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Integral composition operators between weighted Bergman spaces and weighted Bloch type spaces

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ABSTRACT

We characterize boundedness and compactness of integral composition operators acting between weighted Bergman spaces $A_{\nu,p}$ and weighted Bloch type spaces B_w.

RESUMEN

Caracterizamos la acotación y compacidad de operadores integrales compuestos actuando entre espacios de Bergman con peso $A_{\nu,p}$ y espacios B_w de tipo Bloch con peso.

Keywords and Phrases: Weighted Bergman spaces, integral composition operator, weighted Bloch type spaces

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1 Introduction

Let $H(\mathbb{D})$ denote the set of all analytic functions on the open unit disk $\mathbb{D}$ of the complex plane. A map $g \in H(\mathbb{D})$ induces the *Volterra type* or *Riemann-Stieltjes operator*

$$J_g : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \int_0^z f(\xi)g'(\xi)\ d\xi,\ z \in \mathbb{D}.$$

This operator appears naturally in the study of pointwise multiplication operators since with the *companion integral operator*

$$I_g : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \int_0^z f'(\xi)g(\xi)\ d\xi,\ z \in \mathbb{D},$$

we have that

$$J_g f + I_g f = M_g f - f(0)g(0),$$

where M_g denotes the pointwise multiplication operator given by

$$M_g : H(\mathbb{D}) \to H(\mathbb{D}),\ (M_g f)(z) = g(z)f(z),\ z \in \mathbb{D}.$$

See e.g. [1], [2], [3], [17] or [21].
Moreover, let v and w be strictly positive bounded and continuous functions (*weights*) on $\mathbb{D}$. Then the weighted Bergman space $A_{v,p}$ is defined as follows

$$A_{v,p} = \{f \in H(\mathbb{D});\ \|f\|_{v,p} := \left(\int_{\mathbb{D}} |f(z)|^p v(z)\ dA(z)\right)^{\frac{1}{p}} < \infty\}, 1 \leq p < \infty,$$

where $dA(z)$ is the area measure on $\mathbb{D}$ normalized so that area of $\mathbb{D}$ is 1. Furthermore, we consider the weighted Bloch type spaces B_w of functions $f \in H(\mathbb{D})$ satisfying $\|f\|_{B_w} := \sup_{z\in\mathbb{D}} w(z)|f'(z)| < \infty$. Provided we identify functions that differ by a constant, $\|.\|_{B_w}$ becomes a norm and B_w a Banach space.
Let ϕ be an analytic self-map of $\mathbb{D}$. In [13] Li characterized boundedness and compactness of *Volterra composition operators*

$$(J_{g,\phi} f)(z) = \int_0^z (f \circ \phi)(\xi)(g \circ \phi)'(\xi)\ d\xi,\ z \in \mathbb{D},$$

and the *integral composition operators*

$$(I_{g,\phi} f)(z) = \int_0^z (f \circ \phi)'(\xi)(g \circ \phi)(\xi)\ d\xi,\ z \in \mathbb{D},$$

acting between weighted Bergman spaces and weighted Bloch type spaces, both generated by standard weights. In [19] we generalized his results related to the Volterra composition operators $J_{g,\phi}$

to a more general setting. In this article our aim is to characterize boundedness and compactness of the integral composition operators $I_{g,\phi}$ acting between weighted Bergman spaces and weighted Bloch type spaces generated by a quite general class of weights.

2 The setting

This section is devoted to the description of the setting in which we are interested. Let ν be a holomorphic function on $\mathbb{D}$, non-vanishing, strictly positive on $[0,1[$ and satisfying $\lim_{r\to 1}\nu(r)=0$. Then we define the weight ν as follows

$$\nu(z) := \nu(|z|^2) \text{ for every } z \in \mathbb{D}. \tag{2.1}$$

Next, we give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1-z)^\alpha$, $\alpha > 0$. Then the corresponding weight is the so-called standard weight $\nu(z) = (1-|z|^2)^\alpha$.

(ii) Select $\nu(z) = e^{-\frac{1}{(1-z)^\alpha}}$, $\alpha > 0$. Then we obtain the weight $\nu(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$.

(iii) Choose $\nu(z) = \sin(1-z)$ and the corresponding weight is given by $\nu(z) = \sin(1-|z|^2)$.

(iv) Let $\nu(z) = (1-\log(1-z))^\beta$ for some $\beta < 0$. Then we get $\nu(z) = (1-\log(1-|z|^2))^\beta$.

