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Remarks on the Generation of Semigroups of Nonlinear Operators on p-Fréchet Spaces, $0 < p < 1$

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ABSTRACT

In this paper we study the convergence properties of the Crandall-Liggett sequence $J^n_{t/n}(A)(x) = \left(I - \frac{t}{n}A\right)^{-n}(x)$, $n \in \mathbb{N}$, for A a nonlinear operator on some important non-locally convex F-spaces (called p-Fréchet spaces with $0 < p < 1$) and the generation of the corresponding strongly continuous one-parameter nonlinear semigroups.

RESUMEN

En este trabajo se estudian las propiedades de convergencia de la secuencia de Crandall-Liggett $J^n_{t/n}(A)(x) = \left(I - \frac{t}{n}A\right)^{-n}(x)$, $n \in \mathbb{N}$ para A un operador lineal en algunos importantes F-espacios no-localmente convexos (llamado p-Fréchet espacios con $0 < p < 1$) y la generación de los correspondientes semigrupos fuertemente continuos no lineales con un parámetro.

Keywords and phrases:: p-Fréchet space, $0 < p < 1$, Cauchy problem, affine semigroup, nonlinear semigroup, Crandall-Liggett type theorem.

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1. Introduction

It is well known that an F-space $(X, +, \cdot, \|\cdot\|)$ is a linear space (over the field $K = \mathbb{R}$ or $K = \mathbb{C}$) such that $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$, $\|x\| = 0$ if and only if $x = 0$, $\|\lambda x\| \leq \|x\|$, for all scalars λ with $|\lambda| \leq 1, x \in X$, and with respect to the metric $d(x, y) = \|x - y\|$, X is a complete metric space (see e.g. [4, p. 52] or [7]).

In addition, if there exists $0 < p < 1$ with $\|\lambda x\| = |\lambda|^p \|x\|$, for all $\lambda \in K, x \in X$, then $\|\cdot\|$ will be called a p-norm and X will be called p-Fréchet space. (This is only a slight abuse of terminology. Note that in e.g. [1] these spaces are called p-Banach spaces).

It is known that the F-spaces are not necessarily locally convex spaces. Three classical examples of p-Fréchet spaces, non-locally convex, are the Hardy space H^p with $0 < p < 1$ that consists in the class of all analytic functions $f : D \to \mathbb{C}$, $D = \{z \in \mathbb{C}; |z| < 1\}$ with the property

$$\|f\| = \frac{1}{2\pi} \sup\{\int_0^{2\pi} |f(re^{it})|^p dt; r \in [0, 1)\} < +\infty,$$

the sequence space

$$l^p = \{x = (x_n)_n; \|x\| = \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

for $0 < p < 1$, and the $L^p[0, 1]$ space, $0 < p < 1$, given by

$$L^p = L^p[0, 1] = \{f : [0, 1] \to \mathbb{R}; \|f\| = \int_0^1 |f(t)|^p dt < \infty.\}$$

Some important characteristics of the F-spaces are given by the following remarks.

Remarks. 1) Three of the basic results in Functional Analysis hold in F-spaces too : the Principle of Uniform Boundedness (see e.g. [4, p. 52]), the Open Mapping Theorem and the Closed Graph Theorem (see e.g. [7, p. 9-10]).

2) The Hahn-Banach Theorem fails in non-locally convex F-spaces. More exactly, if in an F-space the Hahn-Banach theorem holds, then that space is a necessarily locally convex space (see e.g. [6, Chapter 4]).

The beginning of a theory of semigroups of linear operators on p-Fréchet spaces, $0 < p < 1$, was developed in the very recent paper [5]. One of the main result in [5] is the Chernoff-type formula $e^{tA}(x) = \text{lím}_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n}(x)$, for A a bounded linear operator on a p-Fréchet space with $0 < p < 1$.

The aim of the present paper is to look for similar results, that is for convergence properties of the sequence $J^n_{t/n}(A)(x) = \left(I - \frac{t}{n}A\right)^{-n}(x), n \in \mathbb{N}$, in the case when A is a nonlinear operator on a p-Fréchet space with $0 < p < 1$. A very careful examination of the proofs in [3] shows us that because of the property $\|\lambda x\| = |\lambda|^p \|x\|$ with $0 < p < 1$, the estimate for $\|J^n_{t/n}(A)(x) - J^m_{t/m}(A)(x)\|$ does not converges to zero as $m, n \to \infty$ and in fact the sequence $J^n_{t/n}(A)(x), n \in \mathbb{N}$, is not, in general, a convergent one.

However, by using techniques in Functional Analysis, we will be able to prove that the sequence $(J^n_{t/n}(A)(x))_{n\in\mathbb{N}}$ contains some convergent subsequences in the spaces l^p and H^p with $0 < p < 1$, while this kind of result seems to fail in the space $L^p[0,1]$, $0 < p < 1$. Moreover, in the simplest nonlinear case when A is an affine operator, we prove that the sequence $(J^n_{t/n}(A)(x))_{n\in\mathbb{N}}$ is still convergent and some results in the case of Banach spaces in [6] will be extended to p-Fréchet spaces $(0 < p < 1)$ too.

The plan of the paper goes as follows. In Section 2 we study the case when A is an affine operator on an arbitrary p-Fréchet space, $0 < p < 1$, Section 3 deals with the case when A is a nonlinear Lipschitz operator on l^p, $0 < p < 1$, while the Sections 4 and 5 deal with the similar problem in the spaces H^p and $L^p[0,1]$, respectively, with $0 < p < 1$.

2. Affine Semigroups

As we will see, the affine case is closely connected to the linear case.

First we need a result in operator theory on p-Fréchet spaces (well-known in the case of classical Banach spaces).

Lemma 2.1 *Let* $A, B : X \to X$ *be bounded linear operators on the* p*-Fréchet space* $(X, \|\cdot\|)$, $0 < p < 1$. *If* A *is bijection and* $|||A^{-1}B||| < 1$ *then* $A + B$ *is bounded linear bijection on* X.

Proof. Since A is a bijection, as a consequence of the Open Mapping Theorem it follows that A^{-1} is a bounded linear operator (see e.g. [1, Theorem 14, p. 20 and Corollary 2, p. 23]).

Next we reason as in the case of Banach spaces. Let $y \in X$ be arbitrary fixed and define $T_y(x) = A^{-1}(y) - (A^{-1}B)(x)$. Then the equation $(A + B)(x) = y$ is equivalent to the equation $T_y(x) = x$. But $\|T_y(x_1) - T_y(x_2)\| \leq |||A^{-1}B||| \cdot \|x_1 - x_2\|$, which shows that T_y is a contraction in the complete metric space X (with respect to the metric $d(x_1, x_2) = \|x_1 - x_2\|$). Therefore it has a unique fixed point x, which shows that $A + B$ is bijective and the lemma is proved.

The first result on affine semigroups is the following.

Theorem 2.2 *Let* $(X, \|\cdot\|)$ *be a* p*-Fréchet space,* $0 < p < 1$, $A(x) = B(x) + x_0$, *where* $x_0 \in X$ *is fixed and* $B : X \to X$ *is a bounded, linear and strictly dissipative operator, i.e* $|||(I - \lambda B)^{-1}||| < 1$, *for all* $\lambda > 0$ *sufficiently small. Then* B *is invertible and if we define*

$$J_\lambda(A)(x) = (I - \lambda A)^{-1}(x),$$

(here I *defines the identity operator) then*

$$T(t)(x) = \lim_{n\to+\infty} J^n_{t/n}(A)(x) = e^{tB}(x) + B^{-1}[e^{tB}(x_0)] - B^{-1}(x_0),$$

is a strongly continuous semigroup of nonlinear (affine) operators on X, *(where according to [4],* $e^{tB}(x) = \lim_{n\to+\infty} J^n_{t/n}(B)(x)$ *is a strongly continuous semigroup of linear operators on* X*).*

Proof. By easy calculation we can write

$$J_\lambda(A)(x) = J_\lambda(B)(x+\lambda x_0) = (I-\lambda B)^{-1}(x+\lambda x_0),$$

and in general

$$J_\lambda^n(A)(x) = (I-\lambda B)^{-n}(x) + \lambda\left[\sum_{k=1}^{n}(I-\lambda B)^{-k}(x_0)\right].$$

But it is easy to show that for any operator G we have the identity

$$(I-G)(I+G+G^2+...+G^{n-1}) = I-G^n.$$

Replacing G by $J_\lambda(B)$, by Lemma 2.1 it follows that $I-J_\lambda(B)$ is invertible and we immediately obtain

$$\left[\sum_{k=1}^{n}(I-\lambda B)^{-k}(x_0)\right] = \lambda J_\lambda(B)[I-J_\lambda(B)]^{-1}[I-J_\lambda^n(B)]^{-n}(x_0).$$

