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Existence and stability of almost periodic solutions to impulsive stochastic differential equations

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ABSTRACT

This paper introduces the concept of square-mean piecewise almost periodic for impulsive stochastic processes. The existence of square-mean piecewise almost periodic solutions for linear and nonlinear impulsive stochastic differential equations is established by using the theory of the semigroups of the operators and Schauder fixed point theorem. The stability of the square-mean piecewise almost periodic solutions for nonlinear impulsive stochastic differential equations is investigated.

RESUMEN

Este artículo introduce el concepto de periodicidad cuadrática media por tramos casi periódica para procesos estocásticos impulsivos. La existencia de soluciones de media cuadrática casi periódicas para ecuaciones diferenciales estocásticas impulsivas lineales y no lineales se establece usando la teoría de semigrupos de los operadores y el teorema de punto fijo de Schauder. Se estudia la estabilidad de las soluciones de media cuadrática por tramos casi periódica para ecuaciones diferenciales estocásticas impulsivas no lineales.

Keywords and Phrases: Square-mean piecewise almost periodic; impulsive stochastic differential equation; the semigroups of the operators; Schauder fixed point theorem; stability

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1 Introduction

In recent years, stochastic differential systems have been extensively studied since stochastic modeling plays an important role in physics, engineering, finance, social science and so on. Qualitative properties such as existence, uniqueness and stability for stochastic differential systems have attracted more and more researchers' attention. The existence of periodic, almost periodic(automorphic), asymptotically almost periodic, pseudo almost periodic(automorphic) solutions for stochastic differential equations was obtained. We refer the reader to [14, 6, 7, 17, 16, 10, 8, 1, 11] and references therein.

On the other hand, impulsive phenomenon arises from many different real processes and phenomena which appeared in physics, chemical technology, population dynamics, biotechnology, medicine and economics. There has been a significant development in the theory of impulsive differential equations. For example, the existence of almost periodic (mild) solutions of abstract impulsive differential equations have been considered in [23, 24, 25, 4, 18, 19].

In [26], the authors combined the two directions and derived firstly some sufficient conditions for the existence and uniqueness of almost periodic solutions for a class of impulsive stochastic differential equations with delay. However, these above results quoted concern the case where the activation functions satisfy Lipschitz conditions. There are few authors have considered the problem of almost periodic solutions of impulsive stochastic differential equations without Lipschitz activation functions. On the basis of this, this article is devoted to the discussion of this problem.

Moreover, the stability analysis on impulsive stochastic differential equations has been an important research topic (see [20, 22, 27]). While, because the mild solutions don't have stochastic differentials, Ito's formula fails to deal with the stability of mild solution to stochastic differential equations (see [20, 9, 15]). In [9], the authors gave some properties of the stochastic convolution which ensure the exponential stability of mild solutions.

Motivated by the above discussion, we investigate the existence and stability of almost periodic solutions for impulsive stochastic differential equations. The paper is organized as follows, in Section 2 we recall some definitions, the related notations and some useful lemmas. In Sections 3 and 4, we present some criteria ensuring the existence of almost periodic solutions to some linear and nonlinear impulsive stochastic differential equations, respectively. In Section 5, we discuss the stability of almost periodic solutions to some impulsive stochastic differential equations.

2 Preliminaries

Throughout this paper, R denotes the set of real numbers, R^+ denotes the set of nonnegative real numbers, Z denotes the set of integers, Z^+ denotes the set of nonnegative integers. $(\mathsf{H}, \|\cdot\|)$ is assumed to be a real and separable Hilbert space. Let $(\Omega, \mathsf{F}, \mathsf{P})$ be a complete probability space and $\mathsf{L}^2(\mathsf{P}, \mathsf{H})$ be a space of the H-valued random variables x such that $\mathsf{E}\|x\|^2 = \int_\Omega \|x\|^2 d\mathsf{P} < \infty$.

$L^2(P, H)$ is a Hilbert space equipped with the norm $\|x\|_2 = (\int_\Omega \|x\|^2 dP)^{1/2}$.

Definition 2.1. *A stochastic process* $x : R^+ \to L^2(P, H)$ *is said to be stochastically bounded if there exists* $M > 0$ *such that* $E\|x(t)\|^2 \le M$ *for all* $t \in R^+$.

Definition 2.2. *A stochastic process* $x : R^+ \to L^2(P, H)$ *is said to be stochastically continuous in* $s \in R^+$, *if* $\lim_{t\to s} E\|x(t) - x(s)\|^2 = 0$.

Let $\mathbb{T}$ be the set consisting of all real sequences $\{t_i\}_{i\in Z^+}$ such that $\gamma = \inf_{i\in Z^+}(t_{i+1} - t_i) > 0$, $t_0 = 0$ and $\lim_{i\to\infty} t_i = \infty$. $x(t_i^+)$ and $x(t_i^-)$ represent the right and left limits of $x(t)$ at $t_i, i \in Z^+$, respectively. For $\{t_i\}_{i\in Z^+} \in \mathbb{T}$, let $PC(R^+, L^2(P, H))$ be the space consisting of all stochastically bounded functions $\phi : R^+ \to L^2(P, H)$ such that $\phi(\cdot)$ is stochastically continuous at t for any $t \notin \{t_i\}_{i\in Z^+}$ and $\phi(t_i) = \phi(t_i^-)$ for all $i \in Z^+$; let $PC(R^+ \times L^2(P, H), L^2(P, H))$ be the space formed by all stochastic processes $\phi : R^+ \times L^2(P, H) \to L^2(P, H)$ such that for any $x \in L^2(P, H)$, $\phi(\cdot, x)$ is stochastically continuous at t for any $t \notin \{t_i\}_{i\in Z^+}$ and $\phi(t_i, x) = \phi(t_i^-, x)$ for all $i \in Z^+$ and for any $t \in R^+$, $\phi(t, \cdot)$ is stochastically continuous at $x \in L^2(P, H)$.

Definition 2.3. *For* $\{t_i\}_{i\in Z^+} \in \mathbb{T}$, *the function* $\phi \in PC(R^+, L^2(P, H))$ *is said to be square-mean piecewise almost periodic if the following conditions are fulfilled:*

(1) $\{t_i^j = t_{i+j} - t_i\}$, $j \in Z^+$, *is equipotentially almost periodic, that is, for any* $\epsilon > 0$, *there exists a relatively dense set* Q_ϵ *of* R *such that for each* $\tau \in Q_\epsilon$ *there is an integer* $q \in Z$ *such that* $|t_{i+q} - t_i - \tau| < \epsilon$ *for all* $i \in Z^+$.

(2) For any $\epsilon > 0$, *there exists a positive number* $\delta = \delta(\epsilon)$ *such that if the points* t' *and* t'' *belong to a same interval of continuity of* ϕ *and* $|t' - t''| < \delta$, *then* $E\|\phi(t') - \phi(t'')\|^2 < \epsilon$.

(3) For every $\epsilon > 0$, *there exists a relatively dense set* $\Omega(\epsilon)$ *in* R *such that if* $\tau \in \Omega(\epsilon)$, *then*

$$E\|\phi(t + \tau) - \phi(t)\|^2 < \epsilon$$

for all $t \in R^+$ *satisfying the condition* $|t - t_i| > \epsilon$, $i \in Z^+$. *The number* τ *is called* ϵ*-translation number of* ϕ.

