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Pseudo-Almost Automorphic Solutions to Some Second-Order Differential Equations

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ABSTRACT

In this paper we study and obtain the existence of pseudo-almost automorphic solutions to some classes of second-order abstract differential equations on a Hilbert space. To illustrate our abstract results, we discuss the existence of pseudo almost automorphic solutions to the N-dimensional Sine-Gordon boundary value problem.

RESUMEN

En este trabajo se estudia y obtiene la existencia de soluciones casi-seudo automorfas a algunas clases de ecuaciones diferenciales abstractas de segundo orden en un espacio de Hilbert. Para ilustrar nuestros resultados abstractos, se discute la existencia de soluciones casi-seudo automorfas en el problema de contorno N-dimensional de Sine-Gordon .

Keywords. exponential stability, sectorial operator, hyperbolic semigroup, almost automorphic; pseudo-almost automorphic; autonomous second-order differential equation; Sine-Gordon equation. amenability, Banach modules, module amenability, weak module amenability, semigroup algebra, inverse semigroup.
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1 Introduction

In Leiva [36], the existence of (exponentially stable) bounded solutions and almost periodic solutions to the second-order systems of differential equations given by

$$u''(t) + cu'(t) + dAu + kH(u) = P(t), \quad u \in \mathbb{R}^n, \quad t \quad \in \mathbb{R}, \tag{1.1}$$

where A is an $n \times n$-matrix whose eigenvalues are positive, c, d, k are positive constants, $H : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a locally Lipschitz function, $P : \mathbb{R} \mapsto \mathbb{R}^n$ is a bounded continuous function, were established.

In this paper, using techniques developped in [36], we obtain some reasonable sufficient conditions, which do guarantee the existence of pseudo-almost automorphic solutions to

$$u''(t) + au'(t) + bAu = f(t, u), \quad t \quad \in \mathbb{R}, \tag{1.2}$$

where $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ is a self-adjoint linear operator whose spectrum consists of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < ... < \lambda_n \to \infty$ with each eigenvalue having a finite multiplicity γ_j equals to the multiplicity of the corresponding eigenspace, $a, b >$ are constants, and the function $f : \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$ is pseudo-almost automorphic function satisfying some additional conditions.

For that, the main idea consists of rewriting Eq. (1.2) as a first-order differential equation on $\mathbb{X} := \mathbb{H} \times \mathbb{H}$ involving the 2×2-operator matrix $\mathcal{B}$. Indeed, if u is differential, setting $z := \begin{pmatrix} u \\ u' \end{pmatrix}$, Eq. (1.2) can be rewritten in the Hilbert space $\mathbb{X}$ in the following form

$$z'(t) = \mathcal{B}z(t) + F(t, z(t)), \ t \in \mathbb{R}, \tag{1.3}$$

where $\mathcal{B}$ is the 2×2-operator matrix defined by

$$\mathcal{B} = \begin{pmatrix} 0 & I_{\mathbb{H}} \\ & \\ -bA & -aI_{\mathbb{H}} \end{pmatrix} \tag{1.4}$$

whose domain $D(\mathcal{B})$ is given by $D(\mathcal{B}) = D(A) \times \mathbb{H}$. Moreover, the semilinear term F appearing in Eq. (1.3) is defined on $\mathbb{R} \times \mathbb{X}_\alpha$ for some $\alpha \in (0,1)$ by

$$F(t,u,v) = \begin{pmatrix} 0 \\ f(t,u) \end{pmatrix},$$

where $\mathbb{X}_\alpha$ is an intermediate space (see assumption(H.2)).

Under some reasonable assumptions, it will be shown that the linear operator matrix $\mathcal{B}$ is sectorial and that its associated semigroup is exponentially stable.

The concept of pseudo almost automorphy is a powerful generalization of both the notion of almost automorphy due to Bochner (see [46]) and that of pseudo almost periodicity due to Zhang (see [21]), which has recently been introduced in the literature by Liang et al. [39, 52, 53]. Such a concept has recently generated several developments and extensions, see, e.g., [18], [20], [29], [30], and [40].

The existence of almost periodic solutions to second-order differential equations constitutes one of the most important topics in qualitative theory of differential equations due essentially to their applications such thermoelastic plate equations [12, 37] or telegraph equation [43] or Sine-Gordon equations [36]. Some contributions on the maximal regularity, bounded, almost periodic, asymptotically almost periodic solutions to abstract second-order differential and partial differential equations have recently been made, among them are [9], [10], [18], [20], [29], [30], [39], [40], [52], [53], [54], [55], and [56]. However, to the best of our knowledge, the existence of pseudo-almost automorphic solutions to second-order differential equations of the form Eq. (1.2) is an untreated original question, which in fact is the main motivation of the present paper.

The paper is organized as follows: Section 2 is devoted to preliminaries facts needed in the sequel. In particular, facts related to sectorial operators and hyperbolic semigroups are discussed. In addition, basic definitions and classical results on the concept of pseudo-almost automorphy are also given. In Sections 3 and 4, we prove the main result. In Section 5, we provide the reader with a few examples to illustrate our main result.

2 Preliminaries

In the sequel, $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ stands for a self-adjoint (possibly unbounded) linear operator on the Hilbert space $\mathbb{H}$ whose spectrum consists of isolated eigenvalues

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_n \to \infty$$

with each eigenvalue having a finite multiplicity γ_j equals to the multiplicity of the corresponding eigenspace. Let $\{e_j^k\}$ be a (complete) orthonormal sequence of eigenvectors associated with the eigenvalues $\{\lambda_j\}_{j\geq 1}$. Clearly, for each

$$u \in D(A) := \left\{ u \in \mathbb{H} : \quad \sum_{j=1}^{\infty} \lambda_j^2 \left\| E_j u \right\|^2 < \infty \right\},$$

$$Au = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j u$$

where $E_j u = \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k$.

