

Degree theory for the sum of VMO maps and maximal monotone maps

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ABSTRACT

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, $f : \Omega \rightarrow \mathbb{R}^n$ a VMO map, and $T : D(T) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ a maximal monotone map with $D(T) \cap \Omega \neq \emptyset$. We construct a degree for the sum of $f + T$, which can be viewed as a generalization of the degree both for VMO maps and maximal monotone maps.

RESUMEN

Sea $\Omega \subset \mathbb{R}^n$ un dominio abierto, $f : \Omega \rightarrow \mathbb{R}^n$ un mapa VMO, y $T : D(T) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ un mapa monotono maximal con $D(T) \cap \Omega \neq \emptyset$. Construimos un grado por la suma de $f + T$, que se puede ver como una generalización de la medida, tanto para los mapas de VMO y para los mapas monotono maximal.

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1. Introduction

Degree theory for continuous maps in finite dimensional spaces has a long history and has been extensively studied. In the early 80's of the last century a degree for some classes of non-continuous maps was established (see [8,1,17,18] and the references therein). In 1995 and 1996, H. Brezis and L. Nirenberg [12], [13] invented a degree theory for VMO maps; see [2-6,9-11,19,21,22]. Generally, VMO functions need not be continuous. Another important class of non-continuous maps is the class of maximal monotone maps, and there is no relation between the VMO maps and the maximal monotone maps. In this paper, we consider the sum of a VMO map and a maximal monotone map, and we will define a degree theory for such a map. First we recall some definitions. Let Ω be an open bounded domain in \mathbb{R}^n . The class of bounded mean oscillation functions (see [20]) are defined as

$$\text{BMO}(\Omega) = \{f : \Omega \rightarrow \mathbb{R}^n \text{ is locally integrable, and } |f|_{\text{BMO}} < \infty\},$$

where $|f|_{\text{BMO}} = \sup_{B \subset \Omega} \frac{1}{m(B)} \int_B |f(x) - \bar{f}| dx$, $\bar{f} = \frac{1}{m(B)} \int_B f(x) dx$ (here $m(\cdot)$ represents the Lebesgue measure), and the class of vanishing mean oscillation functions (see [23]) are defined as

$$\text{VMO}(\Omega) = \{f : \Omega \rightarrow \mathbb{R}^n \text{ is locally integrable, and } \lim_{m(B) \rightarrow 0} \frac{1}{m(B)} \int_B |f(x) - \bar{f}| dx = 0\},$$

where $B \subset \mathbb{R}^n$ is an open ball with its closure contained in Ω . It is well known that if $f \in \text{VMO}$, then $f_\epsilon(x) = \frac{1}{m(B_\epsilon(x))} \int_{B_\epsilon(x)} f(y) dy$ is continuous in ϵ and x where it is defined. Let $T : D(T) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function. If $(h - g, x - y) \geq 0$ for all $x, y \in D(T)$ and $h \in Tx, g \in Ty$, then T is said to be monotone. If T is monotone and T has no monotone extension in \mathbb{R}^n , then T is said to be maximal monotone. It is well known that T is maximal monotone iff T is monotone and $T + \epsilon I$ is surjective for all $\epsilon > 0$. If T is maximal monotone, we use $T_\epsilon = (T^{-1} + \epsilon I)^{-1}$ to represent the Yosida approximation, and $R_\epsilon = I - \epsilon T_\epsilon$, the resolvent with respect to T_ϵ . For maximal monotone maps we refer the reader to [7]. Let $f : \Omega \rightarrow \mathbb{R}^n$ be a VMO map, $T : D(T) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ a maximal monotone map, $p \in \mathbb{R}^n$, and $D(T) \cap \Omega \neq \emptyset$. Under appropriate assumptions, see (2.1) below, we define the degree $\deg(f + T, \Omega \cap D(T), p)$. If $T = 0$, this degree coincides with the degree for VMO

maps in [13], and if $f = 0$, then it coincides with the degree for maximal monotone maps (see [14-16]).