We denote by $AP_T(R^+, L^2(P, H))$ *the collection of all the square-mean piecewise almost periodic processes, it thus is a Banach space with the norm* $\|x\|_\infty = \sup_{t\in R^+} \|x(t)\|_2 = \sup_{t\in R^+}(E\|x(t)\|^2)^{\frac{1}{2}}$ *for* $x \in AP_T(R^+, L^2(P, H))$.

Lemma 2.4. *Let* $f \in AP_T(R^+, L^2(P, H))$, *then,* $R(f)$, *the range of* f *is a relatively compact set of* $L^2(P, H)$.

Refer to [18] for the detailed proof of Lemma 2.4.

Definition 2.5. *For* $\{t_i\}_{i\in Z^+} \in \mathbb{T}$, *the function* $f(t, x) \in PC(R^+ \times L^2(P, H), L^2(P, H))$ *is said to be square-mean piecewise almost periodic in* $t \in R^+$ *and uniform on compact subset of* $L^2(P, H)$ *if for every* $\epsilon > 0$ *and every compact subset* $K \subseteq L^2(P, H)$, *there exists a relatively dense subset* Ω *of* R *such that*

$$E\|f(t + \tau, x) - f(t, x)\|^2 < \epsilon,$$

for all $x \in K, \tau \in \Omega, t \in R^+$ *satisfying* $|t - t_i| > \epsilon$. *The collection of all such processes is denoted by* $AP_T(R^+ \times L^2(P, H), L^2(P, H))$.

Lemma 2.6. *Suppose that* $f(t, x) \in AP_T(R^+ \times L^2(P, H), L^2(P, H))$ *and* $f(t, \cdot)$ *is uniformly continuous on each compact subset* $K \subseteq L^2(P, H)$ *uniformly for* $t \in R$. *That is, for all* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $x, y \in K$ *and* $E\|x - y\|^2 < \delta$ *implies that* $E\|f(t, x) - f(t, y)\|^2 < \epsilon$ *for all* $t \in R^+$. *Then* $f(\cdot, x(\cdot)) \in AP_T(R^+, L^2(P, H))$ *for any* $x \in AP_T(R^+, L^2(P, H))$.

Proof. Since $x \in AP_T(R^+, L^2(P, H))$, by Lemma 2.4, $R(x)$ is a relatively compact subset of $L^2(P, H)$. Because $f(t, \cdot)$ is uniformly continuous on each compact subset $K \subseteq L^2(P, H)$ uniformly for $t \in R$. Then for any $\epsilon > 0$, there exists number $\delta : 0 < \delta \leq \frac{\epsilon}{4}$, such that

$$E\|f(t, x_1) - f(t, x_2)\|^2 < \frac{\epsilon}{4}, \tag{1}$$

where $x_1, x_2 \in R(x)$ and $E\|x_1 - x_2\|^2 < \delta,\ t \in R$. By square-mean piecewise almost periodic of f and x, there exists a relatively set Ω of R such that the following conditions hold:

$$E\|f(t + \tau, x_0) - f(t, x_0)\|^2 < \frac{\epsilon}{4}, \tag{2}$$

$$E\|x(t + \tau) - x(t)\|^2 < \frac{\epsilon}{4}, \tag{3}$$

for every $x_0 \in R(x)$ and $t \in R^+,\ |t - t_i| > \epsilon,\ i \in Z^+,\ \tau \in \Omega$. Note that $(a + b)^2 \leq 2(a^2 + b^2)$ and

$$\begin{aligned}&E\|f(t + \tau, x(t + \tau)) - f(t, x(t))\|^2\\ \leq& 2E\|f(t + \tau, x(t + \tau)) - f(t + \tau, x(t))\|^2 + 2E\|f(t + \tau, x(t)) - f(t, x(t))\|^2.\end{aligned}$$

Combing (1), (2) and (3), it follows that

$$E\|f(t + \tau, x(t + \tau)) - f(t, x(t))\|^2 < \epsilon,\ t \in R^+,\ |t - t_i| > \epsilon,\ i \in Z^+,\ \tau \in \Omega.$$

The proof is complete. □

We obtain the following corollary as an immediate consequence of Lemma 2.6.

Corollary 2.7. *Let* $f(t, x) \in AP_T(R^+ \times L^2(P, H), L^2(P, H))$ *and* f *is Lipschitz, i.e., there is a number* $L > 0$ *such that*

$$E\|f(t, x) - f(t, y)\|^2 \leq LE\|x - y\|^2,$$

for all $t \in R^+$ *and* $x, y \in L^2(P, H)$, *if for any* $x \in AP_T(R^+, L^2(P, H))$, *then* $f(\cdot, x(\cdot)) \in AP_T(R^+, L^2(P, H))$.

Definition 2.8. *A sequence* $x : Z^+ \to L^2(P, H)$ *is called a square-mean almost periodic sequence if the* ϵ*-translation set of* x

$$\mathfrak{T}(x; \epsilon) = \{\tau \in Z : E\|x(n + \tau) - x(t)\|^2 < \epsilon,\ \textit{for all}\ n \in Z^+\}$$

is a relatively dense set in Z *for all* $\epsilon > 0$.

The collection of all square-mean almost periodic sequences $x : Z^+ \to L^2(P, H)$ *will be denoted by* $AP_T(Z^+, L^2(P, H))$.

Remark 2.9. *If* $x(n) \in AP_T(Z^+, L^2(P,H))$*, then* $\{x(n) : n \in Z^+\}$ *is stochastically bounded, that is,* $\sup_{n\in Z^+} E\|x(n)\|^2 < \infty$.

In order to obtain our main results, we introduce the following lemmas.

Let $h : R^+ \to R$ be a continuous function such that $h(t) \geq 1$ for all $t \in R^+$ and $h(t) \to \infty$ as $t \to \infty$. We consider the space

$$(PC)_h^0(R^+, L^2(P,H)) = \left\{ u \in PC(R^+, L^2(P,H)) : \lim_{t\to\infty} \frac{E\|u(t)\|^2}{h(t)} = 0 \right\}.$$

Endowed with the norm $\|u\|_h = \sup_{t\in R^+} \frac{E\|u(t)\|^2}{h(t)}$, it is a Banach space.

Lemma 2.10. *A set* $B \subseteq (PC)_h^0(R^+, L^2(P,H))$ *is a relatively compact set if and only if*

(1) $\lim_{t\to\infty} \frac{E\|x(t)\|^2}{h(t)} = 0$ *uniformly for* $x \in B$.

(2) $B(t) = \{x(t) : x \in B\}$ *is relatively compact in* $L^2(P,H)$ *for every* $t \in R^+$.

(3) The set B *is equicontinuous on each interval* $(t_i, t_{i+1})(i \in Z^+)$.

