Approximate solution of fractional integro-differential equation by Taylor expansion and Legendre wavelets methods

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ABSTRACT

This paper, deals with the approximate solution of fractional integro-differential equations of the type

$$D_*^q y(t) = f(t) + p(t)y(t) + \int_0^t k(t,s)y(s)ds, \qquad t \in I = [0,1]$$

by Taylor expansion and Legendre wavelet methods.In addition, illustrative example are presented to demonstrate the efficiency and accuracy of this methods.

RESUMEN

Este artículo considera la solución aproximada de ecuaciones integro-diferenciales fraccionales del tipo

$$D_*^q y(t) = f(t) + p(t)y(t) + \int_0^t k(t,s)y(s)ds, \qquad t \in I = [0,1]$$

por expansiones de Taylor y métodos de Ondeletas de Legendre. Además, un ejemplo ilustrativo se presenta para mostrar la eficiencia y precisión de este método.

Keywords and Phrases: Fractional integro-differential equation, Caputo fractional derivative, Taylor expansion method, Legendre wavelets method.

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1. Introduction

We study the approximate solution of an integro-differential equation with fractional derivative of the type

$$D_*^q y(t) = f(t) + p(t)y(t) + \int_0^t k(t,s)y(s)ds, \qquad t \in I = [0,1], \quad y(0) = \alpha , \qquad (1.1)$$

where $0 < q < 1, \alpha \in Re$ and the functions f, p, k are assumed to be sufficiently smooth on their domains I and S ($S = \{(t,s) : 0 \le s \le t \le 1\}$.This kinds of equations arise in many modeling problems in mathematical physics such as heat conduction in materials with memory .The existence and uniqueness of solution of fractional differential equation have been investigated in [5,7]. Recently some attentions have been paid to the numerical solution of equation (1.1) . Rawashdeh [4] applied the collocation method to find a spline approximation, in [11] used the decomposition method to find an analytic solution.

2. Basic definitions

Definition2.1. The Riemann-Liouville fractional integral of order $q \ge 0$ of a function $f \in c_\alpha, \alpha \ge -1$ is defined by :

$$J^q f(x) = \frac{1}{\Gamma(q)} \int_0^x (x-s)^{q-1} f(s)ds$$

where the real function $f(x) \in C_\alpha, \alpha \in Re, x > 0$ is said to be in space if there exist a real number $P > \alpha$ such that $f(x) = x^P f_1(x)$ where $f_1(x) \in C[0,\infty)$.

Definition2.2. Let $f \in C_{-1}^k, k \in N$. Then the Caputo fractional derivative of f is defined by :

$$D_*^q f(x) = \begin{cases} J^{k-q} f^{(k)}(x) & if k-1 < q < k, \\ \\ \\ f^{(k)}(x) & if k = q \end{cases}$$

. To obtain a numerical scheme for the approximation of Caputo derivative , we can use a representation that has been introduced by Elliots [2];

$$D_*^q f(x) = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(s) - f(0)}{(x-s)^{1+q}} ds, \qquad (1.2)$$

, where the integral in equation (1.2) is a Hadamard finite-part integral

Definition2.3. D^q, denotes the fractional differential operator of order q, defined by [5] as:

$$D^q y(x) = \begin{cases} \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{y(s)}{(x-s)^{q-n+1}} ds, & 0 \le n-1 < q < n, \\ \\ \\ \frac{d^n y(x)}{dx^n}, & q = n \end{cases}$$

Definition2.4. The following functions

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - n b_0)$$

, form a family of discrete wavelets , where $a_0 > 1, b_0 > 0$ and n, k are positive integers and ψ is given function called mother wavelet . Moreover, the functions

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m + \frac{1}{2}} \; 2^{\frac{k}{2}} \; p_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \le t < \frac{\hat{n}+1}{2^k}, \\ \\ \\ = 0 & \text{otherwise} \end{cases} \tag{1.3}$$

are called legendre wavelets polynomials where $\hat{n} = 2n-1, \quad n = 1, \ldots, 2^k - 1, \quad k \in N, \quad t \in [0,1]$ and m is the order of the legendre polynomials p_m . some basic properties of the caputo and fractional operator can be found in [5].