For a fixed point $a \in \mathbb{D}$ we introduce a function $\nu_a(z) := \nu(\overline{a}z)$ for every $z \in \mathbb{D}$. Since ν is holomorphic on $\mathbb{D}$, so is the function ν_a.
We say that a weight ν is *radial* if $\nu(z) = \nu(|z|)$ for every $z \in \mathbb{D}$. Moreover, radial weights are *typical* if additionally $\lim_{|z|\to 1}\nu(z) = 0$ holds. Thus, we introduced a class of typical weights. In [15] Lusky studied weights satisfying the following condition $(L1)$ which was renamed after the author:

$$(L1) \qquad \inf_{n\in\mathbb{N}} \frac{\nu(1-2^{-n-1})}{\nu(1-2^{-n})} > 0.$$

Among others examples of weights satisfying condition $(L1)$ are the standard weights (see Example (i)) and the logarithmic weights (Example (iv)). Throughout this work condition $(L1)$ will play a great role, and we will need the following condition (A) which is equivalent to $(L1)$:

$$(A) \qquad \text{there are } 0<r<1 \text{ and } 1<C<\infty \text{ with } \frac{\nu(z)}{\nu(p)} \le C \text{ for all } p,z\in\mathbb{D} \text{ with } \rho(p,z)\le r.$$

The equivalence of the conditions $(L1)$ and (A) was shown in [10]. See also [14].

3 Basic facts

We need some geometric data of the open unit disk. Fix $a \in \mathbb{D}$ and consider the authomorphism $\varphi_a(z) := \frac{z-a}{1-\overline{a}z}$, $z \in \mathbb{D}$, which interchanges 0 and a. Moreover, we use the fact that

$$\varphi_a'(z) = \frac{|a|^2 - 1}{(1-\overline{a}z)^2},\ z \in \mathbb{D}.$$

Now, the *pseudohyperbolic metric* is given by

$$\rho(z,a) = |\varphi_a(z)|,\ z, a \in \mathbb{D}.$$

One of the most important properties of the pseudohyperbolic metric is that it is *Möbius invariant*, that is,

$$\rho(\sigma(z), \sigma(a)) = \rho(z,a) \text{ for every automorphism } \sigma \text{ of } \mathbb{D},\ z, a \in \mathbb{D}.$$

The pseudohyperbolic metric is a true metric. In fact, it even satisfies a stronger version of the triangle inequality, more precisely, for every $z, a, b \in \mathbb{D}$ we have that

$$\rho(z,a) \le \frac{\rho(z,b) + \rho(b,a)}{1 + \rho(z,b)\rho(b,a)}.$$

4 Results

Before we are able to treat boundedness and compactness of operators $I_{g,\phi}$ we need a number of auxiliary lemmas. The first lemma is taken from [18].

Lemma 1. *Let* v *be a weight as defined in (2.1) such that* $\sup_{a\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \le C < \infty$. *Then*

$$|f(z)| \le \frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}} \|f\|_{v,p}$$

for all $z \in \mathbb{D}$, $f \in A_{v,p}$.

Calculations show that the examples (i) -(iv) which were listed up above satisfy the assumptions of the previous lemma. The next lemma was shown in [20].

Lemma 2. *Let* v *be a radial weight as defined in (2.1) such that* v *additionally satisfies condition* (L1). *Then for every* $f \in A_{v,p}$ *there is* $C_v > 0$ *such that*

$$|f(z) - f(w)| \le C_v \|f\|_{v,p} \max\left\{ \frac{1}{(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}} v(w)^{\frac{1}{p}}} \right\} \rho(z,w)$$

for every $z, w \in \mathbb{D}$.

Lemma 3. *Let v be a radial weight as defined in (2.1) such that v additionally satisfies condition* (L1) *and* $\sup_{a\in\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))}\leq C<\infty$. *Then*

$$|f'(z)|\leq\frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}(1-|z|^2)^{\frac{2}{p}+1}v(z)^{\frac{1}{p}}}\|f\|_{v,p}$$

for every $z\in\mathbb{D}$ *and every* $f\in A_{v,p}$.

