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More on Approximate Operators

Philip J. Maher
Mathematics And Statistics Group,
Middlesex University, Hendon Campus, The Burrough,
London Nw4 4 Bt, United Kingdom.
email: p.maher@mdx.ac.uk
and
Mohammad Sal Moslehian
Department Of Pure Mathematics,
Centre Of Excellence In Analysis On Algebraic Structures,
(CEAAS), Ferdowsi University Of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.
email: moslehian@ferdowsi.um.ac.ir, moslehian@member.ams.org

ABSTRACT

This note is a continuation of the work on (p, ϵ)–approximate operators studied by Mirzavaziri, Miura and Moslehian. [4]. We investigate approximate partial isometries and approximate generalized inverses. We also prove that if T is an invertible contraction satisfying $\|TT^*T - T\| < \epsilon < \frac{2}{3\sqrt{3}}$. Then there exists a partial isometry V such that $\|T - V\| < K\epsilon$ for $K > 0$.

RESUMEN

Esta trabajo es una continuación del trabajo sobre operadores (p, ϵ)–aproximados estudiados por Mirzavaziri, Miura y Moslehian [4]. Investigamos isometrías parciales

aproximadas e inversas aproximadas generalizadas. También probamos que si T es una contracción invertible que satisface $\|TT^*T - T\| < \epsilon < \frac{2}{3\sqrt{3}}$ entonces existe una isometría parcial V tal que $\|T - V\| < K\epsilon$ para $K > 0$.

Keywords and Phrases: Hilbert space; approximation; unitary; partial isometry; polar decomposition; (p, ϵ)-approximate operator

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1 Introduction

This note is a continuation of the work on (p, ϵ)–approximate operators and operator approximation studied in [4]. Mirzavaziri et al investigated (p, ϵ)–approximate (co) isometries and (p, ϵ)–approximate unitaries. For example, a (p, ϵ)–approximate isometry is defined as an operator T in $\mathcal{L}(H)$ for which

$$\| [T^*T - I] f\| \leq \epsilon \|f\|^p \tag{1.1}$$

where p is a real number and ϵ a fixed positive number. They also proved, for example, the following result on unitary approximation: if to each $0 < \epsilon < 1$ an operator T in $\mathcal{L}(H)$ satisfies $\|T^*T - I\| \leq \epsilon$ and $\|TT^* - I\| \leq \epsilon$ there corresponds a unitary operator U such that $\|T - U\| < \epsilon$.

In section 2 we investigate approximate partial isometries and approximate generalized inverses. In section 3 we investigate operator approximation. We prove (Theorem 3.2 below) that an invertible contraction T satisfying $\|TT^*T - T\| < \epsilon < \frac{2}{3\sqrt{3}}$ can be approximated by a partial isometry.

Recall that a contraction T in $\mathcal{L}(H)$ is an operator such that $\|T\| \leq 1$. Recall that the polar decomposition of an operator T says that T can be expressed uniquely as $T = U|T|$, provided $\mathrm{Ker}U = \mathrm{Ker}|T|$, where U is a partial isometry. By definition, a partial isometry U is a isometric on $(\mathrm{Ker}U)^+$; and $|T|$ denotes the positive square root of T^*T.

2 Approximate Operators

In (1.1) (the example of a (p, ϵ)–approximate isometry) there is no question of letting $\epsilon \to 0$; for otherwise, the subject would collapse into triviality. For fixed ϵ the upshot of this section is that the (p, ϵ)–approximate operators considered here coincide with their ordinary (exact) counterparts provided $p \neq 1$. In the cases studied here the operator T we are concerned with must satisfy an operator equation of the form

$$F(T, T^*, T^-) = 0$$

(where T^- is a generalized inverse of T: see Example 2.2 below). Our results hinge on the following lemma.

Lemma 2.1. *Let p be a real number such that $p \neq 1$ and let $\epsilon > 0$. If*

$$\|F(T, T^*, T^-)\| \leq \epsilon \|f\|^p \tag{2.1}$$

then $\|F(T, T^*, T^-)\| = 0$.