3. Taylor expansion method

We consider the following fractional integro-differential equation

$$D^q y(t) = f(t) + p(t) y(t) + \lambda \int_0^t k(t,s) y(s) ds, \tag{3.1}$$

subjected to the initial conditions $y^{(k)}(0) = c_k, k = 0, 1, \ldots n-1, \quad n-1 < q \le n, \quad n \in N$. To find the solution of Eq.(3.1) , we integrate both sides of Eq.(3.1) with respect to s for n times by using definitions (2.2) , (2.3) .

$$I^{n-q} y(t) = I^n f(t) + I^n(p(t) y(t)) + \lambda I^n(\int_0^t k(t,s) y(s) ds), \tag{3.2}$$

further

$$\int_0^t \frac{(t-s)^{n-q-1}}{\Gamma(n-q)} y(s) ds = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds$$

$$+ \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \; p(s) y(s) ds + \frac{\lambda}{(n-1)!} \int_0^t y(s) \int_s^t k(x,s)(x-s)^{n-1} ds dx + q_n(t). \tag{3.3}$$

Next , we assume that the desired solution $y(s)$ is $m+1$ times continuously differentiable on the interval I. Consequently , for $y \in c^{m+1}, y(s)$ can be represented in terms of the mth order Taylor expansion as

$$y(s) = y(t) + y'(t)(s-t) + ... + y^{(m)}(t)\frac{(s-t)^m}{m!} + y^{(m+1)}(\xi)\frac{(s-t)^{m+1}}{(m+1)!}, \tag{3.4}$$

where ξ is between s and t. The Lagrange reminder $y^{(m+1)}(\xi)\frac{(s-t)^{m+1}}{(m+1)!}$ is small for a large enough m provided that $y^{(m+1)}(s)$ is uniformly bounded function for any m on the interval I. Consequently we will neglect the reminder and the truncated Taylor expansions $y(x)$ as

$$y(s) \approx \sum_{j=0}^{m} y^{(j)}(t)\frac{(x-t)^j}{j!}. \tag{3.5}$$

We notice that the lagrange reminder vanishes for a polynomial of degree equal to or less than m, this is implying that the above m^{th} order Taylor expansion is exact . Substituting the approximate expression (3.5) for $y(t)$ into Eq(3.2) , we get

$$\sum_{j=0}^{m}\int_0^t \frac{(t-s)^{n-q-1}}{\Gamma(n-q)}y^j(t)\frac{(s-t)^j}{j!}ds = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}f(s)ds + \sum_{j=0}^{m}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}p(s)y^j(t)\frac{(s-t)^j}{j!}ds$$

$$+ \sum_{j=0}^{m}\frac{\lambda}{(n-1)!}\int_0^t y^j(t)\frac{(s-t)^j}{j!}\int_0^t k(x,s)(x-s)^{n-1}dsdx + Q_n(t), \tag{3.6}$$

or

$$K_{00}(t)y(t) + K_{01}(t)y'(t) + ... + K_{0m}(t)y^{(m)}(t) = f_n(t), \tag{3.7}$$

where

$$K_{0j}(t) = \frac{(-1)^j t^{n+j-q}}{(n+j-q)\Gamma(n-q)j!} - \frac{\lambda}{(n-1)!j!}\int_0^t(s-t)^j\int_0^t k(x,s)(x-s)^{n-1}dsdx$$

$$- \frac{(-1)^j}{(n-1)!j!}\int_0^t p(s)(t-s)^{n+j-1}ds, \quad j=0,1,...,m. \tag{3.8}$$

$$f_{(n)}(t) = \frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}f(s)ds + Q_n(t). \tag{3.9}$$

Thus Eq.(3.3) becomes an mth order , linear , ordinary differential equation with variable coefficients for $y(t)$ and its derivatives up to m . We will determine $y(t),...,y^m(t)$ by solving linear equations instead of solving analytically ordinary differential equation . By integrating both sides of Eq.(3.3) with respect to s and changing the order of the integrations we shall obtain m independent linear equation for $y(s),...,y^m(s)$.