2. Results

In this section, $\Omega \subset \mathbb{R}^n$ is an bounded open domain, $f \in \text{VMO}(\Omega)$, $T : D(T) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a maximal monotone map, $p \in \mathbb{R}^n$, and $\Omega \cap D(T) \neq \emptyset$. Suppose there exists an open neighborhood \mathcal{U} of $\partial\Omega$ in Ω and a constant $\beta > 0$ such that

$$\frac{1}{m(B_\epsilon(\mathbf{y}))} \int_{B_\epsilon(\mathbf{y})} |f(\mathbf{x}) + g - p| dx \geq \beta \tag{2.1}$$

for all $0 < \epsilon < \frac{1}{2}d(\mathbf{y}, \partial\Omega)$, $g \in Tz$, $z \in D(T) \cap B_\epsilon(\mathbf{y})$, where $B_\epsilon(\mathbf{y})$ is an open ball centered at \mathbf{y} with radius ϵ such that $\overline{B_\epsilon(\mathbf{y})} \subset \mathcal{U}$, and $d(\mathbf{y}, \partial\Omega)$ is the distance between \mathbf{y} and $\partial\Omega$.

We remark that if $T = 0$, then (2.1) was first used in [13]. If $f = 0$, then (2.1) is equivalent to $|g - p| \geq \beta$ for all $z \in D(T) \cap \mathcal{U}$ and $g \in Tz$, and in this case Proposition 2.1 below shows that the assumption $p \notin T(\partial\Omega \cap D(T))$ will guarantee (2.1) holds.

Proposition 2.1. If $p \notin T(\partial\Omega \cap D(T))$, then there exists $d_0 > 0, \alpha_0 > 0$ such that $d(p, T\mathbf{x}) \geq d_0$ for all $\mathbf{x} \in \Omega \cap D(T)$ with $d(\mathbf{x}, \partial\Omega) < \alpha_0$.

Proof. Suppose the conclusion is not true. There exist $\mathbf{x}_n \in \Omega \cap D(T)$, $g_n \in T\mathbf{x}_n$ such that $d(\mathbf{x}_n, \partial\Omega) \rightarrow 0$, and $g_n - p \rightarrow 0$. Without loss of generality, we may assume that $\mathbf{x}_n \rightarrow \mathbf{x}_0 \in \partial\Omega$.

Since $(g_n - g, \mathbf{x}_n - \mathbf{x}) \geq 0$ for all $\mathbf{x} \in D(T)$, $g \in T\mathbf{x}$, we have

$$(p - g, \mathbf{x}_0 - \mathbf{x}) \geq 0, \text{ for all } \mathbf{x} \in D(T), g \in T\mathbf{x}.$$

Therefore $\mathbf{x}_0 \in \partial\Omega \cap D(T)$, $p \in T\mathbf{x}_0$, which is a contradiction.

As in [13], we define $\Omega_\epsilon = \{\mathbf{x} \in \Omega : d(\mathbf{x}, \partial\Omega) > 2\epsilon\}$ for each $\epsilon > 0$. By definition of VMO functions, there exists $\epsilon_0 > 0$ such that

$$\frac{1}{m(B_\epsilon(\mathbf{x}))} \int_{B_\epsilon(\mathbf{x})} |f(\mathbf{y}) - \bar{f}| d\mathbf{y} < \frac{\beta}{2} \tag{2.2}$$

for all $\epsilon < \epsilon_0$, $\mathbf{x} \in \Omega$ and $\epsilon < \frac{d(\mathbf{x}, \partial\Omega)}{2}$. We may also take ϵ_0 such that $\{\mathbf{x} \in \Omega : d(\mathbf{x}, \partial\Omega) \leq 3\epsilon_0\} \subset \mathcal{U}$, where \mathcal{U} is the same as in (2.1). Now for $0 < \epsilon < \epsilon_0$, and $\mathbf{x} \in \partial\Omega_\epsilon \cap D(T)$, $g \in T\mathbf{x}$, by (2.1) and (2.2), we obtain

$$|f_\epsilon(\mathbf{x}) + g - p| \geq \frac{\beta}{2}, \tag{2.3}$$

where $f_\epsilon(\mathbf{x}) = \frac{1}{m(B_\epsilon(\mathbf{x}))} \int_{B_\epsilon(\mathbf{x})} f(\mathbf{y}) d\mathbf{y}$.

