isometrically isomorphic to X^*, where $X = (\Delta(E)\hat{\otimes}A)^{**}/\mathrm{Im}(\pi^*)^\perp$, so by assumption, there is $Q \in \mathrm{Ker}(\pi^{**})$ with $D_P \circ \Delta = D_Q \circ \Delta$. It is easy to see that $M := P - Q$ is a module virtual diagonal for E.

$(ii) \to (iii)$. Let M be a module virtual diagonal and let $\{e_\alpha\}$ be a net in $A\hat{\otimes}A$ which w^*-clusters to M. Then clearly $e_\alpha.\Delta(x) - \Delta(x).e_\alpha \to 0$, $\Delta \circ \pi(e_\alpha).\Delta(x) - \Delta(x) \to 0$ as $\alpha \to \infty$ for each $x \in E$ in the w^*-topology of $(A\hat{\otimes}A)^{**}$. A standard argument based on Mazur's theorem shows that the same holds in the norm topology for a net consisting of appropriate convex combinations of elements of $\{e_\alpha\}$.

$(iii) \to (ii)$. Just take any w^*-cluster point.

$(iii) \to (i)$. Now assume that Δ has a dense range. Let $\{m_\alpha\}$ be a module approximate diagonal for E with w^*-cluster point M, then $\{\pi(m_\alpha)\}$ is a bounded approximate identity for E. By Lemma 2.5(i), it is enough to show that for each essential A-module Y, all module derivation D from A to Y^* are inner. Each $y \in Y$ could be regarded as a bounded linear functional $\hat{y}$ on $A\hat{\otimes}A$ via

$$\langle \hat{y}, b \otimes a\rangle := \langle b.D(a), y\rangle \quad (a, b \in A).$$

Then for each $x, x^{'} \in E$, $a \in A$, and $y \in Y$

$$\langle (y.\Delta(x) - \Delta(x).y)\hat{}, \Delta(x^{'}) \otimes a\rangle = \langle \hat{y}.\Delta(x) - \Delta(x).\hat{y}, \Delta(x^{'}) \otimes a\rangle + \langle \Delta(x^{'})a.D \circ \Delta(x), y\rangle.$$

It follows that

$$\langle (y.\Delta(x) - \Delta(x).y)\hat{}, m\rangle = \langle \hat{y}.\Delta(x) - \Delta(x).\hat{y}, m\rangle + \langle \pi(m).D \circ \Delta(x), y\rangle,$$

for each $m \in A\hat{\otimes}A$. If we identify M with an element of Y^* with $M(y) = \langle \hat{y}, M\rangle$, for $y \in Y$, then

$$\begin{aligned}\langle D_M \circ \Delta(x), y\rangle &= w^*\text{-}\lim_{\alpha}\langle y.\Delta(x) - \Delta(x).y, m_\alpha\rangle \\ &= \langle M, \hat{y}.\Delta(x) - \Delta(x).\hat{y}\rangle + w^*\text{-}\lim_{\alpha}\langle \pi(m_\alpha).D \circ \Delta(x), y\rangle.\end{aligned}$$

Now in the last equation, the first term is zero, as M is a module virtual diagonal, and the second term is easily seen to be equal to $\langle D \circ \Delta(x), y\rangle$, using the fact that $y = z.\Delta(x^{'})$, for some $z \in Y$ and $x^{'} \in E$. Therefore $D = D_M$ on $\Delta(E)$, as required. □

3. Weak Module Amenability

In this section we study weak module amenability of Banach modules. All over this section E is a commutative Banach A-module (that is $a.x = x.a$, for each $a \in A, x \in E$) and $\Delta : E \to A$ is a bounded Banach A-module homomorphism. A Banach A-module X is called Δ-*commutative* (or more specifically $\Delta(E)$-commutative) if

$$a.x = x.a \quad (a \in \Delta(E), x \in X).$$

Definition 3.1. E is called *weak module amenable* (or more specifically weak Δ-amenable as an A-module) if each Δ-derivation from A to $\Delta(E)^*$ is inner on $\Delta(E)$.

Clearly A is weak A-module amenable (with $\Delta = id$) if and only if it is weakly amenable as a Banach algebra. The following result could be proved as in the classical case.