$$\int_0^t \frac{(t-s)^{n-q}}{\Gamma(n+1-q)}y(s)ds = \int_0^t \frac{(t-s)^n}{n!}y(s)ds + \int_0^t \frac{(t-s)^n}{n!}f(s) + Q_n(s)ds$$

$$+\frac{\lambda}{n!}\int_0^t y(s)\int_0^t k(x,s)(x-s)^n ds dx \qquad (3.10)$$

Where we have replaced x with t . Applying the Taylor expansion again and substituting (3.5) for $y(s)$ into Eq. (3.10) gives

$$K_{10}(t)y(t) + K_{11}(t)y'(t) + ... + K_{1m}(t)y^{(m)}(t) = f_{n+1}(t), \qquad (3.11)$$

$$K_{1j}(t) = \frac{(-1)^j t^{n+j+1-q}}{(n+j+1-q)\Gamma(n+1-q)j!} - \frac{\lambda}{n!j!}\int_0^t (s-t)^j \int_0^t k(x,s)(x-s)^n ds dx$$

$$-\frac{(-1)^j}{n!j!}\int_0^t p(s)(t-s)^{n+j}ds, \quad j=0,1,...,m. \qquad (3.12)$$

$$f_{(n+1)}(t) = \int_0^t \frac{(t-s)^n}{n!}f(s) + Q_n(t)ds. \qquad (3.13)$$

Now we have another linear equation for $y^{(j)}(t)$, $j = 0,...,m$ with $y^{(0)}(t) = y(t)$. By repeating the above integration process for i $(i \in N^+, 1 < i \le m)$ times, we get

$$K_{i0}(t)y(t) + K_{i1}(t)y'(t) + ... + K_{im}(t)y^{(m)}(t) = f_{n+i}(t), \qquad , i \le m \qquad (3.14)$$

where

$$K_{ij}(t) = \frac{(-1)^j t^{n+j+i-q}}{(n+j+i-q)\Gamma(n+i-q)j!} - \frac{(-1)^j}{(n+i-1)!j!}\int_0^t p(s)(t-s)^{n+j+i-1}ds$$

$$-\frac{\lambda}{(n+i-1)!j!}\int_0^t (s-t)^j \int_0^t k(x,s)(x-s)^{n+j-1}ds dx, \quad j=0,1,...,m. \qquad (3.15)$$

$$f_{(r)}(t) = \int_0^t f_{r-1}(s)ds, \qquad r > n+1, \ \ r \in N^+. \qquad (3.16)$$

Consequently, Eqs. (3.7), (3.11) and (3.14) form a system of m+1 unknown functions $y(s),....y^{(m)}(s)$. This system can be written as

$$K_{mm}(t)Y_m(t) = F_m(t), \qquad (3.17)$$

where $K_{mm}(t)$ is an $(m+1) \times (m+1)$ matrix function in t , $Y_m(t)$ and $F_m(t)$ are two vector defined as

$$K_{mm} = \begin{pmatrix} K_{00}(t) & K_{01}(t) & \ldots & K_{0m}(t) \\ K_{10}(t) & K_{11}(t) & \ldots & K_{1m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ K_{m0}(t) & K_{m1}(t) & \ldots & K_{mm}(t) \end{pmatrix} , \qquad (3.18)$$

$$Y_m(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(m)} \end{pmatrix} , \qquad F_m(t) = \begin{pmatrix} f_{(n)}(t) \\ f_{(n+1)}(t) \\ \vdots \\ f_{(n+m)}(t) \end{pmatrix} . \qquad (3.19)$$

Using cramer's rule , we obtain the m th-order approximate solution as

$$y(t) = \frac{\det M_{mm}(t)}{\det K_{mm}(t)} \qquad . \qquad (3.20)$$

where

$$M_{mm} = \begin{pmatrix} f_n(t) & K_{01}(t) & \dots & K_{0m}(t) \\ \\ f_{(n+1)}(t) & K_{11}(t) & \dots & K_{1m}(t) \\ \\ \vdots & \vdots & \ddots & \vdots \\ \\ f_{n+m}(t) & K_{m1}(t) & \dots & K_{mm}(t) \end{pmatrix}. \tag{3.21}$$