Lemma 2.11. *Assume that* $f \in AP_T(R^+, L^2(P,H))$*, the sequence* $\{x_i : i \in Z^+\}$ *is almost periodic in* $L^2(P,H)$ *and* $\{t_i^j\}$*,* $j \in Z^+$*, is equipotentially almost periodic. Then for each* $\epsilon > 0$ *there are relatively dense sets* Ω_{ϵ,f,x_i} *of* R *and* Q_{ϵ,f,x_i} *of* Z *such that the following conditions hold:*

(i) $E\|f(t+\tau) - f(t)\|^2 < \epsilon$ *for all* $t \in R^+$*,* $|t - t_i| > \epsilon$*,* $\tau \in \Omega_{\epsilon,f,x_i}$ *and* $i \in Z^+$.

(ii) $E\|x_{i+q} - x_i\|^2 < \epsilon$ *for all* $q \in Q_{\epsilon,f,x_i}$ *and* $i \in Z^+$.

(iii) For every $\tau \in \Omega_{\epsilon,f,x_i}$*, there exists at least one number* $q \in Q_{\epsilon,f,x_i}$ *such that*

$$|t_i^q - \tau| < \epsilon, \ i \in Z^+.$$

Lemma 2.10 and Lemma 2.11 are stochastic generalized versions of Lemma 4.1 in [12] and Lemma 35 in [23], respectively, and one may refer to [23, 18, 19, 26, 2, 13, 12] for more details. Here we omit the proofs.

Lemma 2.12. ([9]) *For any* $r \geq 1$ *and for arbitrary* $L^2(P,H)$*-valued process* $\phi(\cdot)$ *such that*

$$\sup_{s\in[0,t]} E\left\| \int_0^s \phi(u)dw(u) \right\|^{2r} \leq C_r \left(\int_0^t (E\|\phi(s)\|^{2r})^{\frac{1}{r}} ds \right)^r, \ t \geq 0,$$

where $C_r = (r(2r-1))^r$.

3 Almost periodic solutions for linear impulsive stochastic differential equations

To begin, consider the following linear impulsive stochastic differential equation:

$$\begin{cases} dx(t) & = [Ax(t) + f(t)]dt + g(t)dw(t), \ t \geq 0, t \neq t_i, i \in Z^+, \\ \triangle x(t_i) & = x(t_i^+) - x(t_i^-) = \beta_i, i \in Z^+, \end{cases} \tag{4}$$

where A is an infinitesimal generator which generates a C_0-semigroup $\{T(t) : t \geq 0\}$ such that for all $t \geq 0$, $\|T(t)\| \leq Me^{-\delta t}$ with $M, \delta > 0$ and $\{T(t) : t > 0\}$ is compact. Furthermore, $f, g : R \to L^2(P, H)$ are two stochastic processes, β_i is a square-mean almost periodic sequence and $w(t)$ is a two-sided standard one-dimensional Brownian motion, which is defined on the filtered probability space (Ω, F, P, F_σ) with $F_t = \sigma\{w(u) - w(v) : u, v \leq t\}$.

Definition 3.1. *An F_t-progressive process $x(t)$ is called a mild solution of system (4) if it satisfies the following stochastic integral equation*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)g(s)dw(s) + \sum_{0<t_i<t} T(t-t_i)\beta_i.$$

for all $t \geq 0$.

Theorem 3.2. *Assume $f, g \in AP_T(R^+, L^2(P, H))$, $\{\beta_i, i \in Z^+\}$ is a square-mean almost periodic sequence, then system (4) has a square-mean piecewise almost periodic mild solution.*

Proof. From semigroup theory, we know

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)g(s)dw(s), t \geq 0,$$

is a mild solution to

$$dx(t) = [Ax(t) + f(t)]dt + g(t)dw(t), t \geq 0.$$

So for system (4), if $t \in [t_0, t_1)$,

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)g(s)dw(s),$$

which implies

$$x(t_1^-) = T(t_1)x_0 + \int_0^{t_1} T(t_1-s)f(s)ds + \int_0^{t_1} T(t_1-s)g(s)dw(s),$$

by using $x(t_1^+) = x(t_1^-) + \beta_1$, for $t \in (t_1, t_2)$, we get

$$\begin{aligned} x(t) =& T(t-t_1)x(t_1^+) + \int_{t_1}^t T(t-s)f(s)ds + \int_{t_1}^t T(t-s)g(s)dw(s) \\ =& T(t-t_1)[x(t_1^-) + \beta_1] + \int_{t_1}^t T(t-s)f(s)ds + \int_{t_1}^t T(t-s)g(s)dw(s) \\ =& T(t-t_1)[T(t_1)x_0 + \int_0^{t_1} T(t_1-s)f(s)ds + \int_0^{t_1} T(t_1-s)g(s)dw(s) + \beta_1] \\ &+ \int_{t_1}^t T(t-s)f(s)ds + \int_{t_1}^t T(t-s)g(s)dw(s) \\ =& T(t)x_0 + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)g(s)dw(s) + T(t-t_1)\beta_1, \end{aligned}$$

reiterating this procedure, we can prove that

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)g(s)dw(s) + \sum_{0<t_i<t} T(t-t_i)\beta_i, \tag{5}$$

and by Definition 3.1, (5) is a mild solution of system (4), to finish the proof, we need to prove the above process (5) is a square-mean piecewise almost periodic process.

Since $f, g \in AP_T(R^+, L^2(P,H))$, from Lemma 31 in [23], for the two almost periodic functions f, g, there exists a relatively dense set of their common ϵ-translation number. Moreover, $\{\beta_i, i \in Z^+\}$ is a square-mean almost periodic sequence, then by Lemma 2.11, for each $\epsilon > 0$, there exist relatively dense sets $\Omega_{\epsilon,f,g,x_i}$ of R and Q_{ϵ,f,g,x_i} of Z such that the following relations hold:

(1) $E\|f(t+\tau) - f(t)\|^2 < \epsilon, t \in R^+, |t-t_i| > \epsilon, i \in Z^+, \tau \in \Omega_{\epsilon,f,g,x_i}$.

(2) $E\|g(t+\tau) - g(t)\|^2 < \epsilon, t \in R^+, |t-t_i| > \epsilon, i \in Z^+, \tau \in \Omega_{\epsilon,f,g,x_i}$.

(3) $E\|x_{i+q} - x_i\|^2 < \epsilon, i \in Z^+, q \in Q_{\epsilon,f,g,x_i}$.

(4) For each $\tau \in \Omega_{\epsilon,f,g,x_i}$, $\exists\ q \in Q_{\epsilon,f,g,x_i}$, s.t. $|t_{i+q} - t_i - \tau| < \epsilon,\ i \in Z^+$.