But

$$J_\lambda(B)[I-J_\lambda(B)]^{-1} = (I-\lambda B)^{-1}[I-J_\lambda(B)]^{-1} =$$

$$\{[I-(I-\lambda B)^{-1}][I-\lambda B]\}^{-1} =$$

$$\{-\lambda B\}^{-1} = -\frac{1}{\lambda}B^{-1},$$

which implies that

$$\left[\sum_{k=1}^{n}(I-\lambda B)^{-k}(x_0)\right] = -B^{-1}[(I-\lambda B)^{-n}](x_0).$$

Taking $\lambda = \frac{t}{n}$, passing to limit with $n \to +\infty$ and taking into account the important Remark after the Theorem 2.11 in [4] which says that

$$e^{tB}(x) = \lim_{n\to+\infty}(I-\frac{t}{n}B)^{-n},$$

we arrive at

$$\lim_{n\to+\infty} J_{t/n}^n(A)(x) = e^{tB} + B^{-1}[e^{tB}(x_0)] - B^{-1}(x_0).$$

Also, simple calculations show that if we denote $T(t)(x) = e^{tB}(x) + B^{-1}[e^{tB}(x_0)] - B^{-1}(x_0)$, then $T(0) = I$, $\{T(t), t \geq 0\}$ has the semigroup property, $T(\cdot)(x)$ is continuous as function of t, $A(x) = \lim_{h\searrow 0}\frac{T(h)(x)-x}{h}$, for all $x \in X$ and $T'(t)(x) = B[T(t)(x)] + x_0$, which proves the theorem.

Remarks. 1) According to Theorem 2.2, $T(t)(u_0)$ is the unique solution of the Cauchy problem

$$u'(t) = B[u(t)] + x_0, u(0) = u_0.$$

(The uniqueness of the solution follows from Lemma 2.12 in [5] concerning the uniqueness of the solution for the inhomogeneous Cauchy problem in p-Fréchet spaces, $0 < p < 1$.)

2) Let us give a simple example satisfying Theorem 2.2. Let $(X, \|\cdot\|)$ be a p-Fréchet space, $0 < p < 1$, and define $A : X \to X$ by $A(x) = B(x) + x_0$, where $B(x) = -x$ for all $x \in X$ and $x_0 \in X$ is fixed. B obviously is strictly dissipative and A obviously is nonlinear, strictly dissipative, with

$$\|(I - \lambda A)^{-1}\|_{Lip} = \frac{1}{1+\lambda} < 1,$$

for all $\lambda > 0$.

We see that $B^{-1} = B$, $e^{tB}(x) = xe^{-t}$ and $T(t)(x) = (x - x_0)e^{-t} + x_0$ and in this case $T(t)(u_0)$ is the unique solution to the nonlinear Cauchy problem

$$\frac{du}{dt} = -u(t) + x_0, u(0) = u_0.$$

3) From the proof of Theorem 2.2, it easily follows the following.

Corollary 2.3 *In the case when* $(X, \|\cdot\|)$ *is a Banach space (i.e. a* p*-Fréchet space with* $p = 1$*), the statement of Theorem 2.2 still remains true.*

In what follows, let us consider some concepts introduced in [6] for Banach spaces. They remain unchanged for the case of p-Fréchet spaces too.

Definition 2.4 By an affine semigroup $(S(t) : t \geq 0)$ on a p-Fréchet space X, $0 < p < 1$, we mean a family of continuous affine transformations on X with the properties :

(i) S(0)=I, S(t+s)=S(t)[S(s)], for all $t, s \geq 0$;

(ii) For each $x \in X$, $t \to S(t)(x)$ is a continuous function from $[0, +\infty)$ into X.

(iii) Any family $(S(t) : t \geq 0)$ of affine transformations on X can be written in the form $S(t)(x) = T(t)(x) + z(t)$, for all $t \geq 0$, $x \in X$, where $T(t)(x) = S(t)(x) - S(t)(0)$ is its linear part and $z(t) = S(t)(0)$ is its translation part $(z : [0, +\infty) \to X)$.

(iv) Let us denote by $\overline{X} = X \times \mathbb{R}$. It is a p-Fréchet space, endowed with the p-norm $\|(x, r)\| = \text{máx}\{\|x\|, |r|^p\}$. If $(S(t) : t \geq 0)$ is a family of affine transformations on X of the form $S(t)(x) = T(t)(x)+z(t)$, for all $t \geq 0$, $x \in X$, where $T(t)(x)$ is its linear part and $z(t)$ is its translation part, the augmented family associated with $(S(t) : t \geq 0)$, is a family $(\overline{T}(t); t \geq 0)$ of linear transformations on $\overline{X}$, defined by

$$\overline{T}(t)[x, r] = [T(t)(x) + rz(t), r].$$

Having introduced these concepts, Propositions 1.1 and 1.2 proved in [6] for Banach spaces, hold (with the same proofs) for p-Fréchet spaces too, summarized as follows.

Theorem 2.5 *(i) Let* $(S(t) : t \geq 0)$ *be a family of affine transformations on the* p*-Fréchet space* X*,* $0 < p < 1$*, with its linear part* $(T(t) : t \geq 0)$ *and its translation part* $z(t); t \geq 0)$*. Then* $(S(t) : t \geq 0)$ *is an affine semigroup on* X *if and only if* $(T(t) : t \geq 0)$ *is a linear semigroup on* X *and* $z(\cdot)$ *is a continuous map from* $[0, +\infty)$ *into* X *satisfying*

$$z(t + s) = T(t)[z(s)] + z(t), s, t \geq 0.$$

(ii) Let $(S(t) : t \geq 0)$ *be a family of affine transformations on the* p*-Fréchet space* X, $0 < p < 1$, *and let* $(\overline{T}(t) : t \geq 0)$ *be the augmented family on* $\overline{X}$*, associated with* $S(\cdot)$*. Then* $(S(t) : t \geq 0)$ *is an affine semigroup on* X*, if and only if* $(\overline{T} : t \geq 0)$ *is a linear semigroup on* $\overline{X}$*.*

Remark. While Proposition 2.1 in [6] remains valid in the case of p-Fréchet spaces too, $0 < p < 1$, the other results in [6, Section 2] (i.e. Corollary 2.2, Proposition 2.3, Proposition 2.4 and Corollary 2.5) seem to be not valid. The reason is that they use the Fundamental Theorem of Calculus in Banach spaces, which, as it was pointed out in [5], does not hold in p-Fréchet spaces, $0 < p < 1$.

It would be of interest to see what other results for affine semigroups on Banach spaces in [6], would remain valid for p-Fréchet spaces too, $0 < p < 1$.

3. Nonlinear Semigroups on l^p, $0 < p < 1$

Before to starting the study in the concrete l^p-case, $0 < p < 1$, let us briefly recall the problem and make a useful remark, valid in any p-Fréchet space, $0 < p < 1$.

For $(X, \|\cdot\|_X)$ a p-Fréchet space, $0 < p \leq 1$ (the case $p = 1$ means that X is a Banach space), let $A : X \to X$ be a nonlinear operator and let us consider the abstract Cauchy problem

$$\frac{d}{dt}u(t) = A[u(t)], t \geq 0,$$

$$u(0) = x,$$

where the solution is $u : \mathbb{R}_+ \to X$ and $x \in X$ is fixed. The nonlinear operator A is considered a Lipschitz mapping, that is

$$\|A(x) - A(y)\|_X \leq |||A|||_{Lip}\|x - y\|_X, \text{ for every } x, y \in X,$$

where $|||A|||_{Lip} = \sup\{\|A(x)-A(y)\|_X/\|x-y\|_X; x, y \in X, x \neq y\} < +\infty$. If we replace this differential equation by the difference equation

$$\frac{1}{\varepsilon}[u_\varepsilon(t) - u_\varepsilon(t - \varepsilon)] = A[u_\varepsilon(t)], t \geq 0,$$

with initial condition $u_\varepsilon(s) = x, -\varepsilon \leq s \leq 0$, then we easily get by recurrence that $u_\varepsilon(t) = \left(I - \frac{t}{n}A\right)^{-n}(x)$, for $\varepsilon = \frac{t}{n}$.

Remark. Without loss of generality, we may suppose $A(0) = 0$. Indeed, if we suppose that $A(0) \neq 0$, then denoting $B(u) = A(u) - A(0)$ we get $B(0) = 0$ and if $v(t)$ is solution of the abstract Cauchy problem

$$\frac{d}{dt}v(t) = B[v(t)], v(0) = u_0,$$

then $u(t) = v(t)+tA(0)$ is a solution of the above (in A) mentioned problem. Moreover, if for a fixed $\omega \in \mathbb{R}$, the operator $A - \omega I$ is dissipative, then $B - \omega I$ also is dissipative. Indeed, from $B - \omega I =$

$(A - \omega I) - A(0)$, since $A - \omega I$ is injective and surjective, it easily follows that $B - \omega I$ is injection and surjection and , in addition, from the relationship $(B - \omega I)^{-1}(y) = (A - \omega I)^{-1}(y + A(0))$, we get $|||(B - \omega I)^{-1}|||_{Lip} = |||(A - \omega I)^{-1}|||_{Lip} \leq 1$.