Proof. In (2.1) substitute rf for f where $r > 0$. Then, by the linearity of T,

$$\|F(T, T^*, T^-)\| \leq \epsilon r^{p-1} \|f\|^p. \tag{2.2}$$

If $p < 1$ so that $r^{p-1} = r^{-k}$ where $k > 0$ then $\epsilon r^{p-1} \|f\|^p = \dfrac{\epsilon \|f\|^p}{r^k} \to 0$ as $r \to \infty$. If $p > 1$ then $\epsilon r^{p-1} \|f\|^p \to 0$ as $r \to 0$.

□

Example 2.2. (Partial isometries). There is the following (equivalent) algebraic definition of a partial isometry: T is a partial isometry if $T = TT^*T$ [3, Problem 127, Corollary 3]. Given a real number p and $\epsilon > 0$, a (p, ϵ)–approximate partial isometry is an operator T in $\mathbb{B}(H)$ for which

$$\| [TT^*T - T] f\| \leq \|f\|^p.$$

Let $F(T, T^*) = TT^*T - T$; if $p \neq 1$ then, by Lemma 2.1, $F(T, T^*) = 0$ i.e. T is an (exact) partial isometry.

Counterexample 2.3. This shows that the condition $p \neq 1$ in Lemma 2.1 cannot be dropped. Let

$$T = \begin{bmatrix} 0 & 0 & \sqrt{\epsilon} \\ 0 & \sqrt{\epsilon} & 0 \\ \sqrt{\epsilon} & 0 & 0 \end{bmatrix} \in M_3(\mathbb{C}).$$

Then, for $1 < \epsilon < 2$,

$$\| [TT^*T - T] f\| = |\epsilon - 1| \sqrt{\epsilon} \, \|f\| < \sqrt{\epsilon} \, \|f\| < \epsilon \, \|f\|$$

yet T is not a partial isometry since $\|Tf\| = \sqrt{\epsilon} \, \|f\|$ for all f in H.

A (p, ϵ)–approximate normal partial isometry is an operator T in $\mathbb{B}(H)$ for which

$$\left\| \left[T^*T^2 - T \right] f \right\| \leq \epsilon \|f\|^p \qquad (a)$$

and

$$\| \left[T^2T^* - T \right] f\| \leq \epsilon \|f\|^p \qquad (b)$$

for given $\epsilon > 0$ and a real number p. Let $F_1(T, T^*) = T^*T^2 - T$ and $F_2(T, T^*) = T^2T^* - T$; then if $p \neq 1$ Lemma 2.1 applied to F_1 and F_2 yields (a) $T^*T^2 = T$ and (b) $T^2T^* = T$. Therefore, from (a), $T^*T^2T^* = TT^*$ and, from (b), $T^*T^2T^* = T^*T$. Thus, T is normal and hence by, say (a), $TT^*T = T$.

Example 2.4. (Generalized inverses). An operator T^- is said to be a generalized inverse of the operator T if $TT^-T = I$. An operator T in $\mathbb{B}(H)$ has a generalized inverse if and only if $\mathrm{Ran}T$ is closed [7, p. 261]. For an operator T, with closed range, its Moore – Penrose inverse T^+ has range $\mathrm{Ran}T^+ = (\mathrm{Ker}T)^\perp$ and satisfies

$$\left.\begin{array}{ll} TT^+T = T & (i) \\ T^+TT^+ = T^+ & (ii) \\ (TT^+)^* = TT^+ & (iii) \\ (T^+T)^* = T^+T & (iv) \end{array}\right\} \quad (MP)$$

and, further, T^+ is uniquely determined by these properties. If an operator T^- satisfies properties (i), (iii) [(i), (iv)] it will be called a (i), (iii) [(i), (iv)] inverse of T. A (p, ϵ)–approximate generalized inverse of T is an operator T^- in $\mathbb{B}(H)$ for which

$$\|[TT^-T - T]f\| \leq \epsilon\|f\|^p$$

for $\epsilon > 0$ and real p. Let $F_1(T, T^+) = TT^+T - T$, $F_2(T, T^+) = T^+TT^+ - T^+$, $F_3(T, T^+) = (TT^+)^* - TT^* - T$ and $F_4(T, T^+) = (T^+T)^* - T^+T$; then a (p, ϵ)–approximate Moore–Penrose inverse pf T is an operator T^+ in $\mathbb{B}(H)$ for which