Note that $\{E_j\}_{j\geq 1}$ is a sequence of orthogonal projections on $\mathbb{H}$. Moreover, each $u \in \mathbb{H}$ can written as follows:

$$u = \sum_{j=1}^{\infty} E_j u.$$

It should also be mentioned that the operator $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$, which is explicitly expressed in terms of those orthogonal projections E_j by, for all $u \in \mathbb{H}$,

$$S(t)u = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j u.$$

In addition, the fractional powers A^r $(r \geq 0)$ of A exist and are given by

$$D(A^r) = \left\{ u \in \mathbb{H} : \sum_{j=1}^{\infty} \lambda_j^{2r} \left\| E_j u \right\|^2 < \infty \right\}$$

and

$$A^r u = \sum_{j=1}^{\infty} \lambda_j^{2r} E_j u, \quad \forall u \in D(A^r).$$

Let $(\mathbb{X}, \left\| \cdot \right\|)$ be a Banach space. If L is a linear operator on the Banach space $\mathbb{X}$, then, $D(L)$, $\rho(L)$, $\sigma(L)$, $N(L)$, and $R(L)$, stand respectively for the domain, resolvent, spectrum, null-space or kernel, and range of the operator L. Moreover, one sets $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\langle \in \rho(A)$. Furthermore, we set $Q = I - P$ for a projection P. If $Y, \mathbb{Z}$ are Banach spaces, then the space $B(\mathcal{Y}, \mathbb{Z})$ denotes the collection of all bounded linear operators from $\mathcal{Y}$ into $\mathbb{Z}$ equipped with its natural topology. This is simply denoted by $B(\mathcal{Y})$ when $\mathcal{Y} = \mathbb{Z}$.

3 Sectorial Linear Operators

Definition 3.1. A linear operator $L : D(L) \subset \mathbb{X} \mapsto \mathbb{X}$ (not necessarily densely defined) is said to be sectorial if the following hold: there exist constants $\omega \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, and $M > 0$ such that

$\rho(L) \supset S_{\theta,\omega}$,

$$S_{\theta,\omega} := \left\{\lambda \in \mathbb{C} : \lambda \neq \omega, \quad \left|\arg(\lambda-\omega)\right| < \theta\right\}, \tag{3.1}$$

$$\text{and} \quad \|R(\lambda, L)\| \leq \frac{M}{\left|\lambda-\omega\right|}, \quad \lambda \in S_{\theta,\omega}. \tag{3.2}$$

The class of sectorial operators is very rich and contains most of classical operators encountered in the literature. Two examples of sectorial operators are given below.

Example 3.1. Let $p \geq 1$ and let $\mathbb{X} = L^p(0,1)$ be the Lebesgue space equipped with its norm $\left\|\cdot\right\|_p$ defined by

$$\|\varphi\|_p = \left(\int_0^1 \left|\varphi(x)\right|^p dx\right)^{1/p}.$$

Define the linear operator A on $L^p(0,1)$ by

$$D(A) = \left\{u \in W^{2,p}(0,1) : u'(0) = u'(1) = 0\right\}, \quad A(\varphi) = \varphi'', \quad \forall \varphi \in D(A).$$

It can be checked that the operator A is sectorial on $L^p(0,1)$.

Example 3.2. Let $p \geq 1$ and let $\Omega \subset \mathbb{R}^d$ be open bounded subset with C^2 boundary $\partial\Omega$. Let $\mathbb{X} := L^p(\Omega)$ be the Lebesgue space equipped with the norm, $\|\cdot\|_p$ defined by,

$$\|\varphi\|_p = \left(\int_\Omega \left|\varphi(x)\right|^p dx\right)^{1/p}.$$

Define the operator A as follows:

$$D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad A(\varphi) = \Delta\varphi, \quad \forall \varphi \in D(A),$$

where $\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator.

It can be checked that the operator A is sectorial on $L^p(\Omega)$.

It is well-known that [41] if A is sectorial, then it generates an analytic semigroup $(T(t))_{t\geq 0}$, which maps $(0,\infty)$ into $B(\mathbb{X})$ and such that there exist $M_0, M_1 > 0$ with

$$\left\|T(t)\right\| \leq M_0 e^{\omega t}, \quad t > 0, \tag{3.3}$$

$$\left\|t(A-\omega)T(t)\right\| \leq M_1 e^{\omega t}, \quad t > 0. \tag{3.4}$$

In this paper, we suppose that the semigroup $(T(t))_{t\geq 0}$ is hyperbolic, that is, there exist a projection P and constants $M, \delta > 0$ such that $T(t)$ commutes with P, $N(P)$ is invariant with respect to $T(t)$, $T(t) : R(Q) \mapsto R(Q)$ is invertible, and the following hold

$$\left\|T(t)Px\right\| \leq Me^{-\delta t}\left\|x\right\| \qquad \text{for } t \geq 0, \tag{3.5}$$

$$\left\|T(t)Qx\right\| \leq Me^{\delta t}\left\|x\right\| \qquad \text{for } t \leq 0, \tag{3.6}$$

where $Q := I - P$ and, for $t \leq 0$, $T(t) := (T(-t))^{-1}$.

Recall that the analytic semigroup $(T(t))_{t\geq 0}$ associated with A is hyperbolic if and only if

$$\sigma(A) \cap i\mathbb{R} = \emptyset,$$

see details in [28, Prop. 1.15, pp.305].

Definition 3.2. Let $\alpha \in (0,1)$. A Banach space $(\mathbb{X}_\alpha, \left\|\cdot\right\|_\alpha)$ is said to be an intermediate space between $D(A)$ and $\mathbb{X}$, or a space of class $\mathcal{J}_\alpha$, if $D(A) \subset \mathbb{X}_\alpha \subset \mathbb{X}$ and there is a constant $c > 0$ such that

$$\left\|x\right\|_\alpha \leq c\left\|x\right\|^{1-\alpha}\left\|x\right\|_A^\alpha, \qquad x \in D(A), \tag{3.7}$$

where $\left\|\cdot\right\|_A$ is the graph norm of A.