In what follows, we denote the p-norm in l^p by $\|\cdot\|_p$. The first main result of this section is the following.

Theorem 3.1 *Let* $A : l^p \to l^p$, $0 < p < 1$, *be nonlinear and Lipschitz, such that* $A(0) = 0$ *and there exists* $\omega \in \mathbb{R}$ *with* $A - \omega I$ *dissipative. Then the sequence in* l^p *defined by* $J^n_{t/n}(x) = (I - \frac{t}{n}A)^{-n}(x), n \in \mathbb{N}, t \geq 0, x \in l^p$, *contains a subsequence* $J^{n_k}_{t/n_k}(x), k \in \mathbb{N}$ *(the same subsequence for all* $x \in l^p$ *and all* $t \in \mathcal{R}_+$ *= the set of all rational numbers* ≥ 0*), convergent to an element of* l^p *in the weak topology of* l^p.

Proof. By $A(0) = 0$ we get $(I - \frac{t}{n}A)^{-n}(0) = 0$, for all $n \in \mathbb{N}$. The dissipative property implies $|||[I - t(A - \omega I)]^{-1}|||_{Lip} \leq 1$, for all $t \geq 0$, which is equivalent to $|||(I - \frac{t}{1+t\omega}A)^{-1}|||_{Lip} \leq |1 + t\omega|^p$, for all $t \geq 0$ with $1 + t\omega \neq 0$. For $\lambda = \frac{t}{1+t\omega}$ we get $|||(I - \lambda A)^{-1}|||_{Lip} \leq (1 - \lambda\omega)^{-p}$, in particular for all $\lambda > 0$ with $\lambda\omega < 1$. (Note that for n sufficiently great, depending on t and ω, we have $\frac{t}{n}\omega < 1$.) Therefore,

$$|||(I - \frac{t}{n}A)^{-1}|||_{Lip} \leq (1 - \frac{t}{n}\omega)^{-p}$$

and by mathematical induction

$$|||(I - \frac{t}{n}A)^{-n}|||_{Lip} \leq (1 - \frac{t}{n}\omega)^{-np}.$$

But it is known that the sequence $(1 + \frac{s}{n})^n$ converges (for $n \to +\infty$) to e^s, and for any $s \in \mathbb{R}$ it is monotonically increasing, for all $n \geq [|s|]+1$ (see e.g. [10, p. 263]), which implies that $(1 - \frac{t}{n}\omega)^{-np}$ converges to $e^{t\omega p}$, monotonically decreasing, for all $n \geq [|tw|] + 1$. Therefore, the greatest value of $(1 - \frac{t}{n}\omega)^{-np}$ is for $n = [|t\omega|] + 1$, which means that there exists $M = M(t, p, \omega) > 0$ (depending only on t, p and ω) such that

$$|||(I - \frac{t}{n}A)^{-n}|||_{Lip} \leq M,$$

for all $n \in \mathbb{N}$.

We obtain

$$\|J^n_{t/n}(x) - J^n_{t/n}(y)\|_p \leq (1 - \frac{t}{n}\omega)^{-np}\|x - y\|_p \leq M\|x - y\|_p, \tag{3.1}$$

for all $x, y \in l^p$. Taking $y = 0$ we have $J^n_{t/n}(0) = 0$ and denoting $J^n_{t/n}(x) = (g_{n,r}(t)(x))_r \in l^p$, we obtain $\|J^n_{t/n}(x)\|_p \leq M\|x\|_p$, i.e.

$$\sum_{r=1}^{+\infty} |g_{n,r}(t)(x)|^p \leq M\|x\|_p < +\infty, \tag{3.2}$$

for all $n \in \mathbb{N}$.

Now, since l^p, $0 < p < 1$, has a Schauder basis (see e.g. [7, p. 20]), it follows that it is separable, denote by Y a countable dense subset of l^p. Also, denote by $\mathcal{R}_+$, the set of all positive nonnegative rational numbers and define $G_n : \mathbb{N} \times \mathcal{R}_+ \times Y \to K$ (where $K = \mathbb{R}$ or $\mathbb{C}$), by $G_n(r, t, y) = g_{n,r}(t)(y)$.

Since $E := \mathbb{N} \times \mathcal{R}_+ \times Y$ is countable and by (2) the sequence $(G_n)_n$ is pointwise bounded on E, by the Cantor's diagonal process (see e.g. [11, p. 156-157]), there exists a subsequence $G_{n_k}, k \in \mathbb{N}$, pointwise convergent on E. Denote $g_r(t)(y) = \lim_{k\to+\infty} g_{n_k,r}(t)(y)$, for all $(r, t, y) \in E$.

We will show that in fact there exists the limit (in $\mathbb{R}$), $\lim_{k\to\infty} g_{n_k,r}(t)(x)$, for all $r \in \mathbb{N}$, $t \in \mathcal{R}_+$ and $x \in l^p$. For this purpose, we will show that $(g_{n_k,r}(t)(x))_k$ is a Cauchy sequence in $\mathbb{R}$ (i.e. it is convergent).

For this purpose, let $x \in l^p$ and $y \in Y$. We have

$$|g_{n_k,r}(t)(x) - g_{n_s,r}(t)(x)| \le |g_{n_k,r}(t)(x) - g_{n_k,r}(t)(y)| +$$

$$|g_{n_k,r}(t)(y) - g_{n_s,r}(t)(y)| + |g_{n_s,r}(t)(y) - g_{n_s,r}(t)(x)|.$$

Taking into account (1) too, we immediately obtain

$$|g_{n_k,r}(t)(x) - g_{n_s,r}(t)(x)| \le 2M^{1/p} \cdot \|x - y\|_p^{1/p} + |g_{n_k,r}(t)(y) - g_{n_s,r}(t)(y)|.$$

Now, since Y is dense in l^p, for $x \in l^p$ and $\varepsilon > 0$, let $y \in Y$ such that $2M^{1/p}\|x - y\|_p^{1/p} < \frac{\varepsilon}{2}$, which implies

$$|g_{n_k,r}(t)(x) - g_{n_s,r}(t)(x)| < \frac{\varepsilon}{2} + |g_{n_k,r}(t)(y) - g_{n_s,r}(t)(y)|.$$

But for this y, the sequence $(g_{n_k,r}(t)(y))_k$ is convergent, i.e. it is a Cauchy sequence, which implies that there exists l_0 such that for all $k, s \ge l_0$ we have

$$|g_{n_k,r}(t)(y) - g_{n_s,r}(t)(y)| < \frac{\varepsilon}{2}.$$

This leads to

$$|g_{n_k,r}(t)(x) - g_{n_s,r}(t)(x)| < \varepsilon,$$

for all $k, s \ge l_0$, i.e. $(g_{n_k,r}(t)(x))_k$ is a Cauchy sequence in $\mathbb{R}$. Therefore, we can write

$$g_r(t)(x) = \lim_{k\to\infty} g_{n_k,r}(t)(x),$$

for all $t \in \mathcal{R}_+$ and $x \in l^p$.

By (2) it follows

$$\sum_{r=1}^{m} |g_{n_k,r}(t)(x)|^p \le M\|x\|_p < +\infty,$$

for all $k, m \in \mathbb{N}$. Passing here to limit with $k \to \infty$, we get

$$\sum_{r=1}^{m} |g_r(t)(x)|^p \le M\|x\|_p < +\infty,$$

for all $m \in \mathbb{N}$, which obviously implies

$$\sum_{r=1}^{\infty} |g_r(t)(x)|^p \leq M\|x\|_p < +\infty,$$

i.e. $g(t)(x) := (g_r(t)(x))_r$ belongs to l^p.

Now, we will show that for any $x^* \in (l^1)^*$, i.e. of the form (see e.g. [8, pp. 36-37]) $x^*(z) = \sum_{j=1}^{\infty} u_j z_j$, for all $z \in l^1$, where $(u_j)_j \in m$, with m denoting the space of all bounded sequences, we have $x^*(J^{n_k}_{t/n_k}(x)) \to x^*(g(t)(x))$, when $k \to \infty$, for any fixed $t \in \mathcal{R}_+, x \in l^p$. Note that $x^*(g(t)(x))$ has sense for $g(t)(x) \in l^p$, because $l^p \subset l^1$.

It is obvious that each functional of the form $x_i^*(x) = x_i$, for all $x = (x_i)_i \in l^1$, is linear and continuous on l^1, since $|x_i^*(x)| = |x_i| \leq \sum_{j=1}^{\infty} |x_j| = \|x\|_{l^1}$ and for $k \to \infty$, $x_i^*[J^{n_k}_{t/n_k}(x)] = g_{n_k,i}(t)(x) \to g_i(t)(x) = x_i^*(g(t)(x))$, for all $i \in \mathbb{N}$.