We write $x(t)$ of (5) as

$$x(t) = T(t)x_0 + x_1(t) + x_2(t) + x_3(t)$$

where

$$x_1(t) = \int_0^t T(t-s)f(s)ds,\ x_2(t) = \int_0^t T(t-s)g(s)dw(s),\ x_3(t) = \sum_{0<t_i<t} T(t-t_i)\beta_i.$$

(i) $x_1 \in AP_T(R^+, L^2(P,H))$. By (1), for $\tau \in \Omega_{\epsilon,f,g,x_i}, t \in R^+, |t-t_i| > \epsilon, i \in Z^+$, one obtains

$$\begin{aligned} E\|x_1(t+\tau) - x_1(t)\|^2 =& E\left\|\int_0^t T(t-s)[f(s+\tau) - f(s)]ds\right\|^2 \\ \le& E\left[\int_0^t Me^{-\delta(t-s)}\|f(s+\tau) - f(s)\|ds\right]^2 \\ \le& E\left[\int_0^t M^2e^{-\delta(t-s)}ds \int_0^t e^{-\delta(t-s)}\|f(s+\tau) - f(s)\|^2 ds\right] \\ \le& \frac{M^2}{\delta}\int_0^t e^{-\delta(t-s)}E\|f(s+\tau) - f(s)\|^2 ds \\ \le& \frac{M^2}{\delta}\int_0^t e^{-\delta(t-s)}\epsilon ds \le \frac{M^2}{\delta^2}\epsilon. \end{aligned}$$

(ii) $x_2 \in AP_T(R^+, L^2(P,H))$. Let $\tilde{w}(s) = w(s+\tau) - w(\tau)$ for each $s \in R^+$. Note that $\tilde{w}$ is also a Brownian motion and has the same distribution as w. By Lemma 2.12 and (2), we have

$$\begin{aligned}E\|x_2(t+\tau)-x_2(t)\|^2 =&E\left\|\int_0^t T(t-s)g(s+\tau)dw(s+\tau)-\int_{-\infty}^t T(t-s)g(s)dw(s)\right\|^2\\ =&E\left\|\int_0^t T(t-s)[g(s+\tau)-g(s)]d\tilde{w}(s)\right\|^2\\ \le&\int_0^t E\|T(t-s)[g(s+\tau)-g(s)]\|^2ds\\ \le&\int_0^t M^2e^{-2\delta(t-s)}E\|g(s+\tau)-g(s)\|^2ds\\ \le&\int_0^t M^2e^{-2\delta(t-s)}\epsilon ds=\frac{M^2}{2\delta}\epsilon.\end{aligned}$$

(iii) $x_3\in AP_T(R^+,L^2(P,H))$. Define

$$r(t)=T(t-t_i)\beta_i,\ \ t_i<t\le t_{i+1}, i\in Z^+.$$

For $t_i<t\le t_{i+1},\ |t-t_i|>\epsilon,\ |t-t_{i+1}|>\epsilon, i\in Z^+$, by (4), we can get

$$t+\tau>t_i+\epsilon+\tau>t_{i+q},$$

and

$$t_{i+q+1}>t_{i+1}+\tau-\epsilon>t+\tau,$$

that is, $t_{i+q+1}>t+\tau>t_{i+q}$. Since $(a+b)^2\le 2(a^2+b^2)$, one has

$$\begin{aligned}&E\|r(t+\tau)-r(t)\|^2\\ =&E\|T(t+\tau-t_{i+q})\beta_{i+q}-T(t-t_i)\beta_i\|^2\\ =&E\|[T(t+\tau-t_{i+q})-T(t-t_i)]\beta_{i+q}+T(t-t_i)[\beta_{i+q}-\beta_i]\|^2\\ \le&2E\|[T(t+\tau-t_{i+q})-T(t-t_i)]\beta_{i+q}\|^2+2E\|T(t-t_i)[\beta_{i+q}-\beta_i]\|^2\\ \le&2\|T(t+\tau-t_{i+q})-T(t-t_i)\|^2E\|\beta_{i+q}\|^2+2\|T(t-t_i)\|E\|\beta_{i+q}-\beta_i\|^2\\ \le&2\|T(t+\tau-t_{i+q})-T(t-t_i)\|^2E\|\beta_{i+q}\|^2+2M^2\epsilon,\end{aligned}$$

since $\{T(t):t\ge 0\}$ is a C_0-semigroup (see [21, 3]), for the above ϵ, there exists $0<\mu<\epsilon<1$ such that $0<s<\mu$ implies $\|T(t-t_i+s)-T(t-t_i)\|<\epsilon$. Note that $M_0=\sup_{i\in Z}E\|\beta_i\|^2<\infty$, so

$$E\|r(t+\tau)-r(t)\|^2\le 2M_0\epsilon^2+2M^2\epsilon.$$

Next we will prove that r is uniformly continuous on each interval $(t_i,t_{i+1})(i\in Z^+)$. Let $t,h\in R^+$ such that $t_i<t,t+h<t_{i+1}$, then

$$E\|r(t+h)-r(t)\|^2\le\|T(t+h-t_i)-T(t-t_i)\|^2E\|\beta_i\|^2.$$

Since $\{T(t) : t \geq 0\}$ is a C_0-semigroup and $M_0 = \sup_{i \in Z^+} E\|\beta_i\|^2 < \infty$, we conclude that $E\|r(t+h) - r(t)\|^2 \to 0$ as $h \to 0$ independent of t and i.

Finally, by Cauchy-Schwarz inequality and (3),

$$
\begin{aligned}
& E\left\| \sum_{0<t_i<t+\tau} T(t+\tau-t_i)\beta_i - \sum_{0<t_i<t} T(t-t_i)\beta_i \right\|^2 \\
\leq & E\left[\sum_{0<t_i<t} \|T(t-t_i)[\beta_{i+q} - \beta_i]\| \right]^2 \\
\leq & E\left[\sum_{0<t_i<t} M e^{-\delta(t-t_i)} \|\beta_{i+q} - \beta_i\| \right]^2 \\
\leq & E\left[\Big(\sum_{0<t_i<t} M^2 e^{-\delta(t-t_i)} \Big) \sum_{0<t_i<t} e^{-\delta(t-t_i)} \|\beta_{i+q} - \beta_i\|^2 \right] \\
\leq & \frac{M^2}{1-e^{-\delta\gamma}} \sum_{0<t_i<t} e^{-\delta(t-t_i)} E\|\beta_{i+q} - \beta_i\|^2 \\
\leq & \frac{M^2}{1-e^{-\delta\gamma}} \sum_{0<t_i<t} e^{-\delta(t-t_i)} \epsilon \\
\leq & \frac{M^2}{(1-e^{-\delta\gamma})^2} \epsilon.
\end{aligned}
$$

In view of the above, it is clear that $x_3 \in AP_T(R^+, L^2(P, H))$.

Furthermore, since $\{T(t) : t \geq 0\}$ is a bounded C_0-semigroup and $\{T(t) : t > 0\}$ is compact, by Theorem 2.1 in [5], $T(\cdot)x_0 \in AP_T(R^+, L^2(P, H))$. By combing (i), (ii) and (iii), it follows that (5) is a square-mean piecewise almost periodic process, so system (4) has a square-mean piecewise almost periodic solution. The proof is complete. □

4 Almost periodic solutions for nonlinear impulsive stochastic differential equations

Consider the following nonlinear impulsive stochastic differential equation

$$
\begin{cases}
dx(t) & = [Ax(t) + f(t, x(t))]dt + g(t, x(t))dw(t), \ t \geq 0, t \neq t_i, i \in Z^+, \\
\triangle x(t_i) & = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), i \in Z^+,
\end{cases}
\tag{6}
$$

where $f, g : R^+ \times L^2(P, H) \to L^2(P, H)$, $I_i : L^2(P, H) \to L^2(P, H), i \in Z^+$ and $w(t)$ is a two-sided standard one dimensional Brownian motion defined on the filtered probability space (Ω, F, P, F_σ) with $F_t = \sigma\{w(u) - w(v) : u, v \leq t\}$.