Concrete examples of $\mathbb{X}_\alpha$ include $D((-A^\alpha))$ for $\alpha \in (0,1)$, the domains of the fractional powers of A, the real interpolation spaces $D_A(\alpha,\infty)$, $\alpha \in (0,1)$, defined as the space of all $x \in \mathbb{X}$ such

$$\left[x\right]_\alpha = \sup_{0<t\leq 1}\left\|t^{1-\alpha}AT(t)x\right\| < \infty.$$

with the norm

$$\left\|x\right\|_\alpha = \left\|x\right\| + \left[x\right]_\alpha,$$

the abstract Hölder spaces $D_A(\alpha) := \overline{D(A)}^{\|\cdot\|_\alpha}$ as well as the complex interpolation spaces $[\mathbb{X}, D(A)]_\alpha$, see Lunardi [41] for details.

For a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, one can easily check that similar estimations as both Eq. (3.5) and Eq. (3.6) still hold with the α-norms $\left\|\cdot\right\|_\alpha$. In fact, as the part of A in $R(Q)$ is bounded, it follows from Eq. (3.6) that

$$\left\|AT(t)Qx\right\| \leq C'e^{\delta t}\left\|x\right\| \ \text{ fort} \leq 0.$$

Hence, from Eq. (3.7) there exists a constant $c(\alpha) > 0$ such that

$$\left\|T(t)Qx\right\|_\alpha \leq c(\alpha)e^{\delta t}\left\|x\right\| \quad \text{for } t \leq 0. \tag{3.8}$$

In addition to the above, the following holds

$$\left\|T(t)Px\right\|_\alpha \leq \left\|T(1)\right\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\left\|T(t-1)Px\right\|, \quad t \geq 1,$$

and hence from Eq. (3.5), one obtains

$$\left\|T(t)Px\right\|_\alpha \leq M'e^{-\delta t}\left\|x\right\|, \qquad t \geq 1,$$

where M' depends on α. For $t \in (0,1]$, by Eq. (3.4) and Eq. (3.7),

$$\left\|T(t)Px\right\|_{\alpha} \leq M'' t^{-\alpha}\left\|x\right\|.$$

Hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\left\|T(t)Px\right\|_{\alpha} \leq M(\alpha) t^{-\alpha} e^{-\gamma t}\left\|x\right\| \qquad \text{for } t > 0. \tag{3.9}$$

3.1 Pseudo-Almost Automorphic Functions

Let $BC(\mathbb{R}, \mathbb{X})$ (respectively,
$BC(\mathbb{R} \times \mathcal{Y}, \mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R} \times \mathcal{Y} \mapsto \mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm defined by

$$\left\|u\right\|_{\infty} = \sup_{t \in \mathbb{R}} \left\|u(t)\right\|,$$

is a Banach space. Furthermore, $C(\mathbb{R}, \mathcal{Y})$ (respectively, $C(\mathbb{R} \times \mathcal{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathcal{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times \mathcal{Y} \mapsto \mathbb{X}$).

Definition 3.3. *A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that*

$$g(t) := \lim_{n \to \infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \to \infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic in the classical Bochner's sense. Denote by $AA(\mathbb{X})$ the collection of all almost automorphic functions $\mathbb{R} \mapsto \mathbb{X}$. Note that $AA(\mathbb{X})$ equipped with the sup-norm turns out to be a Banach space.

Among other things, almost automorphic functions satisfy the following properties.

Theorem 3.1. *[46] If $f, f_1, f_2 \in AA(\mathbb{X})$, then*

(i) $f_1 + f_2 \in AA(\mathbb{X})$,

(ii) $\lambda f \in AA(\mathbb{X})$ *for any scalar* λ,

(iii) $f_\alpha \in AA(\mathbb{X})$ *where* $f_\alpha : \mathbb{R} \to \mathbb{X}$ *is defined by* $f_\alpha(\cdot) = f(\cdot + \alpha)$,

(iv) the range $\mathcal{R}_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* $\mathbb{X}$, *thus* f *is bounded in norm,*

(v) if $f_n \to f$ *uniformly on* $\mathbb{R}$ *where each* $f_n \in AA(\mathbb{X})$, *then* $f \in AA(\mathbb{X})$ *too.*

In addition to the above-mentioned properties, we have the the following property due to Bugajewski and Diagana [15]:

(vi) if $g \in L^1(\mathbb{R})$, then $f * g \in AA(\mathbb{R})$, where $f * g$ is the convolution of f with g on $\mathbb{R}$.

Let $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be another Banach space.

Definition 3.4. *A jointly continuous function* $F : \mathbb{R} \times \mathcal{Y} \mapsto \mathbb{X}$ *is said to be almost automorphic in* $t \in \mathbb{R}$ *if* $t \mapsto F(t, x)$ *is almost automorphic for all* $x \in K$ *(*$K \subset \mathcal{Y}$ *being any bounded subset). Equivalently, for every sequence of real numbers* $(s'_n)_{n\in\mathbb{N}}$*, there exists a subsequence* $(s_n)_{n\in\mathbb{N}}$ *such that*

$$G(t, x) := \lim_{n\to\infty} F(t + s_n, x)$$

is well defined in $t \in \mathbb{R}$ *and for each* $x \in K$*, and*

$$\lim_{n\to\infty} G(t - s_n, x) = F(t, x)$$

for all $t \in \mathbb{R}$ *and* $x \in K$*.*

The collection of such functions will be denoted by $AA(\mathcal{Y}, \mathbb{X})$*.*

For more on almost automorphic functions and related issues, we refer the reader to the excellent book by N'Guérékata [46].