Then, obviously that for any $y^* \in span\{x_1^*, ..., x_i^*, ...,\} =: Y^*$ we also have $y^*[J^{n_k}_{t/n_k}(x)] \to y^*[g(t)(x)]$, for $k \to \infty$.

We show that Y^* is dense in $(l^1)^*$ in the weak topology on $(l^1)^*$. Indeed, let $x^* \in (l^1)^*$ be arbitrary, $x^*(u) = \sum_{i=1}^{\infty} \alpha_i u_i$, for all $u = (u_j)_j \in l^1$, where $\alpha = (\alpha_j)_j \in m$. Since $z_n^*(u) = \sum_{j=1}^{n} \alpha_j u_j = \sum_{j=1}^{n} \alpha_j x_j^*(u)$, it follows $z_n^* \in Y^*$ and we get

$$|x^*(u) - z_n^*(u)| \leq \sum_{j=n+1}^{+\infty} |\alpha_j u_j| \leq \|\alpha\|_m \sum_{i=n+1}^{+\infty} |u_i| \leq M_0 \sum_{i=n+1}^{+\infty} |u_i| \to 0,$$

for $n \to \infty$.

This implies that $z_n^* \to x^*$ in the weak topology (i.e. the density of Y^* in $(l^1)^*$ in the weak topology) and that for any $\varepsilon > 0$ and any $u_1, u_2 \in l^1$, $x^* \in (l^1)^*$, there exists $y^* \in Y^*$, such that $|x^*(u_j) - y^*(u_j)| < \varepsilon$, $j = 1, 2$.

For $u_1 = J^{n_k}_{t/n_k}(x)$ and $u_2 = g(t)(x)$, we get

$$|x^*[J^{n_k}_{t/n_k}(x)] - x^*[g(t)(x)]| \leq |x^*[J^{n_k}_{t/n_k}(x)] - y^*[J^{n_k}_{t/n_k}(x)]| +$$
$$|y^*[J^{n_k}_{t/n_k}(x)] - y^*[g(t)(x)]| + |y^*[g(t)(x)] - x^*[g(t)(x)]| <$$
$$2\varepsilon + |y^*[J^{n_k}_{t/n_k}(x)] - y^*[g(t)(x)]| < 3\varepsilon,$$

for all $k > k_0$, with k_0 depending on ε, t and x.

This shows that for any $x^* \in (l^1)^*$, if $k \to \infty$ then we have $x^*[J^{n_k}_{t/n_k}(x)] \to x^*[g(t)(x)]$, for any fixed $t \in \mathcal{R}_+$ and $x \in l^p$.

Finally, since according to [7], p. 27, l^1 is the so-called Banach envelope of l^p and $(l^1)^* = (l^p)^*$ (with the same dual norms too), the theorem is proved.

Remarks. 1) We may repeat the reasonings in the proof of Theorem 3.1 for the sequence $(J^n_{t/n}(x), n \in \mathbb{N}, n \neq n_k)$, where n_k is the subsequence in Theorem 3.1, so that by mathematically induction we easily obtain that the sequence $(J^n_{t/n}(x))_{n\in\mathbb{N}}$ has at most a countable set of limit points in the weak topology of l^p, denote that set by $T^*(t)(x)$, where $t \in \mathcal{R}_+, x \in l^p$. For any fixed $t \in \mathcal{R}_+$, an element $a \in T^*(t)$ is in fact a mapping $a : l^p \to l^p$.

2) From the proof of Theorem 3.1, we easily can derive that in addition, the functions $g_{n,r}(t) : l^p \to \mathbb{R}$ are Lipschitz functions, i.e. $|g_{n,r}(t)(x) - g_{n,r}(t)(y)| \le M^{1/p}\|x-y\|_p^{1/p}$, which implies that the family $(g_{n,r}(t))_{n,r\in\mathbb{N}}$ is equicontinuous. Also, for any $x \in l^p$, $t \in \mathcal{R}_+$, the sequence $(g_{n,r}(t)(x))_{n,r\in\mathbb{N}}$ is bounded . Unfortunately we cannot apply the classical Arzela-Ascoli theorem in l^p, because l^p is not locally compact.

However, we may impose some additional properties to the nonlinear operator A, which could imply better convergence results in Theorem 3.1, as follows.

Consider on l^p the so called lexicographic order, i.e. for $x = (x_j)_j, y = (y_j)_j \in l^p$, we write $x \le y$ if and only if $x_j \le y_j$, for all $j \in \mathbb{N}$ and $x < y$ if and only if $x \le y$ and there is a j with $x_j < y_j$.

The following simple result holds.

Lemma 3.2 *Suppose that* $A : l^p \to l^p$ *is a dissipative nonlinear operator, A(0)=0,* A *is convex and non-increasing with respect to the above order, i.e.*

$$A[\alpha x + (1-\alpha)y] \le \alpha A(x) + (1-\alpha)A(y),$$

for all $x, y \in l^p, \alpha \in [0,1]$ *and* $x < y$ *implies* $A(x) \ge A(y)$*. We have :*

(i) $I - \lambda A$ *is concave and non-decreasing, for any* $\lambda > 0$ *;*

(ii) $B := (I-\lambda A)^{-1}$ *is convex and non-decreasing, for any* $\lambda > 0$ *;*

(iii) B^n *is convex and non-decreasing, for any* $\lambda > 0$.

The proof is an easy exercise and it is left to the reader.

Remark. Lemma 3.2 says that if A is convex and non-increasing, then so is $J^n_{t/n}(x)$, which obviously implies that the functions $g_{n,r}(t) : l^p \to \mathbb{R}$ in the proof of Theorem 3.1 are convex and non-decreasing.

Corollary 3.3 *Denote by* $T(t)(x) \in l^p$ *the weak limit in* l^p *of the sequence* $(J^{n_k}_{t/n_k}(x))_k$*, for all* $t \in \mathcal{R}_+$ *and* $x \in l^p$*, where* $(n_k)_k$ *is the subsequence in Theorem 3.1. We have*

(i) $T(0) = I$ *;*

(ii) $\|T(t)(x) - T(t)(y)\|_{l^1} \le e^{t\omega}\|x-y\|_p^{1/p}$*, for all* $t \in \mathcal{R}_+$, $x, y \in l^p \subset l^1$ *;*

(iii) For any $t, s \in R_+$ *and* $a \in T^*(t+s)$*, there exist* $b \in T^*(t)$ *and* $c \in T^*(s)$ *such that* $a(x) = b[c(x)]$*, for all* $x \in l^p$.

Proof. (i) It is obvious by the definition of $J^{n_k}_{t/n_k}(x)$;

(ii) First, passing to limit with $k \to +\infty$ in the following inequality in the proof of Theorem

3.1

$$\|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_p \le (1 - \frac{t}{n_k}\omega)^{-n_k p}\|x - y\|_p,$$

we easily get

$$\lim_{k\to\infty}\|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_p \le \lim_{k\to\infty}(1 - \frac{t}{n_k}\omega)^{-n_k p}\|x - y\|_p = e^{t\omega p}\|x - y\|_p.$$

Let $x^* \in (l^p)^*$ be with $|||x^*|||_{(l^p)^*} \le 1$. According to [7, p. 27], it can be extended to a $x^* \in (l^1)^*$, preserving its norm, i.e. $|||x^*|||_{(l^1)^*} \le 1$. We have

$$|x^*[T(t)(x)] - x^*[T(t)(y)]| = |x^*[T(t)(x) - T(t)(y)]| \le$$

$$|x^*[T(t)(x)] - x^*[J^{n_k}_{t/n_k}(x)]| + |x^*[J^{n_k}_{t/n_k}(x)] - x^*[J^{n_k}_{t/n_k}(y)]| +$$

$$|x^*[J^{n_k}_{t/n_k}(y)] - x^*[T(t)(y)]| :=$$

$$a_k + |x^*[J^{n_k}_{t/n_k}(x)] - x^*[J^{n_k}_{t/n_k}(y)]| + b_k,$$

where $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$ by the definitions of $T(t)(x)$ and $T(t)(y)$.