Definition 4.1. *An F_t-progressive process $x(t)$ is called a mild solution of system (6) if it satisfies* *the corresponding stochastic integral equation*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s,x(s))ds + \int_0^t T(t-s)g(s,x(s))dw(s) + \sum_{0<t_i<t} T(t-t_i)I_i(x(t_i)). \quad (7)$$

for all $t \geq 0$.

In order to obtain the existence of square-mean piecewise almost periodic solution to system (6), we introduce the following assumptions:

(A1) The operator $A : D(A) \subseteq L^2(P,H) \to L^2(P,H)$ is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t) : t \geq 0\}$ on $L^2(P,H)$, i.e., $\|T(t)\| \leq Me^{-\delta t},\ t \geq 0,\ M, \delta > 0$. Moreover, $T(t)$ is compact for $t > 0$.

(A2) $f, g \in AP_T(R^+ \times L^2(P,H), L^2(P,H))$, for each compact set $K \subseteq L^2(P,H)$, $g(t,\cdot), f(t,\cdot)$ are uniformly continuous in each compact set $K \subseteq L^2(P,H)$ uniformly for $t \in R^+$. $I_i(x)$ is almost periodic in $i \in Z^+$ uniformly in $x \in K$ and is a uniformly continuous function defined on the set $K \subseteq L^2(P,H)$ for all $i \in Z^+$.

(A3) $F_L = \sup_{\{t \in R^+,\ E\|x\|^2 \leq L\}} E\|f(t,x)\|^2 < \infty$, $G_L = \sup_{\{t \in R^+,\ E\|x\|^2 \leq L\}} E\|g(t,x)\|^2 < \infty$, $I_L = \sup_{\{i \in Z^+,\ E\|x\|^2 \leq L\}} E\|I_i(x(t_i))\|^2 < \infty$, where L is an arbitrary positive number. Moreover, there exist a number $L_0 > 0$ such that $4M^2L_0 + \frac{4M^2}{\delta^2}F_{L_0} + \frac{2M^2}{\delta}G_{L_0} + \frac{4M^2}{(1-e^{-\delta\gamma})^2}I_{L_0} \leq L_0$.

Theorem 4.2. *Assume that the conditions (A1)-(A3) are satisfied, then the impulsive stochastic differential equation* (6) *admits at least one square-mean piecewise almost periodic solution.*

Proof. Let

$$B = \{x \in AP_T(R^+, L^2(P,H)) : E\|x\|^2 \leq L_0\}.$$

Obviously, B is a closed set of $AP_T(R^+, L^2(P,H))$. Define Γ on $(PC)^0_h(R^+, L^2(P,H))$,

$$\Gamma x(t) = T(t)x_0 + \int_0^t T(t-s)f(s,x(s))ds + \int_0^t T(t-s)g(s,x(s))dw(s) + \sum_{0<t_i<t} T(t-t_i)I_i(x(t_i)).$$

In order to show that the impulsive stochastic differential equation (6) has a square-mean piecewise almost periodic solution, we only need to prove the operator Γ has a fixed point in B.

First we show $\Gamma x \in B, x \in B$. For $x \in B$, by Lemma 2.6 and (A2), we have $f(\cdot, x(\cdot)), g(\cdot, x(\cdot)) \in AP_T(R^+, L^2(P,H))$, by (A2) and Lemma 37 in [23], $I_i(x(t_i))$ is a square-mean almost periodic sequence, analogous to the proof of Theorem 3.2, we can show $\Gamma x \in AP_T(R^+, L^2(P,H))$.

Since $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, by using Cauchy-Schwarz inequality and Lemma

2.12, we obtain

$$
\begin{aligned}
&E\|\Gamma x(t)\|^2\\
\leq &4E\|T(t)x_0\|^2 + 4E\left\|\int_0^t T(t-s)f(s,x(s))ds\right\|^2 + 4E\left\|\int_0^t T(t-s)g(s,x(s))dw(s)\right\|^2\\
&+4E\left\|\sum_{0<t_i<t} T(t-t_i)I_i(x(t_i))\right\|^2\\
\leq &4M^2E\|x_0\|^2 + 4E\left[\int_0^t Me^{-\delta(t-s)}\|f(s,x(s))\|ds\right]^2 + 4\int_0^t E\|T(t-s)g(s,x(s))\|^2ds\\
&+4E\left[\sum_{0<t_i<t} Me^{-\delta(t-s)}\|I_i(x(t_i))\|\right]^2\\
\leq &4M^2L_0 + 4E\left[\int_0^t M^2e^{-\delta(t-s)}ds\int_0^t e^{-\delta(t-s)}\|f(s,x(s))\|^2ds\right]\\
&+4\int_0^t M^2e^{-2\delta(t-s)}E\|g(s,x(s))\|^2ds + 4E\left[\Big(\sum_{0<t_i<t} M^2e^{-\delta(t-t_i)}\Big)\sum_{0<t_i<t} e^{-\delta(t-t_i)}\|I_i(x(t_i))\|^2\right]\\
\leq &4M^2L_0 + \frac{4M^2}{\delta}\int_0^t e^{-\delta(t-s)}E\|f(s,x(s))\|^2ds + 4\int_0^t M^2e^{-2\delta(t-s)}E\|g(s,x(s))\|^2ds\\
&+\frac{4M^2}{1-e^{-\delta\gamma}}\sum_{0<t_i<t} e^{-\delta(t-t_i)}E\|I_i(x(t_i))\|^2\\
\leq &4M^2L_0 + \frac{4M^2}{\delta}\int_0^t e^{-\delta(t-s)}F_{L_0}ds + 4\int_0^t M^2e^{-2\delta(t-s)}G_{L_0}ds + \frac{4M^2}{1-e^{-\delta\gamma}}\sum_{0<t_i<t} e^{-\delta(t-t_i)}I_{L_0}\\
\leq &4M^2L_0 + \frac{4M^2}{\delta^2}F_{L_0} + \frac{2M^2}{\delta}G_{L_0} + \frac{4M^2}{(1-e^{-\delta\gamma})^2}I_{L_0},
\end{aligned}
$$

since $4M^2L_0 + \frac{4M^2}{\delta^2}F_{L_0} + \frac{2M^2}{\delta}G_{L_0} + \frac{4M^2}{(1-e^{-\delta\gamma})^2}I_{L_0} \leq L_0$, then $E\|\Gamma x\|^2 \leq L_0$, that is, $\Gamma x \in B, x \in B$.