4. Legendre wavelets method

We consider the following fractional integro-differential equation

$$D_*^q y(t) = f(t) + p(t)y(t) + \int_0^t k(t,s)y(s)ds, \qquad t \in I = [0,1] \quad y(0) = \alpha \tag{4.1}$$

the exact solution of Eq.(4.1) can be expanded as a legendre wavelets series as

$$y(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{n,m}(t),$$

where $\psi_{n,m}(t)$ is given by Eq.(1.3).We approximate the solution $y(t)$ by the truncated series

$$y_{K,M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{n,m}(t), \tag{4.2}$$

Then a total number of $2^{k-1}M$ conditions exist for determination of $2^{k-1}M$ coefficients

$$c_{10}, c_{11}, ..., c_{1M-1}, c_{20}, ..., c_{2M-1}, ..., c_{2^{k-1}0}, ..., c_{2^{k-1}M-1}.$$

By the initial condition we obtain ,

$$y_{K,M}(0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{n,m}(0) = \alpha. \tag{4.3}$$

We must obtain $2^{k-1}M - 1$ extra conditions to recover the unknown coefficients c_{nm}. These conditions can be obtained by substituting Eq.(4.2) in Eq.(4.1).

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} D_*^q \psi_{n,m}(t) = f(t) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\, p(t)\, \psi_{n,m}(t) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \int_0^t k(t,s)\, \psi_{n,m}(s)ds.$$
(4.4)

Now we assume Eq.(4.4) is exact at $2^{k-1}M - 1$ points x_i as :

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} D_*^q \psi_{n,m}(x_i) = f(x_i) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\, p(x_i)\, \psi_{n,m}(x_i)$$

$$+ \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \int_0^{x_i} k(x_i,s)\, \psi_{n,m}(s)ds.$$
(4.5)

The best choice of the x_i points are the zeros of the shifted chebyshev polynomials of degree $2^{k-1}M - 1$ in the interval $[0,1]$ that is

$$x_i = \frac{s_i + 1}{2}, \quad s_i = \left(\frac{(2i-1)\pi}{2^{k-1}M - 1}\right), \quad i = 1,...,2^{k-1}M - 1.$$

Approximating $D_*^q \psi_{n,m}$ using Diethhelm method [6] on the representation that has been given by Eq.(1.2) , we get

$$D_*^q \psi_{n,m}(x_i) = \frac{1}{\Gamma(-q)} \int_0^{x_i} \frac{\psi_{n,m}(s) - \psi_{n,m}(0)}{(x_i - s)^{1+q}} ds$$

$$= \frac{x_i^{-q}}{\Gamma(-q)} \int_0^1 \frac{\psi_{n,m}(x_i - x_i w) - \psi_{n,m}(0)}{w^{1+q}} ds$$

$$\simeq \sum_{r=0}^l \alpha_r (\psi n, m(x_i - \frac{x_i r}{l}) - \psi_{n,m}(0))$$

where $L \in N$ and the weights α_r is given by

$$q(1-q)L^{-q} \frac{\Gamma(-q)}{x_i^{-q}} \alpha_r = \begin{cases} -1, & \text{if } r = 0, \\ \\ 2r^{1-q} - (r-1)^{1-q} - (r+1)^{1+q}, & \text{if } r = 1, 2, ..., l-1, \\ \\ (q-1)r^{-q} - (r-1)^{1-q} + r^{1-q}, & \text{if } r = l \end{cases}$$

Then Eq.(4.5) becomes

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \sum_{r=0}^{l} \alpha_r (\psi_{n,m}(x_i - \frac{x_i r}{l}) - \psi_{n,m}(0)) c_{nm} = f(x_i) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \ p(x_i) \ \psi_{n,m}(x_i)$$

$$+ \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \int_0^{x_i} k(x_i, s) \ \psi_{n,m}(s) ds. \tag{4.6}$$

Combine Eq.(4.3) and Eq.(4.6) to obtain $2^{k-1}M$ linear equations from which we can compute the unknowns coefficients , c_{nm}

5. Numerical examples

To show the efficiency and the accuracy of the proposed methods , we consider here some fractional integro-differential equations . Now we shall solve some examples by Taylor expansion and Legendre wavelet methods and compare the results in tables . All results are obtained by using Maple 15.