Define

$$PAP_0(\mathbb{R}, \mathbb{X}) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left\| f(s) \right\| ds = 0 \right\}.$$

Similarly, $PAP_0(\mathcal{Y}, \mathbb{X})$ will denote the collection of all bounded continuous functions $F : \mathbb{R} \times \mathcal{Y} \mapsto \mathbb{X}$ such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left\| F(s, x) \right\| ds = 0$$

uniformly in $x \in K$, where $K \subset \mathcal{Y}$ is any bounded subset.

Definition 3.5. *(Liang et al. [39] and Xiao et al. [52]) A function* $f \in BC(\mathbb{R}, \mathbb{X})$ *is called pseudo almost automorphic if it can be expressed as* $f = g + \phi$*, where* $g \in AA(\mathbb{X})$ *and* $\phi \in PAP_0(\mathbb{X})$*. The collection of such functions will be denoted by* $PAA(\mathbb{X})$*.*

The functions g and ϕ appearing in Definition 3.5 are respectively called the *almost automorphic* and the *ergodic perturbation* components of f.

Definition 3.6. *A bounded continuous function* $F : \mathbb{R} \times \mathcal{Y} \mapsto \mathbb{X}$ *belongs to* $AA(Y, \mathbb{X})$ *whenever it can be expressed as* $F = G + \Phi$*, where* $G \in AA(\mathcal{Y}, \mathbb{X})$ *and* $\Phi \in PAP_0(\mathcal{Y}, \mathbb{X})$*. The collection of such functions will be denoted by* $PAA(\mathcal{Y}, \mathbb{X})$*.*

We now collect a few useful properties of pseudo almost automorphic functions.

Proposition 3.1. *If* $g \in L^1(\mathbb{R})$, $f \in PAA(\mathbb{R})$, *then* $f * g \in PAA(\mathbb{R})$, *where* $f * g$ *is the convolution of* f *with* g *on* $\mathbb{R}$.

The proof of Proposition 3.1 is based upon [15] and [16].

A substantial result is the next theorem, which is due to Xiao et al. [52].

Theorem 3.2. *[52] The space* $PAA(\mathbb{X})$ *equipped with the sup norm* $\left\| \cdot \right\|_\infty$ *is a Banach space.*

The next composition result, that is Theorem 3.3, is a consequence of [40, Theorem 2.4] and is crucial for the proof of the main result of the paper.

Theorem 3.3. *Suppose* $f : \mathbb{R} \times \mathcal{Y} \mapsto \mathbb{X}$ *belongs to* $PAA(\mathcal{Y}, \mathbb{X})$; $f = g + h$, *with* $x \mapsto g(t, x)$ *being uniformly continuous on each bounded subset* K *of* $\mathcal{Y}$ *uniformly in* $t \in \mathbb{R}$, *that is, for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $x, y \in K$ *and* $\left\| x - y \right\| < \delta$ *yields* $\left\| g(t, x) - g(t, y) \right\| < \varepsilon$ *for all* $t \in \mathbb{R}$. *Furthermore, we suppose that there exists* $L > 0$ *such that*

$$\left\| f(t, x) - f(t, y) \right\| \leq L \left\| x - y \right\|_{\mathcal{Y}}$$

for all $x, y \in \mathcal{Y}$ *and* $t \in \mathbb{R}$.

Then the function defined by $h(t) = f(t, \varphi(t))$ *belongs to* $PAA(\mathbb{X})$ *provided* $\varphi \in PAA(\mathcal{Y})$.

We also have:

Theorem 3.4. *[52] If* $f : \mathbb{R} \times \mathcal{Y} \mapsto \mathbb{X}$ *belongs to* $PAA(\mathcal{Y}, \mathbb{X})$ *and if* $x \mapsto f(t, x)$ *is uniformly continuous on each bounded subset* K *of* $\mathcal{Y}$ *uniformly in* $t \in \mathbb{R}$, *then the function defined by* $h(t) = f(t, \varphi(t))$ *belongs to* $PAA(\mathbb{X})$ *provided* $\varphi \in PAA(\mathcal{Y})$.

4 Main results

Consider the differential equation

$$u'(t) = Lu(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{4.1}$$

where $L : D(L) \subset \mathbb{X} \mapsto \mathbb{X}$ is sectorial and $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is jointly continuous.

Fix once and for all α, β such that $0 \leq \alpha < \beta < 1$. To study the existence and uniqueness of pseudo-almost automorphic solutions to Eq. (4.1) we make the following additional assumptions

(H.1) The operator L is sectorial on $\mathbb{X}$ and generates a hyperbolic (analytic) semigroup $(T(t))_{t \geq 0}$.

(H.2) Let $0 < \alpha < 1$. Then $\mathbb{X}_\alpha = D((-A^\alpha))$, or $\mathbb{X}_\alpha = D_A(\alpha, p), 1 \leq p \leq +\infty$, or $\mathbb{X}_\alpha = D_A(\alpha)$, or $\mathbb{X}_\alpha = [\mathbb{X}, D(A)]_\alpha$.

(H.3) The function $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is given such that $u \mapsto F(t, u)$ is unformly continuous on each bounded subset B of $\mathbb{X}$ uniformly in $t \in \mathbb{R}$. Furthermore, F is Lipschitz in the following sense: there exists $L > 0$ for which

$$\left\|F(t,u) - F(t,v)\right\|_\beta \leq L\left\|u - v\right\|_\alpha$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

Set $S_1u(t) := S_{11}u(t) - S_{12}u(t)$ and $S_2u = S_{22}u - S_{23}u$, where

$$S_{11}u(t) := \int_{-\infty}^{t} T(t-s)PF_1(s,u(s))ds, \quad S_{12}u(t) := \int_{t}^{\infty} T(t-s)QF_1(s,u(s))ds$$

for all $t \in \mathbb{R}$.