Lemma 2.2. Suppose $|f_\epsilon(x) + g - p| \geq \frac{\beta}{2}$, for $x \in \partial\Omega_\epsilon \cap D(T)$, $g \in Tx$. Then there exists $\lambda_0(\epsilon) > 0$ such that

$$p \neq f_\epsilon(x) + T_\lambda(x), \text{ for all } x \in \partial\Omega_\epsilon, \lambda \in (0, \lambda_0(\epsilon)).$$

Proof. If this is not true, there exist $\lambda_n \rightarrow 0^+$, $x_n \in \partial\Omega_\epsilon$ with $x_n \rightarrow x_0 \in \partial\Omega_\epsilon$, such that

$$f_\epsilon x_n + T_{\lambda_n} x_n = p, \quad n \in \{1, 2, \dots\}.$$

Since $f_\epsilon x_n \rightarrow f_\epsilon x_0$, $R_{\lambda_n} x_n = x_n - \lambda_n T_{\lambda_n} x_n \rightarrow x_0$, the maximal monotonicity of T implies that $x_0 \in D(T)$, and $p - f_\epsilon x_0 \in Tx_0$, which is a contradiction.

Now, assume that (2.1) holds. In view of (2.3) and Lemma 2.2, we define the degree $\deg(f + T, \Omega \cap D(T), p)$ by

$$\deg(f + T, \Omega \cap D(T), p) = \lim_{\epsilon \rightarrow 0^+} \lim_{\lambda \rightarrow 0^+} \deg(f_\epsilon + T_\lambda, \Omega_\epsilon, p). \quad (2.4)$$

We claim this definition is reasonable. First, for each $\epsilon < \epsilon_0$, and $\lambda_1, \lambda_2 \in (0, \lambda_0(\epsilon))$, since $T_{t\lambda_1 + (1-t)\lambda_2} x$ is continuous in (t, x) (see Corollary 2.8 in [15]) we know that $\{f_\epsilon + T_{t\lambda_1 + (1-t)\lambda_2}\}_{t \in [0, 1]}$ is a homotopy, so

$$\deg(f_\epsilon + T_{\lambda_1}, \Omega_\epsilon, p) = \deg(f_\epsilon + T_{\lambda_2}, \Omega_\epsilon, p).$$

Now, for any $\epsilon \in (0, \epsilon_0)$, by the continuity of $f_t(x)$ in (t, x) and (2.3), there exists $\delta > 0$ such that

$$|f_t(x) + g - p| > \frac{\beta}{4},$$

for $|t - \epsilon| \leq \delta$ and $x \in \partial\Omega_\epsilon$ and $g \in Tx$. The same proof as in Lemma 2.2 guarantees that there exists $\lambda_1 > 0$ such that

$$p \neq f_t(x) + T_\lambda(x), \text{ for all } x \in \partial\Omega_\epsilon, |t - \epsilon| \leq \delta, \lambda \in (0, \lambda_1),$$

so $\deg(f_t + T_\lambda, \Omega_\epsilon, p)$ is well defined for $\lambda \in (0, \lambda_1)$, and $|t - \epsilon| \leq \delta$. By homotopy invariance, we have

$$\deg(f_t + T_\lambda, \Omega_\epsilon, p) = \deg(f_\epsilon + T_\lambda, \Omega_\epsilon, p),$$

so the degree in (2.4) is well defined.