Example 1.
Consider the following Fractional integro-differential equation

$$D_*^{0.75} y(t) = \frac{2t^{1.25}}{\Gamma(2.25)} - t^4 - \frac{t^5}{4} + t^2 y(t) + \int_0^t ts y(s) \ ds \ , \tag{5.1}$$

with the initial condition $y(0) = 0$ and the exact solution $y(t) = t^2$, with K=1 and M=6 .

Table 1. The results of Example 1.

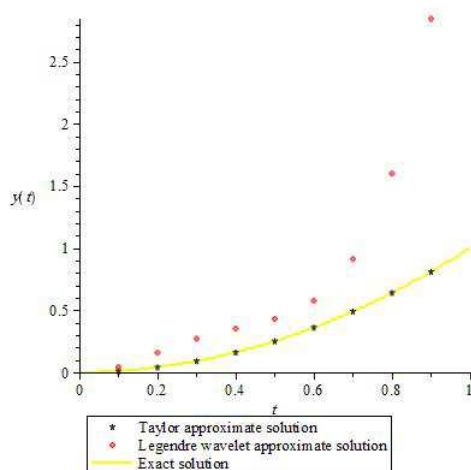| t	Exact	Taylor method	Abs.E	Legendre method	Abs.E
0.1	0.01	0.009999996	$4.086580027 \times 10^{-9}$	0.041204470	$0.312044705 \times 10^{-2}$
0.2	0.04	0.039999686	$3.137931423 \times 10^{-7}$	0.157896393	0.1178963927
0.3	0.09	0.089996776	$3.224235647 \times 10^{-6}$	0.270260094	0.1802600943
0.4	0.16	0.159983397	$1.660282668 \times 10^{-5}$	0.3523832431	0.1923832431
0.5	0.25	0.249941528	$5.847208088 \times 10^{-5}$	0.429628144	0.179628144
0.6	0.36	0.359839516	$1.604836601 \times 10^{-4}$	0.576003047	0.216003047
0.7	0.49	0.489634434	$3.655662632 \times 10^{-4}$	0.911533447	0.421533447
0.8	0.64	0.639288648	$7.113517648 \times 10^{-4}$	1.599633386	0.959633386
0.9	0.81	0.808822770	$1.177230376 \times 10^{-3}$	2.844476755	2.034476755

Figure 1: Results of Example 1

Example 2.

Consider the following Fractional integro-differential equation

$$D_*^{0.25}y(t) = \frac{6t^{2.75}}{\Gamma(3.75)} - t^4 - \frac{t^2 e^t}{5}y(t) + \int_0^t e^t s y(s) \ ds \ , \qquad (5.2)$$

with the initial condition $y(0) = 0$ and the exact solution $y(t) = t^3$, with K=2 and M=2 .

Table 2. The results of Example 2.