On the other hand,

$$|x^*[J^{n_k}_{t/n_k}(x)] - x^*[J^{n_k}_{t/n_k}(y)]| \le |||x^*|||_{(l^1)^*}\|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_{l^1} \le$$

$$\|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_{l^1} \le \|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_p^{1/p}.$$

Passing in the above two inequalities to limit with $k \to \infty$, we get

$$|x^*[T(t)(x)] - x^*[T(t)(y)]| \le e^{t\omega}\|x - y\|_p^{1/p},$$

for all $x^* \in (l^1)^*$, with $|||x^*|||_{(l^1)^*} \le 1$. Passing here to supremum with such x^*, by a classical result in functional analysis for normed spaces, it follows

$$\sup_{|||x^*|||_{(l^1)^*}\le 1}|x^*[T(t)(x)] - x^*[T(t)(y)]| = \|T(t)(x) - T(t)(y)\|_{l^1} \le e^{t\omega}\|x - y\|_p^{1/p}.$$

(iii) Let $q \in \mathbb{N}$, $t \in \mathcal{R}_+, x \in l^p$ be fixed. For any $x^* \in (l^p)^*$, we have

$$\lim_{k\to+\infty} x^*[J^{n_k}_{qt/n_k}(x)] = x^*([T(qt)](x)),$$

which immediately implies

$$\lim_{k\to+\infty} x^*(J^{qn_k}_{qt/qn_k}(x)) = \lim_{k\to+\infty} x^*(J^{qn_k}_{t/n_k}(x)) = \lim_{k\to+\infty} x^*([T(t)]^q(x)).$$

Applying the same reasonings as in the proof of Theorem 3.1, there exists a subsequence of $(qn_k)_k$, let us denote it by $(q_k)_k$, such that

$$\lim_{k\to+\infty} x^*[J^{q_k}_{qt/q_k}(x)] = x^*([T(t)]^q(x)),$$

which shows that for all $t \in \mathcal{R}_+$, if $a \in T^*(qt)$, then there exists $b \in [T^*(t)]^q$ with $a = b$.

Then, for $l, k, r, s \in \mathbb{N}$, we easily get that for any $a \in T^*(\frac{l}{k} + \frac{r}{s}) = T^*\left(\frac{ls+rk}{ks}\right)$, there exists $d \in [T^*(\frac{1}{ks})]^{ls+kr}$ with $a = d$. On the other hand, denoting $ks = t$, we have $J_{t/n}^{n(ls+kr)}(x) = (I - \frac{t}{n}A)^{-n(ls+kr)}(x) = (I - \frac{t}{n}A)^{-nls}[(I - \frac{t}{n}A)^{-nkr}(x)]$, so for the above d, there exists a subsequence $(n_j)_j$ with $d = \text{lím}_{j\to+\infty} x^*(J_{t/n_j}^{n_j(ls+kr)}(x))$ and there exist $b \in [T^*(t)]^{ls}$ and $c \in [T^*(t)]^{kr}$ such that $d(x) = b[c(x)]$. The corollary is proved.

The next result shows that for some particular nonlinear operators, the whole sequence $(J_{t/n}^n(x))_n$ is convergent in the l^p space.

Theorem 3.4 *Let* $A : l^p \to l^p$, $0 < p < 1$, *be a nonlinear operator of the form* $A(x) = (f_k(x_k))_{k\in\mathbb{N}}$, *for all* $x = (x_k)_{k\in l^p}$, *where* $f_k : \mathbb{R} \to \mathbb{R}$ *are non-increasing continuous functions,* $f_k(0) = 0$ *and there exists* $M > 0$ *such that* $|f_k(\alpha) - f_k(\beta)| \leq M|\alpha - \beta|$, *for all* $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$. *Then, for any* $t \geq 0$ *and* $x \in l^p$, *the sequence* $J_{t/n}^n(x) = (I - \frac{t}{n}A)^{-n}(x), n \in \mathbb{N}$ *is strongly convergent to a limit in* l^p.

Proof. First by definition it easily follows that A is a Lipschitz operator with respect to the $\|\cdot\|_p$-norm in l^p. Then, we can write $J_{t/n}^n(x) = (g_{n,k}(t)(x_k))_k$, where $g_{n,k}(t) : \mathbb{R} \to \mathbb{R}$ are given by $g_{n,k}(t)(u) = (I - \frac{t}{n}f_k)^{-n}(u)$. By the hypothesis, it follows that each sequence $(g_{n,k}(t)(u))_k$ is convergent in the Banach space $\mathbb{R}$, denote $g_k(t)(u) = \text{lím}_{n\to+\infty} g_{n,k}(t)(u)$.

We know that l^p has the basis $\{e_1, e_2, ..., e_n, ...,\}$, where $e_i = (\delta_{in})_{n\in\mathbb{N}}$. Due to the particular form of $J_{t/n}^n(x) = (g_{n,k}(t)(x_k))_k$, we have $g_{n,k}(t)(0) = 0$ for all $k \in \mathbb{N}$ and it is obvious that if $x \in span\{e_1, ..., e_i, ...\} =: Y$, then $J_{t/n}^n(x)$ becomes a sequence with only a finite number of non-zero elements. This means that for such x, $J_{t/n}^n(x)$ is convergent in l^p. Also, obviously Y is dense in l^p.

Let $x \in l^p$ and $\varepsilon > 0$ be arbitrary. There exists $y \in Y$ such that $\|x - y\|_p < \varepsilon$. We get

$$\|J_{t/n}^n(x) - J_{t/m}^m(x)\|_p \leq \|J_{t/n}^n(x) - J_{t/n}^n(y)\|_p + \|J_{t/n}^n(y) - J_{t/m}^m(y)\|_p +$$

$$\|J_{t/m}^m(y) - J_{t/m}^m(x)\|_p \leq$$

$$2M\|x - y\|_p + \|J_{t/n}^n(y) - J_{t/m}^m(y)\|_p < 2M\varepsilon + \|J_{t/n}^n(y) - J_{t/m}^m(y)\|_p,$$

where $\|J_{t/n}^n\|_{Lip} \leq M = 1$ (see the proof of Theorem 3.1, where we take $\omega = 0$).

Since $(J_{t/n}^n(y))_n$ is convergent in l^p, it is a Cauchy sequence and therefore given $\delta > 0$, there is a n_0 such that $\|J_{t/n}^n(y) - J_{t/m}^m(y)\|_p < \delta$, for all $m, n > n_0$. Together with the above inequality this implies that $(J_{t/n}^n(x))_n$ is a Cauchy sequence in the complete metric space l^p, i.e. it is convergent in l^p. The theorem is proved.

As an application of Theorem 3.1, we obtain the following

Corollary 3.5 *Let* $A : l^p \to l^p$, $0 < p < 1$, *be nonlinear, Lipschitz, such that* $A(0) = 0$, *there exists* $\omega \in \mathbb{R}$ *with* $A - \omega I$ *dissipative and* A *is weakly continuous (that is for any* $x^* \in (l^p)^*$, *if* $\lim_{n\to\infty} x^*(a_n) = x^*(a)$, *then* $\lim_{n\to\infty} x^*[A(a_n)] = x^*[A(a)]$ *).*

For $x \in l^p$ *and* $t \in \mathcal{R}_+$*, let us consider as in the statement and proof of Theorem 3.1, the sequence in* l^p, $u_k(x)(t) = J^{n_k}_{t/n_k}(x) = (g_{k,r}(x)(t))_{r\in\mathbb{N}}$*, convergent (as* $k \to \infty$*) in the weak topology of* l^p*, to* $u(x)(t) = (g_r(x)(t))_{r\in\mathbb{N}}$.

Let us suppose that for all $r \in \mathbb{N}$, $x \in l^p$*, the real functions* $g_r(x)(t)$ *are left differentiable with respect to* $t \in \mathcal{R}_+$*, that is there exists (finite)*

$$[g_r(x)]'_-(t) = \lim_{h\to 0, h\in\mathcal{R}_+} \frac{g_r(x)(t) - g_r(x)(t-h)}{h}, t \in \mathcal{R}_+,$$

and that for all $k, r \in \mathbb{N}$, $x \in l^p$*, the real functions* $g_{k,r}(x)(t)$ *are differentiable (in the classical sense) with respect to* $t \in [0, \sigma)$*, satisfying in addition the relation*

$$\lim_{t_k \nearrow t}[g_{k,r}(x)]'(t_k) = [g_r(x)]'_-(t),$$

for all $t \in [0, \sigma) \cap \mathcal{R}_+$ *and all* $t_k \in [0, \sigma)$ *with* $t_k \nearrow t$*. Here, for* $s < 0$ *we take by convention* $g_r(x)(s) = g_r(x)(0)$, $g_{k,r}(x)(s) = g_{k,r}(x)(0)$*, which gives sense to* $[g_r(x)]'_-(0) = 0$ *and* $[g_{k,r}(x)]'(s) = 0$, $s \le 0$.

Then, $v(t) = u(x)(t)$ *is a solution of the Cauchy problem*

$$v'_-(t) = A[v(t)], t \in [0, \sigma) \cap \mathcal{R}_+,$$

$$v(0) = x,$$

where $v'_-(t)$ *is defined componentwise as above and v(s)=v(0), for* $s < 0$.