$$\|F_i(T < T^+)f\| \leq \epsilon\|f\|^p$$

for $i = 1, \ldots, 4$ and for $\epsilon > 0$ and real p. Let $F(T, T^-) = TT^-T - T$; then if $p \neq 1$, by Lemma 2.1, $F(T, T^-) = 0$ i.e. T^- is a (exact) generalized inverse of T; and, for $p \neq 1$, applying Lemma 2.1 successively to F_1, F_2, F_3 and F_4 yields $F_1 = F_2 = F_3 = F_4 = 0$ i.e. T^+ satisfies (MP) so that T^+ is the (exact) Moore – Penrose inverse of T.

Counterexample 2.5. Again, we cannot drop the condition $p \neq 1$ in Lemma 2.1. Take $T = \epsilon S$ where

$$S = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and $0 < \epsilon \leq 1$. Let $T' = T$. Then

$$\begin{aligned} \|[TT'T - T]f\| &= |\epsilon^3 - \epsilon|\|Sf\| \leq \epsilon|\epsilon^2 - 1|\|S\|\|f\| \\ &= \epsilon|\epsilon^2 - 1|\|f\| \leq \epsilon\|f\| \end{aligned}$$

yet $T' = \epsilon \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is not a generalized inverse of T (except, as can be verified, if $\epsilon = 1$) for, e.g., if $\epsilon = \frac{1}{2}$ then $TT'T = \frac{1}{4}T$.

Does the algebraic structure of approximate operators mirror that of their exact counterparts? For approximate isometries the answer is " yes". The product of two (exact) isometries is an (exact) isometry. The same is true for approximate isometries.

Proposition 2.6. *The product of two* (p, ϵ)*–approximate isometries is a* (p, ϵ')*–approximate isometry.*

Proof. For $p \neq 1$, by Lemma 2.1, a (p, ϵ)–approximate isometry is an (exact) isometry. Therefore, we need to prove this result in the case of $p = 1$. Accordingly, let T_1 and T_2 be two approximate isometries such that

$$\|[T_1^* T_1 - I]f\| \leq \epsilon_1 \|f\| \quad \text{and} \quad \|[T_2^* T_2 - I]f\| \leq \epsilon_2 \|f\|$$

for $\epsilon_1 > 0$, $\epsilon_2 > 0$ for all f in H. Assertion: if $\|[T^*T - I]f\| \leq \epsilon \|f\|$ for $\epsilon > 0$ and for all f in H then $\|T\|^2 \leq \epsilon + 1$. Proof of assertion:

$$\|T^*Tf\| = \|[T^*T - I]f + f\| \leq \|[T^*T - I]f\| + \|f\| \leq (\epsilon + 1)\|f\|$$

whence the result $\|T\|^2 = \|T^*T\| \leq \epsilon + 1$ follows by taking supremum over unit vectors.

Now,

$$\begin{aligned} \|[(T_1T_2)^*(T_1T_2) - I]f\| &= \|[T_2^*(T_1^*T_1 - I)T_2 - I + T_2^*T_2]f\| \\ &\leq \|T_2^*\| \|[T_1^*T_1 - I]T_2 f\| + \|[T_2^*T_2 - I]f\| \\ &\leq \|T_2^*\| \epsilon_1 \|T_2 f\| + \epsilon_2 \|f\| \\ &\leq ((\epsilon_2 + 1)\epsilon_1 + \epsilon_2)\|f\| = (\epsilon_1 + \epsilon_1\epsilon_2 + \epsilon_2)\|f\|. \end{aligned}$$