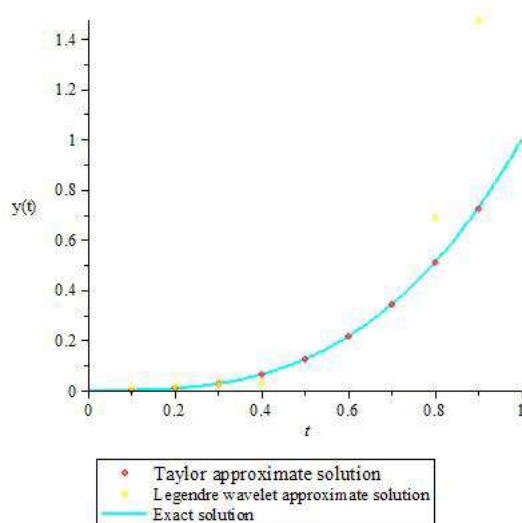| t	Exact	Taylor method	Abs.E	Legendre method	Abs.E
0.1	0.001	0.001000055	5.5082×10^{-8}	$0.7846772718 \times 10^{-2}$	$6.846772718 \times 10^{-3}$
0.2	0.008	0.008001757	1.756874×10^{-6}	0.015693545	$7.69354543 \times 10^{-3}$
0.3	0.027	0.027011518	1.151813×10^{-5}	0.02354031813	$3.45968187 \times 10^{-3}$
0.4	0.064	0.064035232	3.523224×10^{-5}	0.03138709084	$3.261290916 \times 10^{-2}$
0.5	0.125	0.125051836	5.18360×10^{-5}	-1.668332670	1.793332670
0.6	0.216	0.215957399	4.26011×10^{-5}	-0.882599797	1.098599797
0.7	0.343	0.342472414	5.275860×10^{-4}	-0.096866924	0.439866924
0.8	0.512	0.510019153	1.980847×10^{-3}	0.688865948	0.176865948
0.9	0.729	0.723579567	5.4204329×10^{-3}	1.474598821	0.745598821

Figure 2: Results of Example 2.

Example 3.

Consider the following Fractional integro-differential equation

$$D_*^{0.5}y(t) = \frac{2t^{1.5}}{\Gamma(2.5)} + \frac{t^{0.5}}{\Gamma(1.5)} + t(2 - 3\cos t - t\sin t + t^2\cos t) - (\cos t - \sin t)y(t) + \int_0^t e^t sy(s) \, ds \ ,$$
(5.3)

with the initial condition $y(0) = 0$ and the exact solution $y(t) = t^2 + t$ with K=2 and M=2 .

Similarly as in examples 1,2 applying the Taylor expansion method and Legendre wavelet method of the A comparison between the exact solution and the approximate solution and the absolute error (Abs.E) are given in table 3 and Figures 3.

Table 3. The results of Example 3.

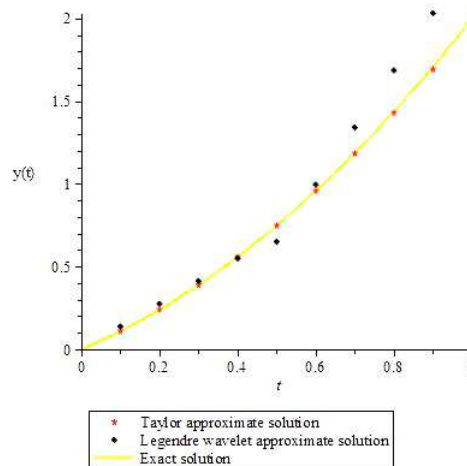| t	Exact	Taylor method	Abs.E	Legendre method	Abs.E
0.1	0.11	0.109999426	5.740×10^{-7}	0.1373123038	$2.73123038 \times 10^{-2}$
0.2	0.24	0.239986839	1.31607×10^{-5}	0.2746246075	$3.46246075 \times 10^{-2}$
0.3	0.39	0.389914938	8.50616×10^{-5}	0.4119369112	$2.19369112 \times 10^{-2}$
0.4	0.56	0.559673632	3.263682×10^{-4}	0.5492492149	$1.07507851 \times 10^{-2}$
0.5	0.75	0.749057525	9.424746×10^{-4}	0.649795587	0.100204413
0.6	0.96	0.957730079	2.2699214×10^{-3}	0.995295259	3.5295259×10^{-2}
0.7	1.19	1.185190201	4.809799×10^{-3}	1.340794930	0.150794930
0.8	1.44	1.430747819	9.252181×10^{-3}	1.686294602	0.246294602
0.9	1.71	1.693507524	1.6492476×10^{-2}	2.031794273	0.321794273

Figure 3: Results of Example 3. with Taylor expansion and Legendre wavelet method.

6. Conclusion

In this paper , we have applied the Legendre wavelet and Taylor expansion methods for solving the fractional integro-differential equation . The integro-differential equations converted to a system of linear equations by two methods . By comparing the errors in two methods we find that Taylor expansion method gives results better than Legendre wavelet method .

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