For a measurable function $f: \Omega \rightarrow \mathbb{R}^n$, we recall that the essential range of f is defined as the smallest closed subset $\text{essR}(f)$ such that $f(x) \in \text{essR}(f)$ a. e. $x \in \Omega$ (see [12]).

Proposition 2.3. If $\deg(f + T, \Omega \cap D(T), p) \neq 0$, then $p \in \overline{\text{essR}(f) + T(\overline{\Omega \cap D(T)})}$.

Proof. Suppose the conclusion is not true. Then exists $r > 0$ such that $B(p, r) \cap \text{essR}(f) + T(\overline{\Omega \cap D(T)}) = \emptyset$. Set $\Sigma = \mathbb{R}^n \setminus (B(p, r) - T(\overline{\Omega \cap D(T)}))$. Clearly, $\text{essR}(f) \subset \Sigma$. Also $f(x) \in \text{essR}(f)$,

a. e. $x \in \Omega$, and $f \in \text{VMO}(\Omega)$, so we deduce that $\lim_{\epsilon \rightarrow 0^+} d(f_\epsilon(x), \Sigma) = 0$ uniformly. Therefore, there exists $\epsilon_1 \in (0, \epsilon_0)$ such that

$$|f_\epsilon(x) - p + g| \geq \frac{r}{2},$$

for all $x \in \Omega$, $z \in D(T) \cap \overline{\Omega}$, $g \in Tz$, $\epsilon \in (0, \epsilon_1)$.

Thus $\deg(f_\epsilon + T_\lambda, \Omega, p) = 0$ for all $\lambda \in (0, \lambda_0(\epsilon))$, and $\epsilon \in (0, \epsilon_1)$. Consequently, it follows from the definition that $\deg(f + T, \Omega \cap D(T), p) = 0$, which is a contradiction.

Proposition 2.4. Let $\{h_t(\cdot)\}_{t \in [0,1]}$ be a family of functions in $\text{VMO}(\Omega)$, and $h_t(\cdot)$ depends continuously on the parameter t in the topology of $\text{BMO} \cap L^1_{\text{loc}}(\Omega)$. Assume that there exists an open neighborhood U of $\partial\Omega$ in Ω and a constant $\beta > 0$ such that

$$\frac{1}{m(B_\epsilon(y))} \int_{B_\epsilon(y)} |h_t(x) + g - p| dx \geq \beta \tag{2.5}$$

for all $0 < \epsilon < \frac{1}{2}d(y, \partial\Omega)$, $g \in Tz$, $z \in D(T) \cap B_\epsilon(y)$, $t \in [0, 1]$, where $B_\epsilon(y)$ is an open ball centered at y with radius ϵ such that $\overline{B_\epsilon(y)} \subset U$. Then $\deg(h_t + T, \Omega \cap D(T), p)$ does not depend on $t \in [0, 1]$.

Proof. Since $h_t(\cdot)$ depends continuously on the parameter t in the topology of $\text{BMO} \cap L^1_{\text{loc}}(\Omega)$, we have

$$\lim_{m(B) \rightarrow 0} \frac{1}{m(B)} \int_B |h_t(x) - \overline{h_t}| = 0, \tag{2.6}$$

uniformly in t . From (2.5), (2.6), and using the same proof as in (2.3), we know that there exists $\epsilon_0 > 0$, such that

$$|h_{t,\epsilon}(x) + g - p| \geq \frac{\beta}{2}, \tag{2.7}$$

for all $x \in \partial\Omega_\epsilon \cap D(T)$, $g \in Tx$, $t \in [0, 1]$, $\epsilon \in (0, \epsilon_0)$. By using the same proof as in Lemma 2.2, we know that there exists $\lambda(\epsilon) > 0$, such that

$$p \neq h_{t,\epsilon}(x) + T_\lambda x,$$

for all $x \in \partial\Omega_\epsilon$, $t \in [0, 1]$, $\lambda \in (0, \lambda(\epsilon))$. Thus $\deg(h_{t,\epsilon} + T_\lambda, \Omega_\epsilon, p)$ does not depend on t for each $\epsilon \in (0, \epsilon_0)$, $\lambda \in (0, \lambda(\epsilon))$. Thus $\deg(h_t + T, \Omega \cap D(T), p)$ does not depend on $t \in [0, 1]$.