Proof. Lemma 2 yields that for every $f\in A_{v,p}$ and every $h,z\in\mathbb{D}$ with $z+h\in\mathbb{D}$, we have

$$|f(z+h)-f(z)|\leq C_v\|f\|_{v,p}\max\left\{\frac{1}{(1-|z+h|^2)^{\frac{2}{p}}v(z+h)^{\frac{1}{p}}},\frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\right\}\frac{|h|}{|1-\overline{z}(z+h)|}.$$

Hence

$$\left|\frac{f(z+h)-f(z)}{h}\right|\leq C_v\|f\|_{v,p}\max\left\{\frac{1}{(1-|z+h|^2)^{\frac{2}{p}}v(z+h)^{\frac{1}{p}}},\frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\right\}\frac{1}{|1-\overline{z}(z+h)|}$$

and finally

$$\begin{aligned}|f'(z)| &= \left|\lim_{h\to0}\frac{f(z+h)-f(z)}{h}\right|\\ &\leq \lim_{h\to0}C_v\|f\|_{v,p}\max\left\{\frac{1}{(1-|z+h|^2)^{\frac{2}{p}}v(z+h)^{\frac{1}{p}}},\frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\right\}\frac{1}{|1-\overline{z}(z+h)|}\\ &= C_v\|f\|_{v,p}\frac{1}{(1-|z|^2)^{\frac{2}{p}+1}v(z)^{\frac{1}{p}}}\end{aligned}$$

for every $z\in\mathbb{D}$, as desired. □

Lemma 4. *Let v be a radial weight as in Lemma 3. Then there exist $0<r<1$ and a constant $M>0$ such that for $f\in A_{v,p}$*

$$|f'(z)-f'(w)|\leq\frac{4MC^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}}\frac{\|f\|_{v,p}}{r(1-|z|^2)^{\frac{2}{p}+1}v(z)^{\frac{1}{p}}}\rho(z,w)$$

for every $z,w\in\mathbb{D}$ with $\rho(z,w)\leq\frac{r}{2}$.

Proof. By hypotesis, v has condition (L1), and, moreover, we know that (L1) is equivalent to condition (A). Since the weight $u(z)=1-|z|^2$ also satisfies condition (L1), we can find $0<r<1$ and constants $M_1<\infty$ and $M_2<\infty$ such that

$$\frac{v(z)}{v(w)}\leq M_1 \text{ and } \frac{1-|z|^2}{1-|w|^2}\leq M_2 \text{ for every } z,w\in\mathbb{D} \text{ with } \rho(z,w)\leq r.$$

Let $w\in\mathbb{D}$ be fixed. Since

$$\varphi_w(\varphi_w(z))=z \text{ and } \varphi_w(0)=w,$$

we get that

$$|f'(z)-f'(w)| = |f'(\varphi_w(\varphi_w(z)) - f'(\varphi_w(\varphi_w(w))|.$$

For $|z| = \rho(\varphi_w(z), w) \le r$ we obtain by using Lemma 3

$$\begin{aligned}|f'(\varphi_w(z))| &\le \frac{C^{\frac{1}{p}}\|f\|_{v,p}}{v(0)^{\frac{1}{p}}(1-|\varphi_w(z)|^2)^{\frac{2}{p}+1}v(\varphi_w(z))^{\frac{1}{p}}}\\ &= \frac{C^{\frac{1}{p}}\|f\|_{v,p}}{v(0)^{\frac{1}{p}}(1-|w|^2)^{\frac{2}{p}+1}v(w)^{\frac{1}{p}}}\frac{(1-|w|^2)^{\frac{2}{p}+1}v(w)^{\frac{1}{p}}}{(1-|\varphi_w(z)|^2)^{\frac{2}{p}+1}v(\varphi_w(z))^{\frac{1}{p}}}\\ &\le \frac{C^{\frac{1}{p}}M_1^{\frac{1}{p}}M_2^{\frac{2}{p}+1}}{v(0)^{\frac{1}{p}}}\frac{\|f\|_{v,p}}{(1-|w|^2)^{\frac{2}{p}+1}v(w)^{\frac{1}{p}}}.\end{aligned}$$

Let us now consider $g_w(z) := f'(\varphi_w(z))$ for every $z \in \mathbb{D}$. Thus, for $\rho(z,w) = |\varphi_w(z)| \le \frac{r}{2}$ we can find $\Theta \in \mathbb{D}$ with $|\Theta| \le |\varphi_w(z)| \le \frac{r}{2}$ such that

$$\begin{aligned}|f'(z)-f'(w)| &= |g_w(\varphi_w(z)) - g_w(0)|\\ &\le |\varphi_w(z)|\left|\int_0^1 \left[\frac{\partial}{\partial t}g_w\right](t\varphi_w(z))\,dt\right|\\ &\le |\varphi_w(z)|\left|\frac{\partial}{\partial z}g_w(\Theta)\right|\\ &= |\varphi_w(z)|\frac{1}{2\pi}\left|\int_{|\xi|=r}\frac{g_w(\xi)}{(\xi-\Theta)^2}\,d\Theta\right|\end{aligned}$$