Definition 4.1. *Under assumption* (H.1)*, a function* $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ *is said to be a mild solution to Eq. (4.1) provided that*

$$u(t) = T(t-s)u(s) + \int_{s}^{t} T(t-r)F(r,u(r))dr \tag{4.2}$$

for each $\forall t \geq s, \ t, s \in \mathbb{R}$.

Consider the differential equation

$$u'(t) = Lu(t) + g(t), \quad t \in \mathbb{R}, \tag{4.3}$$

where $g : \mathbb{R} \mapsto \mathbb{X}$ is continuous.

Theorem 4.1. *Under assumptions* (H.1)-(H.2)*, if* $g \in B(\mathbb{R}, \mathbb{X})$*, then we have:*

(i) Eq.(4.3) has a unique bounded mild solution $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$*, which can be explicitly given by*

$$u(t) = \int_{-\infty}^{t} T(t-s)Pg(s)ds - \int_{t}^{\infty} T(t-s)Qg(s)ds. \tag{4.4}$$

(ii) If $g \in PAA(\mathbb{X}_\alpha)$*, then* $u \in PAA(\mathbb{X}_\alpha)$.

Proof. (i) Since g is bounded, we can easily show that u given above is well-defined. Moreover, u satisfies

$$u(t) = T(t-s)u(s) + \int_{s}^{t} T(t-r)g(r)dr$$

for each $\forall t \geq s, \ t, s \in \mathbb{R}$.

The continuity and uniqueness of u is also clear. For the boundedness in $\mathbb{X}_\alpha$, using (3.8) and (3.9), we obtain

$$\begin{aligned}
\Big\|u(t)\Big\|_\alpha &\leq c\Big\|u(t)\Big\|_\beta \\
&\leq c\int_{-\infty}^{t}\Big\|T(t-s)Pg(s)\Big\|_\beta ds + c\int_t^{+\infty}\Big\|T(t-s)Qg(s)\Big\|_\beta ds \\
&\leq cM(\beta)\int_{-\infty}^{t} e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\beta}\Big\|g(s)\Big\|ds + cc(\beta)\int_t^{+\infty} e^{-\delta(s-t)}\Big\|g(s)\Big\|ds \\
&\leq cM(\beta)\Big\|g\Big\|_\infty\int_0^{+\infty} e^{-\sigma}\left(\frac{2\sigma}{\delta}\right)^{-\beta}\frac{2d\sigma}{\delta} + cc(\beta)\Big\|g\Big\|_\infty\int_0^{+\infty} e^{-\delta\sigma}d\sigma \\
&\leq cM(\beta)\delta^\alpha\Gamma(1-\beta)\Big\|g\Big\|_\infty + cc(\beta)\delta^{-1}\Big\|g\Big\|_\infty,
\end{aligned}$$

and hence

$$\Big\|x(t)\Big\|_\alpha \leq c\Big\|x(t)\Big\|_\beta \leq c'c\Big[M(\beta)\delta^\beta\Gamma(1-\beta) + c(\beta)\delta^{-1}\Big]\Big\|g\Big\|_{\infty,\alpha}. \tag{4.5}$$

It remains to prove (ii). For that, we first consider the first integral in the expression of Eq. (4.4) and denote it Su. Now write $g = \phi + \zeta$ where $\phi \in AA(\mathbb{X}_\alpha)$ and $\zeta \in PAP_0(\mathbb{X}_\alpha)$. Clearly, Su can be rewritten as

$$(Su)(t) \quad = \quad \int_{-\infty}^{t} T(t-s)P\phi(s)ds + \int_{-\infty}^{t} Tt-s)P\zeta(s)ds.$$

Set

$$\Phi(t) = \int_{-\infty}^{t} T(t-s)P\phi(s)ds, \quad \text{and} \quad \Psi(t) = \int_{-\infty}^{t} T(t-s)P\zeta(s)ds$$

for each $t \in \mathbb{R}$.

The next step consists of showing that $\Phi \in AA(\mathbb{X}_\alpha)$ and $\Psi \in PAP_0(\mathbb{X}_\alpha)$. Indeed, since $\phi \in AA(\mathbb{X}_\alpha)$, for every sequence of real numbers $(\tau'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\psi(t) := \lim_{n\to\infty}\phi(t+\tau_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty}\psi(t-\tau_n) = \phi(t)$$

for each $t \in \mathbb{R}$.

Set $\Phi_1(t) = \displaystyle\int_{-\infty}^{t} T(t-s)P\psi(s)ds$ for all $t \in \mathbb{R}$.

Now

$$\begin{aligned}\Phi(t+\tau_n)-\Phi_1(t) &= \int_{-\infty}^{t+\tau_n} T(t+\tau_n-s)P\phi(s)ds-\int_{-\infty}^{t} T(t-s)P\psi(s)ds\\ &= \int_{-\infty}^{t} T(t-s)P\phi(s+\tau_n)ds-\int_{-\infty}^{t} T(t-s)P\psi ds\\ &= \int_{-\infty}^{t} T(t-s)P\Big(\phi(s+\tau_n)-\psi(s)\Big)ds\end{aligned}$$

Using Lebesgue Dominated Convergence Theorem, one can easily see that

$$\Big\|\int_{-\infty}^{t} T(t-s)P\Big(\phi(s+\tau_n)-\psi(s)\Big)ds\Big\|_\alpha \to 0 \text{ as } n\to\infty,\ t\in\mathbb{R}.$$

Thus

$$\Phi_1(t)=\lim_{n\to\infty}\Phi(t+\tau_n),\quad t\in\mathbb{R}.$$

Similarly, one can easily see that

$$\Phi(t)=\lim_{n\to\infty}\Phi_1(t-\tau_n),\quad t\in\mathbb{R}.$$

Therefore, $\Phi\in AA(\mathbb{X}_\alpha)$.