Next we show $B(t) = \{\Gamma x(t) :\ x \in B\}$ is a relatively compact subset of $L^2(P,H)$ for each $t \in R^+$. For each $t \in R^+$, $0 < \epsilon < 1$, $x \in B$, define

$$
\begin{aligned}
\Gamma^\epsilon x(t) =&T(t)x_0 + \int_0^{t-\epsilon} T(t-s)f(s,x(s))ds + \int_0^{t-\epsilon} T(t-s)g(s,x(s))dw(s)\\
&+\sum_{0<t_i<t-\epsilon} T(t-t_i)I_i(x(t_i))\\
=&T(\epsilon)\Big[T(t-\epsilon)x_0 + \int_0^{t-\epsilon} T(t-\epsilon-s)f(s,x(s))ds\\
&+\int_0^{t-\epsilon} T(t-\epsilon-s)g(s,x(s))dw(s) + \sum_{0<t_i<t-\epsilon} T(t-\epsilon-t_i)I_i(x(t_i))\Big]\\
=&T(\epsilon)\Gamma x(t-\epsilon).
\end{aligned}
$$

Since $\{\Gamma x(t-\epsilon) : x \in B\}$ is bounded and $T(\epsilon)$ is compact, $\{\Gamma^\epsilon x(t) : x \in B\}$ is a relatively compact subset of $L^2(P,H)$. Moreover, for ϵ is small enough and the points t and $t-\epsilon$ belong to the same interval of continuity of x, then

$$\Gamma x(t)-\Gamma^{\epsilon}x(t)=\int_{t-\epsilon}^{t}T(t-s)f(s,x(s))ds+\int_{t-\epsilon}^{t}T(t-s)g(s,x(s))dw(s),$$

since $(a+b)^2\le 2(a^2+b^2)$, by using Cauchy-Schwarz inequality and Lemma 2.12, one has

$$\begin{aligned}
&E\|\Gamma x(t)-\Gamma^{\epsilon}x(t)\|^2\\
\le&2\left[E\left\|\int_{t-\epsilon}^{t}T(t-s)f(s,x(s))ds\right\|^2+E\left\|\int_{t-\epsilon}^{t}T(t-s)g(s,x(s))dw(s)\right\|^2\right]\\
\le&2E\left[\int_{t-\epsilon}^{t}Me^{-\delta(t-s)}\|f(s,x(s))\|ds\right]^2+2\int_{t-\epsilon}^{t}E\|T(t-s)g(s,x(s))\|^2ds\\
\le&2E\left[\int_{t-\epsilon}^{t}M^2e^{-\delta(t-s)}ds\int_{t-\epsilon}^{t}e^{-\delta(t-s)}\|f(s,x(s))\|^2ds\right]+2\int_{t-\epsilon}^{t}M^2e^{-2\delta(t-s)}E\|g(s,x(s))\|^2ds\\
\le&2M^2\epsilon\int_{t-\epsilon}^{t}E\|f(s,x(s))\|^2ds+2M^2\int_{t-\epsilon}^{t}E\|g(s,x(s))\|^2ds\\
\le&2M^2\epsilon^2F_{L_0}+2M^2\epsilon G_{L_0},
\end{aligned}$$

so $B(t)=\{\Gamma x(t):\ x\in B\}$ is a relatively compact subset of $L^2(P,H)$ for each $t\in R^+$.

Finally we show $\{\Gamma x : x\in B\}$ is equicontinuous at each interval $(t_i,t_{i+1})(i\in Z^+)$. Let $x\in B,\ t_i<t''<t'<t_{i+1}, i\in Z^+$, and $\rho<\min\left\{\frac{\epsilon}{36M^2F_{L_0}},\frac{\epsilon}{36M^2G_{L_0}},1\right\}$,

$$\begin{aligned}
&\Gamma x(t')-\Gamma x(t'')\\
=&T(t')x_0+\int_0^{t'}T(t'-s)f(s,x(s))ds+\int_0^{t'}T(t'-s)g(s,x(s))dw(s)\\
&+\sum_{0<t_i<t'}T(t'-t_i)I_i(x(t_i))-T(t'')x_0-\int_0^{t''}T(t''-s)f(s,x(s))ds\\
&-\int_0^{t''}T(t''-s)g(s,x(s))dw(s)-\sum_{0<t_i<t''}T(t''-t_i)I_i(x(t_i))\\
=&[T(t')-T(t'')]x_0+\int_{t''}^{t'}T(t'-s)f(s,x(s))ds+\int_{t''}^{t'}T(t'-s)g(s,x(s))dw(s)\\
&+\int_0^{t''}[T(t'-s)-T(t''-s)]f(s,x(s))ds+\int_0^{t''}[T(t'-s)-T(t''-s)]g(s,x(s))dw(s)\\
&+\sum_{0<t_i<t''}[T(t'-t_i)-T(t''-t_i)]I_i(x(t_i)).
\end{aligned}$$

Since $\{T(t):t\ge 0\}$ is a C_0-semigroup, there exists $\mu<\rho$ such that $t'-t''<\mu$ implies that

$$\|T(t)-I\|^2\le\min\left\{\frac{\epsilon}{36M^2L_0},\frac{\epsilon\delta^2}{36M^2F_{L_0}},\frac{\epsilon\delta}{18M^2G_{L_0}},\frac{\epsilon(1-e^{-\delta\gamma})^2}{36M^2I_{L_0}}\right\}.$$

By using Cauchy-Schwarz inequality and Lemma 2.12, we have

$$\begin{aligned}E\|[T(t')-T(t'')]x_0\|^2 &= E\|[T(t'-t'')-I]T(t'')x_0\|^2\\ &\le \|T(t'-t'')-I\|^2M^2E\|x_0\|^2\\ &\le \frac{\epsilon}{36M^2L_0}M^2L_0=\frac{\epsilon}{36},\end{aligned}$$

and,

$$\begin{aligned}E\left\|\int_{t''}^{t'}T(t'-s)f(s,x(s))ds\right\|^2 \le& E\left[\int_{t''}^{t'}Me^{-\delta(t'-s)}\|f(s,x(s))\|ds\right]^2\\ \le& E\left[\int_{t''}^{t'}M^2e^{-\delta(t'-s)}ds\int_{t''}^{t'}e^{-\delta(t'-s)}\|f(s,x(s))\|^2ds\right]\\ \le& M^2(t'-t'')\int_{t''}^{t'}e^{-\delta(t'-s)}E\|f(s,x(s))\|^2ds\\ \le& M^2F_{L_0}(t'-t'')^2\\ \le& M^2F_{L_0}\frac{\epsilon}{36M^2F_{L_0}}=\frac{\epsilon}{36},\end{aligned}$$

and

$$\begin{aligned}E\left\|\int_{t''}^{t'}T(t'-s)g(s,x(s))dw(s)\right\|^2 \le& \int_{t''}^{t'}E\|T(t'-s)g(s,x(s))\|^2ds\\ \le& \int_{t''}^{t'}M^2e^{-2\delta(t'-s)}E\|g(s,x(s))\|^2ds\\ \le& M^2G_{L_0}(t'-t'')\\ \le& M^2G_{L_0}\frac{\epsilon}{36M^2G_{L_0}}=\frac{\epsilon}{36},\end{aligned}$$