Proof. Let $x^* \in (l^p)^*$ be arbitrary. According to [7, p. 27], it can be extended to a $x^* \in (l^1)^*$, preserving its norm. By the considerations from the beginning of this section, it follows that $u_k(x)(t)$ satisfies the difference equation

$$\frac{u_k(x)(t) - u_k(x)(t - t/n_k)}{\frac{t}{n_k}} = A[u_k(x)(t)], t \ge 0.$$

This obviously implies

$$x^* \left[\frac{u_k(x)(t) - u_k(x)(t - t/n_k)}{\frac{t}{n_k}} - A(u_k(x)(t)) \right] = 0, t \ge 0.$$

But by Theorem 3.1 we have $\lim_{k\to\infty} x^*[u_k(x)(t)] = x^*[u(x)(t)]$, for all $t \in \mathcal{R}_+$, $x \in l^p$. Taking into account the weak continuity of A, first we obtain $\lim_{k\to\infty} x^*(A[u_k(x)(t)]) = x^*(A[u(x)(t)])$.

Next we will show that

$$\lim_{k\to\infty} x^* \left[\frac{u_k(x)(t) - u_k(x)(t - t/n_k)}{\frac{t}{n_k}} \right] = x^*([u(x)]'_-(t)) \qquad (3),$$

for all $t \in \mathcal{R}_+$, $x \in l^p$.

For this purpose, we reason as in the proof of Theorem 3.1, that is first we prove (3) for any $x_r^* \in (l^p)^*$, $r \in \mathbb{N}$ of the form $x_r^*(x) = x_r$, for all $x = (x_1, ..., x_r, ...) \in l^p$. This one reduces to

$$\lim_{k\to\infty} \frac{g_{k,r}(x)(t) - g_{k,r}(x)(t - t/n_k)}{\frac{t}{n_k}} = [g_r(x)]'_-(t),$$

for all $t \in \mathcal{R}_+$, $x \in l^p$, $r \in \mathbb{N}$.

By the mean value theorem, there exists $\xi_{t,k} \in (t - t/n_k, t)$ such that $\frac{g_{k,r}(x)(t) - g_{k,r}(x)(t-t/n_k)}{\frac{t}{n_k}} = [g_{k,r}(x)]'(\xi_{t,k})$, which by the hypothesis immediately implies that at the limit with $k \to \infty$ we obtain (3).

Also, it is clear that (3) holds for any $y^* \in span\{x_1^*, ..., x_r^*, ...\} = Y^*$. Reasoning now exactly as at the end of proof in Theorem 3.1 (since Y^* is dense in $(l^1)^*$ in the weak topology on $(l^1)^*$), we easily get that (3) is satisfied for all $x^* \in (l^1)^*$.

In conclusion, we get

$$x^*[(u(x))'_-(t) - A(u(x)(t))] = 0,$$

for all $t \in \mathcal{R}_+$ and all $x^* \in (l^1)^*$. Passing here to supremum with $|||x^*|||_{(l^1)^*} \leq 1$ and taking into account a classical result in functional analysis (since l^1 is a normed space), we obtain

$$||[u(x)]'_-(t) - A(u(x)(t))||_{l^1} = 0, t \in \mathcal{R}_+, x \in l^p,$$

which implies $[u(x)]'_-(t) = A(u(x)(t))$, for all $t \in \mathcal{R}_+$, $x \in l^p$. Also, obviously $u(x)(0) = x$, which proves the corollary.

A consequence of Theorem 3.4 is the following

Corollary 3.6 *For* $x = (x_1, ..., x_k, ...) \in l^p$ *and* $0 < p < 1$, *let us consider as in the statement and proof of Theorem 3.4, the operator* A, *the sequence* $u_n(t) := J_{t/n}^n(x) = (g_{n,k}(t)(x_k))_{k\in\mathbb{N}} \in l^p$, *where* $g_{n,k}(t) : \mathbb{R} \to \mathbb{R}$ *are given by* $g_{n,k}(t)(u) = (I - \frac{t}{n} f_k)^{-n}(u)$ *and* $u(t) = (g_k(t)(x_k))_{k\in\mathbb{N}} \in l^p$ *with* $\lim_{n\to\infty} ||u_n(t) - u(t)||_p = 0$, *for all* $t \geq 0$.

If, in addition, $u_n(t)$, $u(t)$ *are differentiable with respect to* $t \in [0, \sigma]$ *such that*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} ||g'_{n,k}(x_k) - g'_k(x_k)||^p = 0,$$

where $||g'_{n,k}(x_k) - g'_k(x_k)|| := \sup_{t\in[0,\sigma]} |g'_{n,k}(t)(x_k) - g'_k(t)(x_k)|$, *then* $v(t) = u(t)$ *represents the unique solution of the nonlinear Cauchy problem*

$$\frac{d}{dt} v(t) = A[v(t)], t \in [0, \sigma],$$

$$v(0) = x.$$

Proof. By the considerations from the beginning of this section, it follows that $u_n(t) = J^n_{t/n}(x)$ satisfies the difference equation

$$\frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} = A[u_n(t)], t \geq 0.$$

Passing here to limit (with $n \to \infty$) in the $\|\cdot\|_p$-norm in l^p, since A is Lipschitz in l^p (see the proof of Theorem 3.4), it follows that $\lim_{n\to\infty} A(u_n(t)) = A(u(t))$, for all $t \in [0, \sigma]$.

For the left-hand side, we have

$$\left\| u'(t) - \frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} \right\|_p \leq$$

$$\left\| u'(t) - \frac{u(t) - u(t - t/n)}{\frac{t}{n}} \right\|_p + \left\| \frac{u(t) - u(t - t/n)}{\frac{t}{n}} - \frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} \right\|_p,$$

where $\lim_{n\to\infty} \|u'(t) - \frac{u(t)-u(t-t/n)}{\frac{t}{n}}\|_p = 0$ by the definition of derivative, while by the mean value theorem we obtain

$$\left\| \frac{u(t) - u(t - t/n)}{\frac{t}{n}} - \frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} \right\|_p =$$

$$\sum_{k=1}^{\infty} \left| \frac{[g_k(t)(x_k) - g_{n,k}(t)(x_k)] - [g_k(t - t/n)(x_k) - g_{n,k}(t - t/n)(x_k)]}{t/n} \right|^p =$$

$$\sum_{k=1}^{\infty} |g'_k(\xi_{t,k,n})(x_k) - g'_{n,k}(\xi_{t,k,n})(x_k)|^p \leq \sum_{k=1}^{\infty} \|g'_k(x_k) - g'_{n,k}(x_k)\|^p,$$

which by the hypothesis implies

$$\lim_{n\to\infty} \left\| \frac{u(t) - u(t - t/n)}{\frac{t}{n}} - \frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} \right\|_p = 0$$

and proves the corollary.

Example. A simple example satisfying the conditions (and the conclusions) in Corollary 3.6 is given as follows. Define the non-linear strictly decreasing continuous function $f : \mathbb{R} \to \mathbb{R}$, by $f(x) = -x$ if $x < 0$, $f(x) = -2x$ if $x \geq 0$ and $A : l^p \to l^p$ by $A(x) = (f(x_1), ..., f(x_k), ...)$, for all $x = (x_1, ..., x_k, ...) \in l^p$.

It is easy to check that $|f(\alpha) - f(\beta)| \leq 2|\alpha - \beta|$, for all $\alpha, \beta \in \mathbb{R}$, which implies that A is Lipschitz nonlinear operator. Also, it is easy to check that for all $\lambda > 0$, the operator $I - \lambda A$ is invertible, with $x = (x_1, ..., x_k, ...)$, $(I - \lambda A)^{-1}(x) = (g(x_1), ..., g(x_k), ...)$, $g(x_k) = \frac{x_k}{1+\lambda}$ if $x_k < 0$, $g(x_k) = \frac{x_k}{1+2\lambda}$ if $x_k \geq 0$, and

$$\|(I - \lambda A)^{-1}\|_{Lip} \leq \left(\frac{1}{1+\lambda}\right)^p \leq 1,$$

which shows that A is dissipative.

Simple calculation shows that $u_n(t) = (g_n(t)(x_1), ..., g_n(t)(x_k), ...)$, where $g_n(t)(x_k) = \frac{x_k}{(1+(t/n))^n}$ if $x_k < 0$, $g_n(x_k) = \frac{x_k}{(1+2(t/n))^n}$ if $x_k \geq 0$, $u(t) = (g(t)(x_1), ..., g(t)(x_k), ...)$, where $g(t)(x_k) = x_k e^{-t}$ if $x_k < 0$, $g(t)(x_k) = x_k e^{-2t}$ if $x_k \geq 0$.

It is easy to prove that all the conditions in Corollary 3.6 are satisfied with $\sigma = 1$, which shows that $u(t)$ defined as above is the unique solution of the nonlinear Cauchy problem

$$\frac{d}{dt}v(t) = A[v(t)], t \in [0, 1],$$

$$v(0) = x.$$

4. Nonlinear Semigroups on H^p, $0 < p < 1$

In this section we consider the H^p space, $0 < p < 1$, where we denote its p-norm by $\|\cdot\|_p$. The main result is the following.