Finally,

$$\begin{aligned}|f'(z)-f'(w)| \le \frac{C^{\frac{1}{p}}M_1^{\frac{1}{p}}M_2^{\frac{2}{p}+1}}{v(0)^{\frac{1}{p}}}&\frac{|\varphi_w(z)|r\|f\|_{v,p}}{(r-|\varphi_w(z)|)^2(1-|w|^2)^{\frac{2}{p}+1}v(w)^{\frac{1}{p}}}\\ &\le \frac{4C^{\frac{1}{p}}M_1^{\frac{1}{p}}M_2^{\frac{2}{p}+1}}{v(0)^{\frac{1}{p}}}\frac{\rho(z,w)\|f\|_{v,p}}{r(1-|w|^2)^{\frac{2}{p}+1}v(w)^{\frac{1}{p}}}.\end{aligned}$$

We select $M := M_1^{\frac{1}{p}}M_2^{\frac{2}{p}+1}$ and obtain the claim. □

Lemma 5. *Let v be a weight as in Lemma 3. Then, there is $C_v > 0$ such that for every $f \in A_{v,p}$*

$$|f'(z)-f'(w)| \le C_v\|f\|_{v,p}\max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}+1}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}+1}v(w)^{\frac{1}{p}}}\right\}\rho(z,w)$$

for every $z, w \in \mathbb{D}$.

Proof. By Lemma 4 we can find $0 < s < 1$ and a constant $M < \infty$ such that

$$|f'(z)-f'(w)| \le \frac{4MC^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}}\frac{\|f\|_{v,p}}{s(1-|z|^2)^{\frac{2}{p}+1}v(z)^{\frac{1}{p}}}\rho(z,w)$$

for every $z, w \in \mathbb{D}$ with $\rho(z,w) \le \frac{s}{2}$. Next, if $\rho(z,w) > \frac{s}{2}$, then

$$\begin{aligned}|f'(z) - f'(w)| &\le 2\frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}}\|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}+1}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}+1}v(w)^{\frac{1}{p}}}\right\} \\ &\le \frac{4}{s}\frac{C^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}}\|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}+1}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}+1}v(w)^{\frac{1}{p}}}\right\}\rho(z,w).\end{aligned}$$

Hence, with $C_v := \max\left\{\frac{4MC^{\frac{1}{p}}}{v(0)^{\frac{1}{p}}s}, \frac{4C^{\frac{1}{p}}}{sv(0)^{\frac{1}{p}}}\right\}$, we conclude

$$|f'(z) - f'(w)| \le C_v \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}+1}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}+1}v(w)^{\frac{1}{p}}}\right\}\rho(z,w)$$

for every $z, w \in \mathbb{D}$ and the claim follows. □

Inductively, we can show the following lemmas:

Lemma 6. *Let v be a weight as in Lemma 3. Then there is $C_v > 0$ such that for every $f \in A_{v,p}$*

$$|f^{(n)}(z)| \le \frac{C_v}{(1-|z|^2)^{\frac{2}{p}+n}v(z)^{\frac{1}{p}}}\|f\|_{v,p}$$

for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}_0$.

Lemma 7. *Let v be a weight as in Lemma 3. Then there exists $C_v > 0$ such that for every $f \in A_{v,p}$*

$$|f^{(n)}(z) - f^{(n)}(w)| \le C_v\|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}+n}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}+n}v(w)^{\frac{1}{p}}}\right\}\rho(z,w)$$

for every $z, w \in \mathbb{D}$ and every $n \in \mathbb{N}_0$.

Now, we turn our attention to the operators $I_{g,\phi}$ and start with characterizing when they are bounded.