and

$$\begin{aligned}&E\left\|\int_0^{t''}[T(t'-s)-T(t''-s)]f(s,x(s))ds\right\|^2\\ =&E\left\|\int_0^{t''}[T(t'-t'')-I]T(t''-s)f(s,x(s))ds\right\|^2\\ \le&E\left[\int_0^{t''}\|T(t'-t'')-I\|Me^{-\delta(t''-s)}\|f(s,x(s))\|ds\right]^2\\ \le&E\left[\int_0^{t''}\|T(t'-t'')-I\|^2M^2e^{-\delta(t''-s)}ds\int_0^{t''}e^{-\delta(t''-s)}\|f(s,x(s))\|^2ds\right]\\ \le&\|T(t'-t'')-I\|^2\frac{M^2}{\delta}\int_0^{t''}e^{-\delta(t''-s)}E\|f(s,x(s))\|^2ds\\ \le&\|T(t'-t'')-I\|^2\frac{M^2}{\delta}\int_0^{t''}e^{-\delta(t''-s)}F_{L_0}ds\\ \le&\frac{\epsilon\delta^2}{36M^2F_{L_0}}\frac{M^2}{\delta^2}F_{L_0}=\frac{\epsilon}{36},\end{aligned}$$

and

$$
\begin{aligned}
&E\Big|\Big|\int_0^{t''}[T(t'-s)-T(t''-s)]g(s,x(s))dw(s)\Big|\Big|^2\\
\le&\int_0^{t''}E||[T(t'-t'')-I]T(t''-s)g(s,x(s))||^2ds\\
\le&\int_0^{t''}||T(t'-t'')-I||^2M^2e^{-2\delta(t''-s)}E||f(s,x(s))||^2ds\\
\le&||T(t'-t'')-I||^2\int_0^{t''}M^2e^{-2\delta(t''-s)}G_{L_0}ds\\
\le&\frac{\epsilon\delta}{18M^2G_{L_0}}\frac{M^2}{2\delta}G_{L_0}=\frac{\epsilon}{36},
\end{aligned}
$$

and

$$
\begin{aligned}
&E\Big|\Big|\sum_{0<t_i<t''}[T(t'-t_i)-T(t''-t_i)]I_i(x(t_i))\Big|\Big|^2\\
=&E\Big|\Big|\sum_{0<t_i<t''}[T(t'-t'')-I]T(t''-t_i)I_i(x(t_i))\Big|\Big|^2\\
\le&E\Big[\sum_{0<t_i<t''}||T(t'-t'')-I||||T(t''-t_i)||||I_i(x(t_i))||\Big]^2\\
\le&E\Big[\Big(\sum_{0<t_i<t''}||T(t'-t'')-I||^2e^{-\delta(t''-t_i)}\Big)\sum_{0<t_i<t''}e^{-\delta(t''-t_i)}||I_i(x(t_i))||^2\Big]\\
\le&||T(t'-t'')-I||^2\frac{M^2}{1-e^{-\delta\gamma}}\sum_{0<t_i<t''}e^{-\delta(t''-t_i)}E||I_i(x(t_i))||^2\\
\le&||T(t'-t'')-I||^2\frac{M^2}{1-e^{-\delta\gamma}}\sum_{0<t_i<t''}e^{-\delta(t''-t_i)}I_{L_0}\\
\le&\frac{\epsilon(1-e^{-\delta\gamma})^2}{36M^2I_{L_0}}\frac{M^2}{(1-e^{-\delta\gamma})^2}I_{L_0}=\frac{\epsilon}{36},
\end{aligned}
$$

so that, for $x \in B$ and $t' - t'' < \mu,\ t', t'' \in (t_i, t_{i+1}), i \in Z$,

$$\begin{aligned}
&E\|\Gamma x(t') - \Gamma x(t'')\|^2 \\
\leq &6E\|[T(t') - T(t'')]x_0\|^2 + 6E\left\|\int_{t''}^{t'} T(t'-s)f(s,x(s))ds\right\|^2 \\
&+ 6E\left\|\int_{t''}^{t'} T(t'-s)g(s,x(s))dw(s)\right\|^2 + 6E\left\|\int_0^{t''} [T(t'-s) - T(t''-s)]f(s,x(s))ds\right\|^2 \\
&+ 6E\left\|\int_0^{t''} [T(t'-s) - T(t''-s)]g(s,x(s))dw(s)\right\|^2 \\
&+ 6E\left\|\sum_{0<t_i<t''} [T(t'-t_i) - T(t''-t_i)]I_i(x(t_i))\right\|^2 \\
\leq &\ \epsilon,
\end{aligned}$$

which shows that $\{\Gamma x : x \in B\}$ is equicontinuous at each interval $(t_i, t_{i+1})(i \in Z^+)$.

Since $\{\Gamma x :\ x \in B\} \subseteq (PC)_h^0(R^+, L^2(P, H))$ and $\{\Gamma x :\ x \in B\}$ satisfies the conditions of Lemma 2.10, $\{\Gamma x :\ x \in B\}$ is a relatively compact set, moreover, the Lebesgue dominated convergence theorem and our assumptions on f, g and I_i imply that Γ is continuous, then Γ is a compact operator. It follows from Schauder fixed point theorem that Γ has a fixed point in B. Thus x is a square-mean piecewise almost periodic solution of system (6). The proof is complete. □

Note that the uniformly continuous is weaker than the Lipschitz continuous, if (A2) is replaced by the following condition:

(A2') $f, g \in AP_T(R^+, L^2(P, H))$, $I_i(x)$ is almost periodic in $i \in Z^+$ uniformly for $x \in L^2(P, H)$, and there exists positive numbers L_1, L_2, L such that

$$E\|f(t,x) - f(t,y)\|^2 \leq L_1 E\|x - y\|^2,$$

$$E\|g(t,x) - g(t,y)\|^2 \leq L_2 E\|x - y\|^2,$$

$$E\|I_i(x) - I_i(y)\|^2 \leq LE\|x - y\|^2,$$

for all $x, y \in L^2(P, H), t \in R^+$,
then by Lemma 2.7 and Theorem 3.2, we can also get the almost periodic solution of system (6).

Corollary 4.3. *Suppose that the conditions (A1), (A2') and (A3) are satisfied, then the impulsive stochastic differential equation* (6) *has a square-mean piecewise almost periodic solution.*

5 Stability

In this section we consider the stability of square-mean piecewise almost periodic solution to system (6) with Lipschitz activation function. In the sequel, we will need the following lemma.