Theorem 4.1 *Let* $A : (H^p, \|\cdot\|_p) \to (H^p, \|\cdot\|_p)$, $0 < p < 1$, *be nonlinear and Lipschitz, such that* $A(0) = 0$ *and there exists* $\omega \in \mathbb{R}$ *with* $A - \omega I$ *dissipative. Then the sequence in* H^p *defined by* $J^n_{t/n}(x) = (I - \frac{t}{n}A)^{-n}(x), n \in \mathbb{N}, t \geq 0, x \in H^p$, *contains a subsequence* $J^{n_k}_{t/n_k}(x), k \in \mathbb{N}$ *(the same subsequence for all* $x \in H^p$ *and all* $t \in \mathcal{R}_+$ *= the set of all rational numbers* ≥ 0*), uniformly convergent on compacts in* $\mathbb{D}$.

Proof. By $A(0) = 0$ we get $(I - \frac{t}{n}A)^{-n}(0) = 0$, for all $n \in \mathbb{N}$. The dissipative property implies $\|I - t(A - \omega I)^{-1}\|_{Lip} \leq 1$, for all $t \geq 0$, which is equivalent to $\|(I - \frac{t}{1+t\omega}A)^{-1}\|_{Lip} \leq |1 + t\omega|^p$, for all $t \geq 0$ with $1 + t\omega \neq 0$. For $\lambda = \frac{t}{1+t\omega}$ we get $\|(I - \lambda A)^{-1}\|_{Lip} \leq (1 - \lambda\omega)^{-p}$, in particular for all $\lambda > 0$ with $\lambda\omega < 1$. (Note that for n sufficiently great, depending on t and ω, we have $\frac{t}{n}\omega < 1$.) Therefore,

$$\|(I - \frac{t}{n}A)^{-1}\|_{Lip} \leq (1 - \frac{t}{n}\omega)^{-p}$$

and by mathematical induction

$$\|(I - \frac{t}{n}A)^{-n}\|_{Lip} \leq (1 - \frac{t}{n}\omega)^{-np}.$$

But it is known that the sequence $(1 + \frac{s}{n})^n$ converges (for $n \to +\infty$) to e^s, and for any $s \in \mathbb{R}$ is monotonically increasing, for all $n \geq [|s|] + 1$ (see e.g. [10, p. 263]), which implies that $(1 - \frac{t}{n}\omega)^{-np}$ converges to $e^{t\omega p}$, monotonically decreasing, for all $n \geq [|tw|] + 1$. Therefore, the greatest value of $(1 - \frac{t}{n}\omega)^{-np}$ is for $n = [|t\omega|] + 1$, which means that there exists $M = M(t, p, \omega) > 0$ (depending only on t, p and ω) such that

$$\|(I - \frac{t}{n}A)^{-n}\|_{Lip} \leq M,$$

for all $n \in \mathbb{N}$.

We obtain

$$||J^n_{t/n}(x) - J^n_{t/n}(y)||_p \le (1 - \frac{t}{n}\omega)^{-np}||x - y||_p \le M||x - y||_p, \tag{4}$$

for all $x, y \in H^p$. Taking $y = 0$ we have $J^n_{t/n}(0) = 0$ and we obtain

$$||J^n_{t/n}(x)||_p \le M(t, p, \omega)||x||_p < +\infty, \tag{5}$$

for all $n \in \mathbb{N}$.

Since H^p, $0 < p < 1$, has a Schauder basis (see e.g. [9]), it follows that it is separable, denote by Y a countable dense subset of H^p. Also, denote by $\mathcal{R}_+$, the set of all nonnegative rational numbers and define $E = \mathcal{R}_+ \times Y$. Obviously E is a countable set, let us denote it by $E = \{e_1, ..., e_j, ...,\}$ with the distinct elements two by twos, $e_j = (r_j, y_j)$ and for each $e = (t, y) \in E$, denote $S_n(e) = J^n_{t/n}(y) \in H^p$. Obviously, $S_n(e)$ are analytic functions in $\mathbb{D}$, for all $n \in \mathbb{N}$ and all $e \in E$.

According to e.g. [7, p. 35, (3.4)], the point evaluations $\varphi_z(x) = x(z)$, $z \in \mathbb{D}$, are linear and bounded functionals on H^p, $0 < p < 1$ and the following inequality holds

$$|x(re^{i\theta})| \le 2^{1/p}||x||_p(1 - r)^{-1/p}, \text{ for all } x \in H^p \text{ and } z = re^{i\theta}.$$

Together with (5), this implies that for all $z = re^{i\theta}$, $|z| \le r_0 < 1$, i.e. $0 < r \le r_0$, $e = (t, y) \in E$, we obtain

$$|S_n(e)(z)| = |\varphi_z[S_n(e)]| \le 2^{1/p}||S_n(e)||_p \frac{1}{(1 - r)^{1/p}} \le$$

$$2^{1/p} \frac{1}{(1 - r_0)^{1/p}} M(t, p, \omega)||y||_p.$$

In other words, for any fixed $e = (t, y) \in E$, the sequence of analytic functions $(S_n(e))_{n\in\mathbb{N}}$, is uniformly bounded on each compact subset of $\mathbb{D}$, which by the classical Montel's theorem implies that it contains a subsequence uniformly convergent on compact subsets of $\mathbb{D}$.

For $e_1 \in E$, there exists a subsequence of $(S_n(e_1))_{n\in\mathbb{N}}$, denoted by $(S_{1,n}(e_1))_{n\in\mathbb{N}}$, which is uniformly convergent on compact subsets of $\mathbb{D}$.

For $e_2 \in E$, reasoning analogously, the sequence $(S_{1,n}(e_2))_{n\in\mathbb{N}}$ contains in turn, a subsequence denoted by $(S_{2,n}(e_2))_{n\in\mathbb{N}}$, which is uniformly convergent on compact subsets of $\mathbb{D}$.

In general, for $e_m \in E$, there exists a subsequence of the previous one, $(S_{m,n}(e_m))_{n\in\mathbb{N}}$, uniformly convergent on compact subsets of $\mathbb{D}$.

Continuing this process gives rise to the infinite array of analytic functions in $\mathbb{D}$,

$$S_{1,1}, \quad S_{1,2}, \quad S_{1,3}, ...,$$

$$S_{2,1}, \quad S_{2,2}, \quad S_{2,3}, ...,$$

$$S_{1,1}, \quad S_{1,2}, \quad S_{3,3}, ...,$$

....

$$S_{1,m}, \quad S_{2,m}, \quad S_{3,m}, \ldots,$$

and so on, such that the first row means that $(S_{1,n}(e_1))_{n\in\mathbb{N}}$ uniformly converges on compact subsets of $\mathbb{D}$, the second row means that $(S_{2,n}(e_j))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$ for $j = 1, 2$, the third row means that $(S_{3,n}(e_j))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$, for $j = 1, 2, 3$, and so on.

As a consequence, we can consider the diagonal sequence $(S_{n,n})_{n\in\mathbb{N}}$, which has the property that $(S_{n,n}(e_j))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$, for all $j \in \mathbb{N}$, that is $(S_{n,n}(e))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$, for all $e \in E$.

Now, let us denote $A = \mathcal{R}_+ \times H^p$. We will show that in fact $(S_{n,n}(e))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$, for all $e \in A$. Indeed, let $e = (r, x) \in A$ and since Y is dense in H^p, let $y_k \in Y, k \in \mathbb{N}$, satisfying $\|x - y_k\|_p \to 0$, when $k \to \infty$. Denoting $a_k = (r, y_k) \in E$, by (4) we have

$$\|S_{n,n}(e) - S_{n,n}(a_k)\|_p \le M(r, p, \omega)\|x - y_k\|_p,$$

for all $k \in \mathbb{N}$. It is enough to show that $(S_{n,n}(e)(z))_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the uniform norm (denoted by $\|\cdot\|$) in each compact disk, $\overline{D_r}$ in $\mathbb{D}$. Indeed, this is immediate by the inequalities

$$\|S_{n,n}(e) - S_{m,m}(e)\| \le \|S_{n,n}(e) - S_{n,n}(a_k)\| + \|S_{n,n}(a_k) - S_{m,m}(a_k)\| + \|S_{m,m}(a_k) - S_{m,m}(e)\| \le$$

$$2M(r, p, \omega)\|x - y_k\|_p + \|S_{n,n}(a_k) - S_{m,m}(a_k)\|$$

and by the above properties.

The theorem is proved.

Remarks. 1) By relation (3.4) in [7, p. 35], it is evident that if $\lim_{n\to\infty}\|x_n - x\|_p = 0$, then $(x_n)_n$ is uniformly convergent on compact subsets of $\mathbb{D}$. In general, the converse is not valid. As a consequence, if we denote by x the uniform limit of $(x_n)_n$ on compact subsets of $\mathbb{D}$, then x is an analytic function in $\mathbb{D}$, but in general it does not belong to H^p, $0 < p < 1$.