Proposition 3.2. (i) If $E^{'}$ is a commutative A-module and $\Phi : E^{'} \to E$ is a module homomorphism with dense range, and E is weak Δ-amenable then $E^{'}$ is weak $\Delta \circ \Phi$-amenable.

(ii) If I is a closed ideal of A with $IE = EI = \{0\}$ and $q : A \to A/I$ is the quotient map, then E is weak $q \circ \Delta$-amenable as an A/I-module if it is Δ-amenable as an A-bimodule.

Proposition 3.3. If E is weak Δ-amenable, then the closed linear span F of $A\Delta(E)$ is dense in $\Delta(E)$.

Proof If not, there is a nonzero bounded linear functional λ in $\Delta(E)^*$ which vanishes on F. By Hahn-Banach Theorem λ extends to an element of A^*, which we still denote by λ. Define $D : A \to \Delta(E)^*$ by $D(a) = \lambda(a)\lambda$. This is a module derivation which is not inner, a contradiction. □

Now if A is a (commutative) Banach algebra with maximal ideal space $\mathcal{M}_A$ and $\phi \in \mathcal{M}_A$, then $\mathbb{C}$ is a Banach A-module with respect to the module action

$$a.z = z.a = \phi(a)z \quad (a \in A, z \in \mathbb{C}),$$

which is denoted by $\mathbb{C}_\phi$. Each module derivation $D : A \to \mathbb{C}_\phi$ is called a *module point derivation* (at ϕ). Clearly when a commutative Banach A-module E is Δ-weak amenable, all module point derivations vanish on $\Delta(E)$. This holds in general.

Proposition 3.4. If E is weak Δ-amenable, there is no nonzero point derivation on A.

Proof Let $d : A \to \mathbb{C}_\phi$ be a nonzero module point derivation. Let ψ be the restriction of ϕ to $\Delta(E)$ and define $D : A \to \Delta(E)^*$ by $D(a) = d(a)\psi$. Then D is a Δ-derivation and so $D = D_\lambda$ on $\Delta(E)$, for some $\lambda \in \Delta(E)^*$. Choose $e, f \in E$ so that $\psi(\Delta(e)) = 1$, $\psi(\Delta(f)) = 0$, and $d(\Delta(f)) = 1$. Then for $a = \Delta(e) + (1 - d(\Delta(e)))\Delta(f)$, we have $\psi(a) = d(a) = 1$, hence $D(a)(a) = 1$, a contradiction. □

Theorem 3.5. *If E is a commutative A-module and there is a Δ-commutative A-module X and a module derivation $D_0 : A \to X$ which is not identically zero on $\Delta(E)$, then there is a nonzero module derivation $D : A \to \Delta(E)^*$.*

Proof We consider two cases. First assume that $A\Delta(E)$ is not dense in $\Delta(E)$. By Hahn-Banach Theorem, there is a nonzero functional $\lambda \in (\Delta(E))^*$ whose kernel contains $A\Delta(E)$. Extend λ to an element of A^* (still denoted by λ) and define $D : A \to \Delta(E)^*$ by

$$D(a) = \lambda(a)\lambda \quad (a \in A),$$

Next consider the case where $A\Delta(E)$ is dense in $\Delta(E)$. We know that there is a Δ-commutative A-module X and a module derivation $D_0 : A \to X$ which is not identically zero on $\Delta(E)$. Choose $a \in A$, $e \in E$, and $\lambda \in X^*$ such that $D_0(a\Delta(e)) \neq 0$ and $\lambda(D_0(a\Delta(e))) \neq 0$. Define $D : A \to \Delta(E)^*$ by

$$\langle D(a), \Delta(e)\rangle = \lambda(\Delta(e).D_0(a)) \quad (e \in E, a \in A).$$

In both cases D is a nonzero module derivation. □

4. Examples

In this sections we give three examples in which strong and weak module amenability of some Banach modules are demonstrated.