Theorem 8. *Let w be a weight and v be a weight as in Lemma 3 with $M := \sup_{a\in\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{v(z)}{|v(\bar{a}z)|} < \infty$. If*

$$\sup_{z\in\mathbb{D}}\frac{w(z)|\phi'(z)g(\phi(z))|}{(1-|\phi(z)|^2)^{\frac{2}{p}+1}v(\phi(z))^{\frac{1}{p}}} < \infty, \tag{4.1}$$

then the operator $I_{g,\phi} : A_{v,p} \to B_w$ is bounded. If we assume additionally that

$$\sup_{z\in\mathbb{D}}\frac{|v'(|\phi(z)|^2)|w(z)|\phi'(z)g(\phi(z))|}{v(\phi(z))^{\frac{1}{p}+1}(1-|\phi(z)|^2)^{\frac{2}{p}}} < \infty, \tag{4.2}$$

then the converse is also true.

Proof. We start with assuming that the operator $I_{g,\phi}$ is bounded and that the condition (4.2) is satisfied. Fix a point $a \in \mathbb{D}$ and set

$$f_a(z) := \frac{\varphi_a'(z)^{\frac{2}{p}}}{v(\overline{a}z)^{\frac{1}{p}}} \text{ for every } z \in \mathbb{D}.$$

Then

$$\|f\|_{v,p}^p = \int_{\mathbb{D}} \frac{|\varphi_a'(z)|^2}{|v(\overline{a}z)|} v(z)\, dA(z) \le \sup_{z\in\mathbb{D}} \frac{v(z)}{|v(\overline{a}z)|} \int_{\mathbb{D}} |\varphi_a'(z)|^2\, dA(z) \le \sup_{z\in\mathbb{D}} \frac{v(z)}{|v(\overline{a}z)|} \le M,$$

and the constant M is independent of the choice of the point a. For the derivative we have

$$f_a'(z) = \frac{2}{p} \frac{\varphi_a'(z)^{\frac{2}{p}-1}\varphi_a''(z)}{v(\overline{a}z)^{\frac{1}{p}}} - \frac{1}{p} \frac{\overline{a}v'(\overline{a}z)\varphi_a'(z)^{\frac{2}{p}}}{v(\overline{a}z)^{\frac{1}{p}+1}}$$

for every $z \in \mathbb{D}$. Hence we can find a constant $C^* > 0$ such that

$$\begin{aligned}
\left| \frac{w(a)|\phi'(a)||g(\phi(a))|}{(1-|\phi(a)|^2)^{\frac{2}{p}+1}v(\phi(a))^{\frac{1}{p}}} - \frac{|v'(|\phi(a)|^2)|w(a)|\phi'(a)g(\phi(a))|}{v(\phi(a))^{\frac{1}{p}+1}(1-|\phi(a)|^2)^{\frac{2}{p}}} \right| &\le \left| f_{\phi(a)}'(\phi(a))|w(a)|g(\phi(a))||\phi'(a)| \right| \\
&\le |(I_{g,\phi} f_{\phi(a)})'(a)|w(a) \\
&\le C^* \|J_{g,\phi}\| \|f_{\phi(a)}\|_{v,p}.
\end{aligned}$$

Finally, since (4.2) is fulfilled and the operator $I_{g,\phi}$ is bounded, the claim follows. Conversely, an application of Lemma 3 yields for $f \in A_{v,p}$

$$\begin{aligned}
\sup_{z\in\mathbb{D}} |(I_{g,\phi} f)'(z)|w(z) &= \sup_{z\in\mathbb{D}} |f'(\phi(z))||g(\phi(z))||\phi'(z)|w(z) \\
&\le \sup_{z\in\mathbb{D}} \frac{C^{\frac{1}{p}} \|f\|_{v,p} w(z)|g(\phi(z))||\phi'(z)|}{v(0)^{\frac{1}{p}}(1-\phi(z)|^2)^{\frac{2}{p}+1}v(\phi(z))^{\frac{1}{p}}}.
\end{aligned}$$

Hence the claim follows. □

Next, we study, when such operators are compact. To do this we need a lemma which can easily be derived from [9] Proposition 3.11.

Lemma 9. *Let v and w be weights. Then the operator $I_{g,\phi} : A_{v,p} \to B_w$ is compact if and only if it is bounded and for every bounded sequence $(f_n)_n$ in $A_{v,p}$ which converges to zero uniformly on the compact subsets of $\mathbb{D}$, $I_{g,\phi} f_n$ tends to zero in B_w if $n \to \infty$.*

Theorem 10. *Let w be a weight and v be a weight as in Theorem 8. Moreover, we assume that $I_{g,\phi} : A_{v,p} \to B_w$ is bounded. If*

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)g(\phi(z))|}{(1-|\phi(z)|^2)^{\frac{2}{p}+1}v(\phi(z))^{\frac{1}{p}}} = 0, \tag{4.3}$$

then the operator $I_{g,\phi} : A_{\nu,p} \to B_w$ *is compact. If we assume additionally*

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{|\nu'(|\phi(z)|^2)|w(z)|\phi'(z)g(\phi(z))|}{\nu(\phi(z))^{\frac{1}{p}+1}(1-|\phi(z)|^2)^{\frac{2}{p}}} = 0, \tag{4.4}$$

then the converse is also true.