Let us now show that $\Psi\in PAP_0(\mathbb{X}_\alpha)$. First, note that $s\mapsto\Psi(s)$ is a bounded continuous function. It remains to show that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big\|\Psi(t)\Big\|_\alpha dt=0.$$

Again using Eq. (3.9) it follows that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big\|\Psi(t)\Big\|_\alpha dt &\le \lim_{T\to\infty}\frac{M(\alpha)}{2T}\int_{-T}^{T}\int_{0}^{+\infty}s^{-\alpha}e^{-\frac{\delta}{2}s}\Big\|\zeta(t-s)\Big\|_\alpha ds dt\\ &\le \lim_{T\to\infty}M(\alpha)\int_{0}^{+\infty}s^{-\alpha}e^{-\frac{\delta}{2}s}\frac{1}{2T}\int_{-T}^{T}\Big\|\zeta(t-s)\Big\|_\alpha dtds.\end{aligned}$$

Let

$$\Gamma_s(T)=\frac{1}{2T}\int_{-T}^{T}\Big\|\zeta(t-s)\Big\|_\alpha dt.$$

Since $PAP_0(\mathbb{X}_\alpha)$ is translation invariant it follows that $t\mapsto\zeta(t-s)$ belongs to $PAP_0(\mathbb{X}_\alpha)$ for each $s\in\mathbb{R}$, and hence

$$\lim_{T\mapsto\infty}\frac{1}{2T}\int_{-T}^{T}\Big\|\zeta(t-s)\Big\|_\alpha dt=0$$

for each $s\in\mathbb{R}$.

One completes the proof by using the Lebesgue Dominated Convergence Theorem and the fact $\Gamma_s(T) \mapsto 0$ as $T \to \infty$ for each $s \in \mathbb{R}$.

The proof for the second integral in the expression of Eq. (4.4) is similar to that of $Su(\cdot)$ and hence moitted. (In that case, one makes use of Eq. (3.8) rather than Eq. (3.9).) $\square$

Using the composition of pseudo-almost automorphic functions and Theorem 4.1, it is easy to see that the following technical lemmas hold.

Lemma 4.1. *Under assumptions* (H.1)-(H.2)-(H.3)*, then the integral operator* S_1 *defined above maps* $PAA(\mathbb{X}_\alpha)$ *into itself.*

Lemma 4.2. *Under assumptions* (H.1)-(H.2)-(H.3)*, the integral operator* S_1 *defined above is a contraction whenever* L *is small enough.*

Proof. Let $v, w \in PAA(\mathbb{X}_\alpha)$. Now,

$$\begin{aligned}
\left\|S_{11}v(t) - S_{11}w(t)\right\|_\alpha &\leq \int_{-\infty}^{t} M(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \left\|F_1(s, v(s)) - F_1(s, w(s))\right\| ds \\
&\leq c \int_{-\infty}^{t} M(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \left\|F_1(s, v(s)) - F_1(s, w(s))\right\|_\beta ds \\
&\leq LcM(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \left\|v(s) - w(s)\right\| ds \\
&\leq Lc'cM(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \left\|v(s) - w(s)\right\|_\alpha ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\|S_{12}v(t) - S_{12}w(t)\right\|_\alpha &\leq \int_{t}^{\infty} c(\beta) e^{-\delta(t-s)} \left\|F_1(s, v(s)) - F_1(s, w(s))\right\| ds \\
&\leq cc(\beta) \int_{t}^{\infty} e^{-\delta(t-s)} \left\|F_1(s, v(s)) - F_1(s, w(s))\right\|_\beta ds \\
&\leq Lcc(\beta) \int_{t}^{\infty} e^{-\delta(t-s)} \left\|v(s) - w(s)\right\| ds \\
&\leq Lcc'c(\beta) \int_{t}^{\infty} e^{-\delta(t-s)} \left\|v(s) - w(s)\right\|_\alpha ds.
\end{aligned}$$

Consequently,

$$\left\|S_1 v - S_1 w\right\|_{\infty,\alpha} \leq Lcc'\Big(M(\alpha)\Gamma(1-\alpha)(2\delta^{-1})^{1-\alpha} + c(\beta)\delta^{-1}\Big) \left\|v - w\right\|_{\infty,\alpha}$$

and hence S_1 is a contraction whenever L is small enough. $\square$

Theorem 4.2. *Suppose assumptions* (H.1)-(H.2)-(H.3) *and that* L *is small enough, then the autonomous differential equation Eq.* (4.1) *has a unique pseudo almost automorphic solution* u *satisfying* $u = S_1 u$.

Proof. This is an immediate consequence of Lemma 4.1, Lemma 4, Lemma 4.2, and the Banach fixed point theorem.

□

5 Pseudo Almost Automorphic Solutions to Some Second-Order Differential Equations

We have previously seen that each $u \in \mathbb{H}$ can be written in terms of the sequence of orthogonal projections E_n as follows:

$$u = \sum_{n=1}^{\infty} \sum_{k=1}^{\gamma_n} \langle u, e_n^k \rangle e_n^k = \sum_{n=1}^{\infty} E_n u.$$

Moreover, for each $u \in D(A)$,

$$Au = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j u.$$

Therefore, for all $z := \begin{pmatrix} u \\ v \end{pmatrix} \in D = D(\mathcal{B}) = D(A) \times \mathbb{H}$, we obtain the following

$$\begin{aligned}
\mathcal{B}z &= \begin{pmatrix} 0 & I_{\mathbb{H}} \\ -bA & -aI_{\mathbb{H}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \begin{pmatrix} v \\ -bAu - av \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} E_n v \\ -b \sum_{n=1}^{\infty} \lambda_n E_n u - a \sum_{n=1}^{\infty} E_n v \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ -b\lambda_n & -a \end{pmatrix} \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \sum_{n=1}^{\infty} A_n P_n z,
\end{aligned}$$

where

$$P_n := \begin{pmatrix} E_n & 0 \\ \\ 0 & E_n \end{pmatrix}, \ n \geq 1,$$

and

$$A_n := \begin{pmatrix} 0 & 1 \\ \\ -b\lambda_n & -a \end{pmatrix}, \ n \geq 1. \tag{5.1}$$