□

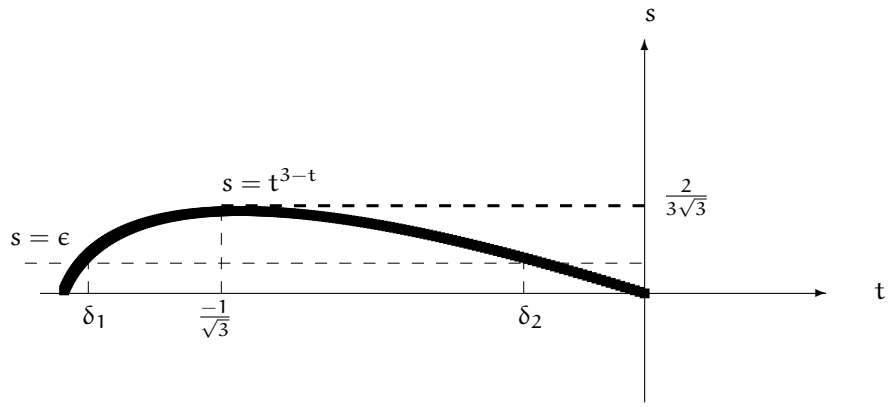
We cannot expect a similar result about product of approximate partial isometries since it is not true that the product of two (exact) partial isometries is an (exact) partial isometry.

3 Approximating Contractions

We need the following lemma.

Lemma 3.1. *Let* $T \leq 0, \|T\| \leq 1$ *and* $\|T^3 - T\| < \epsilon < \dfrac{2}{3\sqrt{3}}$*. Then there is a self–adjoint partial isometry* S *such that* $\|T - S\| < K\epsilon$ *for a certain constant* $K > 0$*.*

Proof. The conditions $T \leq 0, \|T\| \leq 1$ imply that $\mathrm{sp}(T) \subseteq [-1, 0]$. Let $\delta_1, \delta_2 (\delta_1 < \delta_2)$ be the solutions of polynomial equation $t^3 - t = \epsilon$ in $[-1, 0]$. Then $|t^3 - t| < \epsilon$ for all $\mathrm{sp}(T)$, whence $t \in \mathrm{sp}(T) \subseteq [-1, \delta_1] \cup [\delta_2, 0]$.



Therefore,

$$\varphi(t) = \begin{cases} -1 & t \in [-1, \delta_1] \\ 0 & t \in [\delta_2, 0] \end{cases}$$

is a continuous function on $\mathrm{sp}(T)$. Using the functional calculus, we observe that $S = \varphi(T)$ satisfies $S^* = S$ and $SS^*S = S$ and

$$\|T - S\| = \sup_{t \in \mathrm{sp}(T)} |\varphi(t) - t| = \max\{1 + \delta_1, |\delta_2|\} < K\epsilon,$$

for certain $K > 0$.

□

Now we are ready to proof our next result.

Theorem 3.2. *Let T be an invertible contraction and let $\|TT^*T - T\| < \epsilon < \frac{2}{3\sqrt{3}}$. Then there exists a partial isometry V such that $\|T - V\| < K\epsilon$ for a certain constant $K > 0$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. It is known that U is unitary, since T is invertible. Then

$$\||T|^3 - |T|\| = \|U|T|T^*T - U|T|\| = \|TT^*T - T\| < \epsilon$$

since the operator norm is unitarily invariant in the sense that $\|VXW\| = \|X\|$ for all arbitrary operators X and all unitaries V, W in $\mathbb{B}(\mathcal{H})$. Utilizing Lemma 3.1 for $-|T|$ we get a self-adjoint partial isometry S such that $\||T| - S\| < K\epsilon$ for a certain positive number K. Hence

$$\|T - US\| = \|U|T| - US\| = \||T| - S\| < K\epsilon$$

Since $US(US)^*US = US$, the operator US turns into a partial isometry V. □

If T acts on a finite dimensional Hilbert space $\mathcal{H}$, then the partial isometry U appeared in the polar decomposition of T is a unitary. So the proof of Theorem 3.2 above follows the following fact.

Corollary 3.3. *Let* A *be an* $m \times m$ *contractive matrix such that* $\|AA^*A - A\| < \epsilon < \frac{2}{3\sqrt{3}}$. *Then there exists a partial isometry* V *such that* $\|A - V\| < K\epsilon$ *for a certain constant* $K > 0$.

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