Proof. Assume that the operator $I_{g,\phi} : A_{\nu,p} \to B_w$ is compact and that (4.4) is satisfied. To show (4.3) let $(z_n)_n$ be a sequence with $|\phi(z_n)| \to 1$ and put

$$f_k(z) := \frac{\varphi'_{\phi(z_k)}(z)^{\frac{2}{p}}}{\nu(\overline{\phi(z_k)}z)^{\frac{1}{p}}} \text{ for every } z \in \mathbb{D} \text{ and every } k \in \mathbb{N}.$$

Analogously to the proof of Theorem 8 we can show that $(f_n)_n$ is a bounded sequence which tends to zero uniformly on the compact subsets of $\mathbb{D}$. Since $I_{g,\phi}$ is compact, by Lemma 9

$$\|I_{g,\phi} f_n\|_{B_w} \to 0 \text{ if } n \to \infty.$$

Thus,

$$\|I_{g,\phi} f_n\|_{B_w} \geq \left| \frac{w(z_n)|\phi'(z_n)||g(\phi(z_n)|}{(1-|\phi(z_n)|^2)^{\frac{2}{p}+1}\nu(\phi(z_n))^{\frac{1}{p}}} - \frac{|\nu'(|\phi(z_n)|^2)|w(z_n)|\phi'(z_n)g(\phi(z_n))|}{\nu(\phi(z_n))^{\frac{1}{p}+1}(1-|\phi(z_n)|^2)^{\frac{2}{p}}} \right|,$$

and, since (4.4) holds, condition (4.3) follows.
Conversely, suppose that (4.3) is satisfied. Let $(f_n)_n$ be a bounded sequence in $A_{\nu,p}$ such that $\|f_n\|_{\nu,p} \leq M_1 < \infty$ for every $n \in \mathbb{N}$ and such that $(f_n)_n$ converges uniformly to zero on the compact subsets of $\mathbb{D}$ if $n \to \infty$. For a fixed $\varepsilon > 0$ we can find $0 < r_0 < 1$ such that if $|\phi(z)| > r_0$, then

$$\frac{w(z)|g(\phi(z))||\phi'(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}+1}\nu(\phi(z))^{\frac{1}{p}}} < \frac{\varepsilon\nu(0)^{\frac{1}{p}}}{2C^{\frac{1}{p}}M_1}.$$

Moreover, we can find $M_2 > 0$ such that

$$\sup_{|\phi(z)|\leq r_0} w(z)|g(\phi(z))||\phi'(z)| \leq M_2.$$

There is $n_0 \in \mathbb{N}$ such that

$$\sup_{|\phi(z)|\leq r_0} |f'_n(\phi(z))| \leq \frac{\varepsilon}{2M_2} \text{ for every } n \geq n_0.$$

We obtain applying Lemma 3

$$\begin{aligned}
\sup_{z\in\mathbb{D}} |(I_{g,\phi} f_n)'(z)| w(z) &= \sup_{z\in\mathbb{D}} w(z)|f_n'(\phi(z))||g(\phi(z))||\phi'(z)| \\
&\leq \sup_{|\phi(z)|\leq r_0} w(z)|f_n'(\phi(z))||g(\phi(z))||\phi'(z)| \\
&+ \sup_{|\phi(z)|> r_0} w(z)|f_n'(\phi(z))||g(\phi(z))||\phi'(z)| \\
&\leq \sup_{|\phi(z)|\leq r_0} |f_n'(\phi(z))| \sup_{|\phi(z)|\leq r_0} w(z)|g(\phi(z))||\phi'(z)| \\
&+ \sup_{|\phi(z)|> r_0} \frac{C^{\frac{1}{p}} \|f_n\|_{v,p} w(z)|g(\phi(z))||\phi'(z)|}{v(0)^{\frac{1}{p}} (1-|\phi(z)|^2)^{\frac{2}{p}+1} v(\phi(z))^{\frac{1}{p}}} \\
&\leq \varepsilon,
\end{aligned}$$

and the claim follows. □

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