Example 4.1. Let S be an inverse semigroup and E_S be the commutative sub-semigroup of idempotents in S. Then $A = \ell^1(E_S)$ is a commutative Banach algebra and $E = \ell^1(S)$ is a commutative Banach A-bimodule with the module actions

$$\delta_e.\delta_x = \delta_x.\delta_e = \delta_{ex} \quad (e \in E_S, x \in S).$$

Also there is a surjective module homomorphism $\Delta : \ell^1(S) \to \ell^1(E_S)$ defined by

$$\Delta(\delta_x) = \delta_{xx^*} \quad (x \in S).$$

We show that $\ell^1(S)$ is always module weakly amenable. If $D : \ell^1(E_S) \to \ell^\infty(E_S)$ is a Δ-derivation, then for each $e \in E_S$,

$$\begin{aligned} D(\delta_e) &= D(\delta_{ee^*}) = D(\Delta(\delta_e)) \\ &= D(\Delta(\delta_e.\delta_e)) = \Delta(\delta_e).D(\delta_e) + \delta_e.D(\Delta(\delta_e)) = 2\delta_e D(\delta_e). \end{aligned}$$

Applying the same formula to the right hand side,

$$\delta_e.D(\delta_e) = 2\delta_e.(\delta_e.D(\delta_e)) = 2(\delta_e * \delta_e).D(\delta_e)) = 2\delta_e.D(\delta_e),$$

hence $D(\delta_e) = \delta_e.D(\delta_e) = 0$.

Example 4.2. In the above example, if $\ell^1(S)$ is Δ-amenable, then $\ell^1(E_S)$ is amenable (Proposition 2.8). Hence E_S is finite, $\ell^1(S)$ is weakly amenable, and it has a bounded approximate identity [DN].

Example 4.3. It is well known that the disk algebra $A(D)$ is non amenable [BD]. $A(D)$ is a $\mathbb{C}$-module with respect to the scalar product. Now evaluation at zero defines a module epimorphism $\Delta : A(D) \to \mathbb{C}$ and $A(D)$ is Δ-amenable.

Example 4.4. If A is an amenable Banach algebra, the canonical map $\pi : A\hat{\otimes}A \to A$ is an A-module epimorphism (it is surjective, since A has a bounded approximate identity) and $A\hat{\otimes}A$ is π-amenable.

Example 4.5. For a locally compact group G, $L^1(G)$ is a closed two sided ideal in $M(G)$, so we can consider it as a Banach $M(G)$ module. Now if G is a non discrete amenable group, $M(G)$ is not amenable [DGH] but $L^1(G)$ is i-amenable, where $i : L^1(G) \to M(G)$ is the canonical injection.

Example 4.6. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then ℓ^1 is a Banach algebra and ℓ^p is a Banach ℓ^1-bimodule, both with respect to pointwise multiplication. Also each $f \in \ell^q$ defines a module homomorphism $\Delta_f : \ell^p \to \ell^1$ by $\Delta_f(g) = g * f$. If $f = \sum_{k=-\infty}^{\infty} \frac{1}{k}\delta_k$, then Δ_f has dense range(as its range contains all finitely supported element) and ℓ^p is Δ_f-amenable, by Proposition 2.8. This example could also be stated for any discrete group G, where $\ell^p(G)$ is considered as a Banach $\ell^1(G)$-bimodule. Same is true for $L^p(G)$ with convolution, when G is a compact group [D, 3.3.19]. In this case we have the module homomorphism $\Delta_f : L^p(G) \to L^1(G)$ defined by $\Delta_f(g) = g * f$, where $f \in L^q(G)$. If Δ_f has a dense range, then $L^p(G)$ is Δ_f -amenable. This is always the case when G is an abelian compact group. We illustrate this for $G = \mathbb{T}$. The same proof basically works for arbitrary abelian compact groups as well. Take $f = \sum_{k=-\infty}^{\infty} \frac{1}{k} e^{2\pi i k t} \in L^q(\mathbb{T})$ (which is basically the Fourier transform of the above function f used in the discrete case). Then, for each $g \in L^p(G)$,

$$\Delta_f(g) = \sum_{k=-\infty}^{\infty} \frac{1}{k}\hat{g}(k)e^{2\pi i k t},$$

where $\hat{g} \in c_0$ is the Fourier transform of g. In particular, range of Δ_f includes all trigonometric functions which are dense in $L^1(G)$.