Now, the characteristic equation for A_n is given by

$$\lambda^2 + a\lambda + \lambda_n b = 0 \tag{5.2}$$

whose eigenvalues are given by

$$\lambda_1^n := \frac{-a + \sqrt{a^2 - 4\lambda_n b}}{2} \quad \text{and} \quad \lambda_2^n := \frac{-a - \sqrt{a^2 - 4\lambda_n b}}{2}.$$

Since $a > 0$ it follows that all roots of Eq. (5.2) are nonzero. Moreover, the real part of each of its roots is: $-a/2 < 0$. Therefore, there exists $\omega \in \left(\frac{\pi}{2}, \pi\right)$ such that $\rho(L) \supset S_\omega$, where

$$S_\omega := \left\{ z \in \mathbb{C} \setminus \{0\} : \quad \left|\arg z\right| < \omega \right\}.$$

On the other hand, one can show without difficulty that $A_n = K_n^{-1} J_n K_n$, where J_n, K_n and K_n^{-1} are respectively given by

$$J_n = \begin{pmatrix} \lambda_1^n & 0 \\ \\ 0 & \lambda_2^n \end{pmatrix}, \quad K_n = \begin{pmatrix} 1 & 1 \\ \\ \lambda_1^n(t) & \lambda_2^n \end{pmatrix},$$

and

$$K_n^{-1} = \frac{1}{\lambda_1^n - \lambda_2^n} \begin{pmatrix} -\lambda_2^n & 1 \\ \\ \lambda_1^n & -1 \end{pmatrix}.$$

For $\lambda \in S_\omega$ and $z \in \mathbb{X}$, one has

$$\begin{aligned} R(\lambda, \mathcal{B})z &= \sum_{n=1}^{\infty} (\lambda - A_n)^{-1} P_n z \\ &= \sum_{n=1}^{\infty} K_n (\lambda - J_n P_n)^{-1} K_n^{-1} P_n z. \end{aligned}$$

Hence,

$$\begin{aligned}\left\|R(\lambda,\mathcal{B})z\right\|^2 &\le \sum_{n=1}^{\infty}\left\|K_nP_n(\lambda-J_nP_n)^{-1}K_n^{-1}P_n\right\|^2_{B(\mathbb{X})}\left\|P_nz\right\|^2\\ &\le \sum_{n=1}^{\infty}\left\|K_nP_n\right\|^2_{B(\mathbb{X})}\left\|(\lambda-J_nP_n)^{-1}\right\|^2_{B(\mathbb{X})}\left\|K_n^{-1}P_n\right\|^2_{B(\mathbb{X})}\left\|P_nz\right\|^2.\end{aligned}$$

Moreover, for $z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{X}$, we obtain

$$\begin{aligned}\left\|K_nP_nz\right\|^2 &= \left\|E_nz_1+E_nz_2\right\|^2+\left\|\lambda_1^nE_nz_1+\lambda_2^nE_nz_2\right\|^2\\ &\le 3\left(1+\left|\lambda_1^n\right|^2\right)\left\|z\right\|^2.\end{aligned}$$

Thus, there exists $C_1 > 0$ such that

$$\left\|K_nP_nz\right\| \le C_1\left|\lambda_1^n\right|\left\|z\right\| \quad \text{for all} \quad n \ge 1.$$

Similarly, for $z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{X}$, one can show that there is $C_2 > 0$ such that

$$\left\|K_n^{-1}P_nz\right\| \le \frac{C_2}{\left|\lambda_1^n\right|}\left\|z\right\| \quad \text{for all} \quad n \ge 1.$$

Now, for $z \in \mathbb{X}$, we have

$$\begin{aligned}\left\|(\lambda-J_nP_n)^{-1}z\right\|^2 &= \left\|\begin{pmatrix}\frac{1}{\lambda-\lambda_1^n} & 0\\ 0 & \frac{1}{\lambda-\lambda_2^n}\end{pmatrix}\begin{pmatrix} z_1\\ z_2\end{pmatrix}\right\|^2\\ &\le \frac{1}{|\lambda-\lambda_1^n|^2}\left\|z_1\right\|^2+\frac{1}{|\lambda-\lambda_2^n|^2}\left\|z_2\right\|^2.\end{aligned}$$

Let $\lambda_0 > 0$. Define the function

$$\eta(\lambda) := \frac{1+\left|\lambda\right|}{\left|\lambda-\lambda_2^n\right|}.$$

It is clear that η is continuous and bounded on the closed set

$$\Sigma := \left\{\lambda \in \mathbb{C} : \left|\lambda\right| \le \lambda_0, \ \left|\arg\lambda\right| \le \omega\right\}.$$

On the other hand, it is clear that η is bounded for $\left|\lambda\right| > \lambda_0$. Thus η is bounded on S_ω. If we take

$$N = \sup\left\{ \frac{1+\left|\lambda\right|}{\left|\lambda - \lambda_n^j\right|} \ : \ \lambda \in S_\omega, n \geq 1\,;\ j = 1, 2, \right\}.$$

Therefore,

$$\left\|(\lambda - J_n P_n)^{-1} z\right\| \leq \frac{N}{1+\left|\lambda\right|}\|z\|, \quad \lambda \in S_\omega.$$

Consequently,

$$\left\|R(\lambda, \mathcal{B})\right\| \leq \frac{K}{1+\left|\lambda\right|}$$

for all $\lambda \in S_\omega$ and $t \in \mathbb{R}$.

In view of the above, $\mathcal{B}$ is sectorial. Let $(e^{\tau\mathcal{B}})_{\tau\geq 0}$ be the nalytic semigroup associated with it. Let us show that $(e^{\tau\mathcal{B}})_{\tau\geq 0}$ is exponentially stable.