2) We can repeat the reasonings in the proof of Theorem 4.1 for the sequence $(J^n_{t/n}(x), n \in \mathbb{N}, n \ne n_k)$, where n_k is the subsequence in Theorem 4.1, so that by mathematically induction we easily obtain that the sequence $(J^n_{t/n}(x))_{n\in\mathbb{N}}$ has at most a countable set of limit points in the locally convex topology of uniform convergence on compact subsets in $\mathbb{D}$. If we denote that set by $T^*(t)(x)$, where $t \in \mathcal{R}_+, x \in H^p$, then for any fixed $t \in \mathcal{R}_+$, an element $a \in T^*(t)$ is in fact a mapping $a : H^p \to Hol(\mathbb{D})$, where $Hol(\mathbb{D})$ denotes the spaces of all holomorphic (analytic) functions in $\mathbb{D}$.

Corollary 4.2. *Let* $A : (H^p, \|\cdot\|_p) \to (H^p, \|\cdot\|_p)$, $0 < p < 1$*, be nonlinear and Lipschitz, such that* $A(0) = 0$*, there exists* $\omega \in \mathbb{R}$ *with* $A - \omega I$ *dissipative and* $A : (H^p, \mathcal{T}) \to (H^p, \mathcal{T})$ *is continuous, where* $\mathcal{T}$ *represents the locally convex topology of uniform convergence on compact subsets in* $\mathbb{D}$.

For $x \in H^p$ *and* $t \in \mathcal{R}_+$*, let us consider as in the statement of Theorem 4.1, the sequence in* H^p*,* $u_k(x)(t) = J^{n_k}_{t/n_k}(x), k \in \mathbb{N}$*, uniformly convergent on compacts in* $\mathbb{D}$ *(as* $k \to \infty$*) to* $u(x)(t)$*.*

Let us suppose that for $x \in H^p$ *and all* $z \in \mathbb{D}$*, the complex valued function* $[u(x)(t)](z)$ *is left derivable with respect to the real variable* $t \in \mathcal{R}_+$*, that is there exists (finite)*

$$\frac{\partial[u(x)(t)(z)]_-}{\partial t} = \lim_{h\to 0, h\in\mathcal{R}_+} \frac{[u(x)(t)](z) - [u(x)(t-h)](z)}{h}, t \in \mathcal{R}_+,$$

and also suppose that

$$\lim_{k\to\infty} \frac{[u_k(x)(t)](z) - [u_k(x)(t - t/n_k)](z)}{\frac{t}{n_k}} = \frac{\partial[u(x)(t)(z)]_-}{\partial t},$$

for all $t \in \mathcal{R}_+ \cap [0, \sigma)$*,* $x \in H^p$*,* $z \in \mathbb{D}$*.*

Here, for $s < 0$ *we take by convention* $[u(x)(s)](z) = [u(x)(0)](z)$*,* $[u_k(x)(s)](z) = [u_k(x)(0)](z)$*, for all* $z \in \mathbb{D}$*, which gives sense to* $\frac{\partial[u(x)(0)(z)]_-}{\partial t}$*, for all* $z \in \mathbb{D}$*.*

Then, $v(t) = u(x)(t)$ *is a solution (analytic in* $\mathbb{D}$ *but not necessarily in* H^p*) of the Cauchy problem*

$$\frac{\partial[v(t)(z)]_-}{\partial t} = A[v(t)](z), t \in [0, \sigma) \cap \mathcal{R}_+, z \in \mathbb{D}$$

$$v(0)(z) = x(z), z \in \mathbb{D},$$

where $\frac{\partial[v(t)(z)]_-}{\partial t}$ *is defined as above and v(s)=v(0), for* $s < 0$*.*

Proof. By the considerations from the beginning of the Section 3, it follows that $u_k(x)(t)$ satisfies the difference equation

$$\frac{u_k(x)(t) - u_k(x)(t - t/n_k)}{\frac{t}{n_k}} = A[u_k(x)(t)], t \geq 0.$$

But by Theorem 4.1 and by the continuity assumption on A, we have $\lim_{k\to\infty} A[u_k(x)(t)](z) = A[u(x)(t)](z)$, for all $z \in \mathbb{D}$.

Therefore, passing to limit with $k \to \infty$ in the above difference equation, by the hypothesis we immediately obtain

$$\frac{\partial[u(x)(t)(z)]_-}{\partial t} = A[u(x)(t)](z), t \in [0, \sigma) \cap \mathcal{R}_+, z \in \mathbb{D},$$

$z \in \mathbb{D}$.

Also, obviously $u(x)(0) = x$, which proves the corollary.

5. Nonlinear Semigroups on $L^p[0, 1]$, $0 < p < 1$

In this section we consider the $L^p[0, 1]$ space, $0 < p < 1$, where we denote its p-norm by $\|\cdot\|_p$. The main result is the following.

Theorem 5.1. *Let* $A : (L^p[0,1], \|\cdot\|_p) \to (L^p[0,1], \|\cdot\|_p)$, $0 < p < 1$, *be nonlinear and Lipschitz, such that* $A(0) = 0$ *and there exists* $\omega \in \mathbb{R}$ *with* $A - \omega I$ *dissipative. Then, for any fixed* $t \geq 0, x \in L^p[0,1]$, *the sequence in* $L^p[0,1]$ *defined by* $J^n_{t/n}(x) = (I - \frac{t}{n}A)^{-n}(x), n \in \mathbb{N}$, *contains a subsequence* $a_k(t,x) := J^{n_k}_{t/n_k}(x), k \in \mathbb{N}$, *such that*

$$\lim_{k\to\infty} \frac{1}{k^{1/p}} \sum_{i=1}^{k} a_i(t,x)(s) = 0,$$

a.e. $s \in [0,1]$.

Proof. Reasoning exactly as in the proof of Theorem 4.1, relations (4)-(5), we get that

$$\|J^n_{t/n}(x)\|_p \leq M(t,p,\omega)\|x\|_p,$$

where $\|\cdot\|_p$ is the p-norm in $L^p[0,1]$. In other words, for any fixed $t \geq 0$ and $x \in L^p[0,1]$, the sequence $(J^n_{t/n}(x))_n$ is bounded in the p-norm of $L^p[0,1]$, $0 < p < 1$.

According to [2], this implies that for any $t \geq 0$ and $x \in L^p[0,1]$, there exists a subsequence $a_i(t,x) := J^{n_i}_{t/n_i}(x), i \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} a_i(t,x)(s) = 0,$$

a.e. $s \in [0,1]$.

Remark. Unfortunately, a sequence $(a_i(t,x))_{i\in\mathbb{N}}$, satisfying the relation proved by Theorem 5.1, can satisfy (in the sense that does not produce a contradiction) $\lim_{i\to\infty} a_i(t,x,\omega)(s) = \infty$, a.e. $s \in [0,1]$, which is the worst possible divergence result. If to this fact we add that the dual space of $L^p[0,1]$, $0 < p < 1$, is $\{0\}$, then it seems that in this space, in general we cannot derive any result on the convergence of some subsequences of $(J^n_{t/n}(x))_n$.

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Referencias

[1] A. Bayoumi, *Foundations of Complex Analysis in Non-Locally Convex Spaces*, North-Holland Mathematics Studies, Elsevier, vol. 193, Amsterdam, 2003.

[2] S.D. Chatterji, *A general strong law*, Inventiones Math., 9(1970), 235-245.

[3] M.G. Crandall and T.M. Liggett, *Generation of semigroups of non-linear transformations on general Banach spaces*, Amer. J. Math., 93(1971), 265-298.

[4] N. Dunford and J.T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1964.

[5] S.G. Gal and J.A. Goldstein, *Semigroups of linear operators on* p*-Fréchet spaces,* $0 < p < 1$, Acta Math. Hungar., 114(1-2)(2007), 13-36.

[6] J.A. Goldstein, S. Oharu and A. Vogt, *Affine semigroups on Banach spaces*, Hiroshima Math. J., 18(1988), No. 2, 433-450.

[7] N.J. Kalton, N.T. Peck and J.W. Roberts, *An* F*-Space Sampler*, London Mathematical Society Lecture Notes Series, vol. 89, Cambridge University Press, 1984.

[8] I. Muntean, *Course and Problems in Functional Analysis, Vol. II* (Romanian), Faculty of Mathematics, "Babes-BolyaiÜniversity Press, Cluj, 1988.

[9] P. Oswald, *On Schauder bases in Hardy spaces*, Proc. Roy. Soc. Edinburg, Sect. A, 93(1982/83), no. 3-4, 259-263.

[10] T. Popoviciu, *Course of Mathematical Analysis, Part 2, Functions. Limits.* (Romanian), "Babes-BolyaiÜniversity, Faculty of Mathematics-Mechanics, Cluj, 1972.

[11] W. Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill Inc., 1976.