Corollary 2.5. Let $f_1, f_2 \in \text{VMO}(\Omega)$ satisfying (2.1). Suppose there exists $0 < \beta_0 < \beta$ such that

$$\frac{1}{m(B)} \int_B |f_1(x) - f_2(x)| dx < \beta_0,$$

for all $B \subset U$. Then $\deg(f_1 + T, \Omega \cap D(T), p) = \deg(f_2 + T, \Omega \cap D(T), p)$.

Proof. Set $h_t(x) = tf_1(x) + (1-t)f_2(x)$ for $t \in [0, 1]$, $x \in \Omega$. Then it is easy to see that h_t depends continuous on t in the topology of $\text{BMO} \cap L^1_{\text{loc}}(\Omega)$. Also we have

$$\frac{1}{m(B_\epsilon(y))} \int_{B_\epsilon(y)} |h_t(x) + g - p| dx \geq \beta - \beta_0$$

for all $0 < \epsilon < \frac{1}{2}d(y, \partial\Omega)$, $g \in Tz$, $z \in D(T) \cap B_\epsilon(y)$, $t \in [0, 1]$, where $B_\epsilon(y)$ is an open ball centered at y with radius ϵ such that $\overline{B_\epsilon(y)} \subset U$. Therefore the conclusion follows from Proposition 2.4.

Proposition 2.6. Let $T_i : D \subseteq \mathbb{R}^n$, $i = 1, 2$, be two maximal monotone maps. If $tT_1 + (1-t)T_2$ is maximal monotone for each $t \in [0, 1]$, and there exist an open neighborhood U of $\partial\Omega$ in Ω and a constant $\beta > 0$ such that

$$\frac{1}{m(B_\epsilon(y))} \int_{B_\epsilon(y)} |f(x) + g_t - p| dx \geq \beta \quad (2.8)$$

for all $0 < \epsilon < \frac{1}{2}d(y, \partial\Omega)$, $g_t \in [tT_1 + (1-t)T_2]z$, $z \in D \cap B_\epsilon(y)$, $t \in [0, 1]$, where $B_\epsilon(y)$ is an open ball centered at y with radius ϵ such that $\overline{B_\epsilon(y)} \subset U$. Then $\deg(f + [tT_1 + (1-t)T_2], \Omega \cap D, p)$ does not depend on $t \in [0, 1]$.

Proof. By (2.8), using the same proof as in (2.3), we know that there exists $\epsilon_0 > 0$, such that

$$|f_\epsilon(x) + g_t - p| \geq \frac{\beta}{2}, \quad (2.9)$$

for all $x \in \partial\Omega_\epsilon \cap D$, $g_t \in tT_1x + (1-t)T_2x$, $t \in [0, 1]$, $\epsilon \in (0, \epsilon_0)$. From (2.9), and using the same proof as in Lemma 2.2, we know that there exists $\lambda(\epsilon) > 0$, such that

$$p \neq f_\epsilon(x) + T_\lambda^t x,$$

for all $x \in \partial\Omega_\epsilon$, $t \in [0, 1]$, $\lambda \in (0, \lambda(\epsilon))$, where T_λ^t is the Yosida approximation of $tT_1 + (1-t)T_2$. From Lemma 2.7 in [15], we know

$$\deg(f_\epsilon + T_\lambda^t, \Omega_\epsilon, p)$$

does not depend on $t \in [0, 1]$, $\lambda \in (0, \lambda(\epsilon))$. Therefore, $\deg(f + [tT_1 + (1-t)T_2], \Omega \cap D, p)$ does not depend on $t \in [0, 1]$.

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