Now

$$e^{\tau\mathcal{B}} z = \sum_{n=0}^{\infty} K_n^{-1} P_n e^{\tau J_n} P_n K_n P_n z, \ z \in \mathbb{X}.$$

On the other hand, we have

$$\left\|e^{\tau\mathcal{B}} z\right\| \ = \ \sum_{n=0}^{\infty} \left\|K_n^{-1} P_n\right\|_{B(\mathbb{X})} \left\|e^{\tau J_n} P_n\right\|_{B(\mathbb{X})} \left\|K_n P_n\right\|_{B(\mathbb{X})} \left\|P_n z\right\|,$$

with for each $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$\begin{aligned}
\left\|e^{\tau J_n} P_n z\right\|^2 &= \left\| \begin{pmatrix} e^{\lambda_1^n \tau} E_n & 0 \\ 0 & e^{\lambda_2^n \tau} E_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|^2 \\
&\leq \left\|e^{\lambda_1^n \tau} E_n z_1\right\|^2 + \left\|e^{\lambda_2^n \tau} E_n z_2\right\|^2 \\
&\leq e^{-a\tau}\left\|z\right\|^2.
\end{aligned}$$

Therefore,

$$\left\|e^{\tau\mathcal{B}}\right\| \leq C e^{-a\tau}, \quad \tau \geq 0. \tag{5.3}$$

It is now clear that if L is small enough, then the second-order differential equation Eq. (1.3) has at most one solution

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{X}_\alpha = \mathbb{H}_\alpha \times \mathbb{H},$$

which in addition is pseudo almost automorphic. Therefore, Eq. (1.2) has a unique solution $u \in \mathbb{H}_\alpha$, which in addition is pseudo almost automorphic.

6 Examples

6.1 1-Dimensional Sine-Gordon Equation

Let $L > 0$ and and let $J = (0, L)$. Let $\mathbb{H} = L^2(J)$ be equipped with its natural topology. Our main objective here is to study the existence of pseudo almost automorphic solutions to a somewhat general one-dimensional Sine-Gordon equation with Dirichlet boundary conditions, which had been studied in the literature especially by Leiva [36] in the following form

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} - d\frac{\partial^2 u}{\partial x^2} + k\sin u = p(t, x), \qquad t \in \mathbb{R}, \quad x \in J \tag{6.1}$$

$$u(t, 0) = u(t, L) = 0, \qquad t \in \mathbb{R} \tag{6.2}$$

where c, d, k are positive constants, $p : \mathbb{R} \times J \mapsto \mathbb{R}$ is continuous and bounded.

Precisely, we are interested in the following system of second-order partial differential equations

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} - b\frac{\partial^2 u}{\partial x^2} = Q(t, x, u), \qquad t \in \mathbb{R}, \quad x \in J \tag{6.3}$$

$$u(t, 0) = u(t, L) = 0, \qquad t \in \mathbb{R} \tag{6.4}$$

where $a, b > 0$ and $Q : \mathbb{R} \times J \times L^2(J) \mapsto L^2(J)$ is pseudo-almost automorphic.

Let us take

$$Av = -v'' \ \text{ for all } \ u \in D(A) = \mathbb{H}^1_0(J) \cap \mathbb{H}^2(J)$$

and suppose that $Q : \mathbb{R} \times J \times L^2(J) \mapsto \mathbb{H}^\beta_0(J)$ is pseudo-almost automorphic. Moreover, Q is Lipschitz in the following sense: there exists $L'' > 0$ for which

$$\Big\|Q(t, x, u) - Q(t, x, v)\Big\|_{\mathbb{H}^\beta_0(J)} \leq L''\Big\|u - v\Big\|_2$$

for all $u, v \in L^2(J)$, $x \in J$ and $t \in \mathbb{R}$.

Consequently, the system Eq. (6.3) - Eq. (6.4) has at most one solution $u \in PAA(\mathbb{H}^1_0(J))$ when L'' is small enough.

6.2 N-dimensional Sine-Gordon Equation

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a open bounded subset with C^2 boundary $\Gamma = \partial\Omega$ and let $\mathbb{H} = L^2(\Omega)$ equipped with its natural topology $\|\cdot\|_{L^2(\Omega)}$. Here, we are interested in the N-dimensional Sine-Gordon studied in the previous example, that is, the system of second-order partial differential equations given by

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} - b\Delta u = R(t, x, u), \qquad t \in \mathbb{R}, \quad x \in \Omega \tag{6.5}$$

$$u(t, x) = 0, \qquad t \in \mathbb{R}, \quad x \in \partial\Omega \tag{6.6}$$

where $a, b > 0$, and $R : \mathbb{R} \times \Omega \times L^2(\Omega) \mapsto L^2(\Omega)$ is jointly continuous.

Define the linear operator A as follows:

$$Au = -\Delta u \text{ for all } u \in D(A) = \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega).$$

For each $\mu \in (0, 1)$, we take $\mathbb{H}_\mu = D((-\Delta)^\mu) = \mathbb{H}_0^\mu(\Omega) \cap \mathbb{H}^{2\mu}(\Omega)$ equipped with its μ-norm $\| \cdot \|_\mu$.

Suppose that $R : \mathbb{R} \times \Omega \times L^2(\Omega) \mapsto \mathbb{H}_0^\beta(\Omega)$ is pseudo-almost automorphic. Moreover, R is Lipschitz in the following sense: there exists $L''' > 0$ for which

$$\Big\| R(t, x, u) - R(t, x, v) \Big\|_\beta \leq L''' \Big\| u - v \Big\|_2$$

for all $u, v \in L^2(\Omega)$, $x \in \Omega$ and $t \in \mathbb{R}$.

Therefore, the system Eq. (6.5) - Eq. (6.6) has at most one solution $u \in PAA(\mathbb{H}_0^1(\Omega))$ when L''' is small enough.

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