Lemma 5.1. ([23]) *Let a nonnegative piecewise continuous function* $u(t)$ *satisfy for* $t \geq t_0$ *the inequality*

$$u(t) \leq C + \int_{t_0}^{t} \nu(\tau)u(\tau)d\tau + \sum_{t_0<\tau_i<t} \beta_i u(\tau_i),$$

where $C \geq 0,\ \beta_i \geq 0,\ \nu(\tau) > 0$, *and* $\tau_i's$ *are discontinuity points of first type of the function* $u(t)$. *Then the following estimate holds for the function* $u(t)$,

$$u(t) \leq C \prod_{t_0<\tau_i<t} (1+\beta_i)e^{\int_{t_0}^{t} \nu(\tau)d\tau}.$$

Theorem 5.2. *Assume the conditions of Corollary 4.3 are fulfilled. Assume further that* $\frac{1}{\gamma}\ln(1 + \frac{4M^2L}{1-e^{-\delta\gamma}}) + \frac{4M^2L_1}{\delta} + 4M^2L_2 < 0$. *Then system* (6) *has an exponentially stable almost periodic solution.*

Proof. By Corollary 4.3, system (6) has a mild square-mean piecewise almost periodic solution $u(t)$,

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds + \int_0^t T(t-s)g(s,u(s))dw(s) + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)).$$

Let $u(t) = u(t,0,\varphi)$ and $v(t) = v(t,0,\psi)$ be two solutions of equation (6), then

$$u(t) = T(t)\varphi + \int_0^t T(t-s)f(s,u(s))ds + \int_0^t T(t-s)g(s,u(s))dw(s) + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)),$$

$$v(t) = T(t)\psi + \int_0^t T(t-s)f(s,v(s))ds + \int_0^t T(t-s)g(s,v(s))dw(s) + \sum_{0<t_i<t} T(t-t_i)I_i(v(t_i)).$$

Since $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, by (A2'), Cauchy-Schwarz inequality and Lemma

2.12, we have

$$
\begin{aligned}
&E\|u(t)-v(t)\|^2\\
=&E\bigg\|T(t)\varphi - T(t)\psi + \int_0^t T(t-s)[f(s,u(s))-f(s,v(s))]ds\\
&\quad + \int_0^t T(t-s)[g(s,u(s))-g(s,v(s))]dw(s) + \sum_{0<t_i<t} T(t-t_i)[I_i(u(t_i))-I_i(v(t_i))]\bigg\|^2\\
\le& 4E\|T(t)[\varphi-\psi]\|^2 + 4E\bigg\|\int_0^t T(t-s)[f(s,u(s))-f(s,v(s))]ds\bigg\|^2\\
&+4E\bigg\|\int_0^t T(t-s)[g(s,u(s))-g(s,v(s))]dw(s)\bigg\|^2\\
&+4E\bigg\|\sum_{0<t_i<t} T(t-t_i)[I_i(u(t_i))-I_i(v(t_i))]\bigg\|^2\\
\le& 4M^2e^{-2\delta t}E\|\varphi-\psi\|^2 + 4E\left[\int_0^t Me^{-\delta(t-s)}\|f(s,u(s))-f(s,v(s))\|ds\right]^2\\
&+4\int_0^t E\|T(t-s)[g(s,u(s))-g(s,v(s))]\|^2 ds\\
&+4E\left[\sum_{0<t_i<t} Me^{-\delta(t-t_i)}\|I_i(u(t_i))-I_i(v(t_i))\|\right]^2\\
\le& 4M^2e^{-2\delta t}E\|\varphi-\psi\|^2 + 4E\left[\int_0^t M^2e^{-\delta(t-s)}ds\int_0^t e^{-\delta(t-s)}\|f(s,u(s))-f(s,v(s))\|^2 ds\right]\\
&+4\int_0^t M^2e^{-2\delta(t-s)}E\|g(s,u(s))-g(s,v(s))\|^2 ds\\
&+4E\left[\Big(\sum_{0<t_i<t} M^2e^{-\delta(t-t_i)}\Big)\sum_{0<t_i<t} e^{-\delta(t-t_i)}\|I_i(u(t_i))-I_i(v(t_i))\|^2\right]\\
\le& 4M^2e^{-\delta t}E\|\varphi-\psi\|^2 + \frac{4M^2}{\delta}\int_0^t e^{-\delta(t-s)}E\|f(s,u(s))-f(s,v(s))\|^2 ds\\
&+4M^2\int_0^t e^{-\delta(t-s)}E\|g(s,u(s))-g(s,v(s))\|^2 ds\\
&+\frac{4M^2}{1-e^{-\delta\gamma}}\sum_{0<t_i<t} e^{-\delta(t-t_i)}E\|I_i(u(t_i))-I_i(v(t_i))\|^2\\
\le& 4M^2e^{-\delta t}E\|\varphi-\psi\|^2 + \left(\frac{4M^2L_1}{\delta}+4M^2L_2\right)\int_0^t e^{-\delta(t-s)}E\|u(s)-v(s)\|^2 ds\\
&+\frac{4M^2L}{1-e^{-\delta\gamma}}\sum_{0<t_i<t} e^{-\delta(t-t_i)}E\|u(t_i)-v(t_i)\|^2.
\end{aligned}
$$

Then,

$$e^{\delta t}E\|u(t)-v(t)\|^2 \leq 4M^2E\|\varphi-\psi\|^2+\left(\frac{4M^2L_1}{\delta}+4M^2L_2\right)\int_0^t e^{\delta s}E\|u(s)-v(s)\|^2ds$$
$$+\frac{4M^2L}{1-e^{-\delta\gamma}}\sum_{0<t_i<t}e^{\delta t_i}E\|u(t_i)-v(t_i)\|^2.$$

Let $\Upsilon(t)=e^{\delta t}E\|u(t)-v(t)\|^2$, then

$$\Upsilon(t)\leq 4M^2\Upsilon(0)+\left(\frac{4M^2L_1}{\delta}+4M^2L_2\right)\int_0^t\Upsilon(s)ds+\frac{4M^2L}{1-e^{-\delta\gamma}}\sum_{0<t_i<t}\Upsilon(t_i).$$

By Lemma 5.1, we have

$$\begin{aligned}\Upsilon(t)\leq & 4M^2\Upsilon(0)\prod_{0<t_i<t}\left(1+\frac{4M^2L}{1-e^{-\delta\gamma}}\right)e^{\int_0^t\left(\frac{4M^2L_1}{\delta}+4M^2L_2\right)ds}\\ = & 4M^2\Upsilon(0)\prod_{0<t_i<t}\left(1+\frac{4M^2L}{1-e^{-\delta\gamma}}\right)e^{\left(\frac{4M^2L_1}{\delta}+4M^2L_2\right)t}\\ \leq & 4M^2\Upsilon(0)\left(1+\frac{4M^2L}{1-e^{-\delta\gamma}}\right)^{\frac{t}{\gamma}}e^{\left(\frac{4M^2L_1}{\delta}+4M^2L_2\right)t}\\ = & 4M^2\Upsilon(0)e^{\left(\frac{1}{\gamma}\ln(1+\frac{4M^2L}{1-e^{-\delta\gamma}})+\frac{4M^2L_1}{\delta}+4M^2L_2\right)t},\end{aligned}$$

that is,

$$E\|u(t)-v(t)\|^2\leq 4M^2E\|\varphi-\psi\|^2e^{\left(\frac{1}{\gamma}\ln(1+\frac{4M^2L}{1-e^{-\delta\gamma}})+\frac{4M^2L_1}{\delta}+4M^2L_2\right)t}.$$

Since $\frac{1}{\gamma}\ln(1+\frac{4M^2L}{1-e^{-\delta\gamma}})+\frac{4M^2L_1}{\delta}+4M^2L_2<0$. The square-mean piecewise almost periodic solution of system (6) is exponentially stable. This completes the proof. □

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