Example 4.7. If G is a discrete group (with identity e) which acts on a C^*-algebra A, then $A \times \{e\}$ could be identified with A and $A \times \{g\}$ is a Banach A-module under

$$a.(b, g) = ((g.a)b, g) \ , \ (b, g).a = (ba, g) \quad (a, b \in A, g \in G),$$

and there is a natural surjective module homomorphism $\Delta_g : A \times \{g\} \to A$ which sends (a, g) to a. The crossed product C^*-algebra $A \rtimes G$ is nuclear iff A is nuclear and G is amenable [Ro] iff G is amenable and modules $A \times \{g\}$ are Δ_g-amenable, for each $g \in G$.

Example 4.8. If A is a Banach algebra such that $A^* \subseteq A$ and A^* is a dense subspace of A, then A^* is a Banach A-bimodule (with canonical Arens actions) and $\Delta = \mathtt{id} : A^* \to A$ has dense range. Therefore A^* is Δ-amenable as a Banach A-bimodule iff A is amenable as a Banach algebra. There are many examples of this type. If G is a compact group, then the Fourier algebra $A(G)$ is dense in the group C^*-algebra $C^*(G)$. Indeed

$$A(G) \subseteq C(G) \subseteq L^1(G) \subseteq C^*(G),$$

and each space in this chain is dense in the subsequent space (with respect to the norm of the bigger space). But the norms of the last three spaces satisfy $\|.\|_{C^*(G)} \leq \|.\|_1 \leq \|.\|_\infty$ [Ey]. Hence $A(G)$ is dense in $C^*(G)$. Also, since G is compact, $A(G) = B(G) \simeq C^*(G)^*$, where $B(G)$ is the Fourier-Stieltjes algebra [Ey]. But $C^*(G)$ is amenable when G is compact [Ru]. Hence $A(G)$ is $\mathtt{id}$-amenable in this case. This becomes more interesting when we recall that there are compact groups for which the Fourier algebra $A(G)$ is not amenable [J2]. Another example is ℓ^1 which is dense in c_0. It follows that $\ell^1 \simeq c_0^*$ is $\mathtt{id}$-amenable as a c_0-bimodule. Finally, for a compact group G, $L^1(G)$ is an amenable Banach algebra with convolution, and so $L^1(G)$ is $\mathtt{id}$-amenable as a Banach $L^1(G)$-bimodule.

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Referencias

[A] M. Amini, *Module amenability for semigroup algebras*, Semigroup Forum 69 (2004) 243-254.

[AE] M. Amini and D. Ebrahimi Bagha, *Weak module amenability for semigroup algebras*, Semigroup Forum 71 (2005) 18-26.

[BD] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, New York, 1973.

[D] H.G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monographs, Volume 24, Clarendon Press, Oxford, 2000.

[DGH] H.G. Dales, F. Ghahramani and A. Ya. Helemskii, *The amenability of measure algebras*, J. London Math. Soc. (2) 66 (2002), no. 1, 213-226.

[DLS] H.G. Dales, A.T.M. Lau and D. Strauss, *Banach algebras on semigroups and their compactification*, to appear in Memoirs Amer. Math. Soc.

[DN] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proceedings of the Royal Society of Edinburgh 80A (1978), 309-321.

[Ey] P. Eymard, *L'algebre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181-236.

[FD] J. M. G. Fell and R. S. Doran, *Representations of *-algebras*, locally compact groups, and Banach *-algebraic bundles, Pure and Applied Mathematics, Vols. 125 & 126, Academic Press Inc., Boston, MA, 1988.

[J] B.E. Johnson, *Cohomology in Banach algebras.* Memoirs of the American Mathematical Society, No. 127, American Mathematical Society, Providence, 1972.

[J2] B.E. Johnson,*Non-amenability of the Fourier algebra of a compact group*, J. London Math. Soc. (2), 50 (1994), 361-374.

[Ro] J. Rosenberg, *Amenability of crossed products of C^*-algebras*, Comm. Math. Phys. 57 (1977), no. 2, 187-191.

[Ru] V. Runde, *Lectures on amenability*, Lecture Notes in Mathematics 1774, Springer-Verlag, Berlin, 2002.