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# Coupled Coincidence Points for Generalized (ψ, φ)-Pair Mappings in Ordered Cone Metric Spaces

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**ABSTRACT**

The existence of coupled coincidence points for mappings satisfying generalized contractive conditions related to ψ and φ-maps in an ordered cone metric space is proved. Our results extend and generalize some well-known comparable results in the existing literature.

**RESUMEN**

Se prueba la existencia de puntos coincidentes acoplados para aplicaciones que satisfacen las condiciones de contractividad generalizada relacionada a las aplicaciones ψ y φ en un espacio métrico cono ordenados. Nuestro resultado extiende y generaliza algunos resultados comparables conocidos en la literatura.

**Keywords and Phrases:** Cone metric space, ψ-map, φ-map, coupled coincidence point.

**2010 AMS Mathematics Subject Classification:** 54H25, 47H10.

# 1 Introduction

Fixed point theory plays a major role in mathematics because of its applications in many important areas such as optimization, mathematical models, nonlinear and adaptive control systems. Over the past two decades a considerable amount of research work for the development of metric fixed point theory have executed by numerous mathematicians. The fixed points for certain mappings in ordered metric spaces has been studied by Ran and Reurings [16]. In [11] Nieto and López extended the result of Ran and Reurings [16] for nondecreasing mappings and applied their results to obtain a unique solution for a first order differential equation. The existence of coupled fixed points in partially ordered metric spaces was first investigated by Bhaskar and Laksmikantham [3]. So far, many mathematicians have studied coupled fixed point results for mappings under various contractive conditions in different metric spaces. In 2007, Huang and Zhang [5] introduced the concept of cone metric spaces and proved some important fixed point theorems. Afterwards, Sabetghadam and Masiha [17] obtained some fixed point results for generalized $\varphi$-pair mappings in cone metric spaces. The purpose of this paper is to obtain sufficient conditions for existence of coupled coincidence points for mappings satisfying generalized contractive conditions related to $\psi$ and $\varphi$ -maps in ordered cone metric spaces.

# 2 Preliminaries

In this section we need to recall some basic notations, definitions, and necessary results from existing literature.

**Definition 1.** *[3] Let* $(X, \sqsubseteq)$ *be a partially ordered set and* $F : X \times X \to X$ *be a self-map. One can say that* $F$ *has the mixed monotone property if* $F(x, y)$ *is monotone nondecreasing in* $x$ *and is monotone nonincreasing in* $y$*, that is, for all* $x_1, x_2 \in X,\ x_1 \sqsubseteq x_2$ *implies* $F(x_1, y) \sqsubseteq F(x_2, y)$ *for any* $y \in X$*, and for all* $y_1, y_2 \in X,\ y_1 \sqsupseteq y_2$ *implies* $F(x, y_1) \sqsubseteq F(x, y_2)$ *for any* $x \in X$*.*

**Definition 2.** *[4] Let* $(X, \sqsubseteq)$ *be a partially ordered set and* $F : X \times X \to X$ *and* $g : X \to X$ *be two self-mappings.* $F$ *has the mixed* $g$*-monotone property if* $F$ *is monotone* $g$*-nondecreasing in its first argument and is monotone* $g$*-nonincreasing in its second argument, that is, for all* $x_1, x_2 \in X,\ gx_1 \sqsubseteq gx_2$ *implies* $F(x_1, y) \sqsubseteq F(x_2, y)$ *for any* $y \in X$*, and for all* $y_1, y_2 \in X,\ gy_1 \sqsubseteq gy_2$ *implies* $F(x, y_1) \sqsupseteq F(x, y_2)$ *for any* $x \in X$*.*

**Definition 3.** *[3] An element* $(x, y) \in X \times X$ *is called a coupled fixed point of the mapping* $F : X \times X \to X$ *if* $x = F(x, y)$ *and* $y = F(y, x)$*.*

**Definition 4.** *[8] An element* $(x, y) \in X \times X$ *is called*

**(i)** *a coupled coincidence point of the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *if* $gx = F(x, y)$ *and* $gy = F(y, x)$*,*

**(ii)** *a common coupled fixed point of the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *if* $x = gx = F(x, y)$ *and* $y = gy = F(y, x)$.

**Definition 5.** *[4] Let* $X$ *be a nonempty set. One can say that the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are commutative if* $g(F(x, y)) = F(gx, gy)$, *for all* $x, y \in X$.

Let $E$ be a real Banach space and $\theta$ denote the zero element in $E$. A cone $P$ is a subset of $E$ such that

$$\begin{array}{ll} (i) & P \text{ is closed, nonempty and } P \neq \{\theta\}; \\ (ii) & a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \ \Rightarrow\ ax + by \in P; \\ (iii) & P \cap (-P) = \{\theta\}. \end{array}$$

For any cone $P \subseteq E$, we can define a partial ordering $\preceq$ on $E$ with respect to $P$ by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of $P$. The cone $P$ is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq k\,\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of $P$. Rezapour and Hamlbarani [13] proved that there are no normal cones with normal constant $k < 1$.

**Definition 6.** *[2] Let* $P$ *be a cone. A nondecreasing mapping* $\varphi : P \to P$ *is called a* $\varphi$*-map if*

$(\varphi_1)$ $\varphi(\theta) = \theta$ and $\theta \prec \varphi(w) \prec w$ for $w \in P \setminus \{\theta\}$,

$(\varphi_2)$ $w - \varphi(w) \in int(P)$ for every $w \in int(P)$,

$(\varphi_3)$ $\lim\limits_{n\to\infty} \varphi^n(w) = \theta$ for every $w \in P \setminus \{\theta\}$.

**Definition 7.** *[17] Let* $P$ *be a cone and let* $(w_n)$ *be a sequence in* $P$. *One says that* $w_n \to \theta$ *if for every* $\epsilon \in P$ *with* $\theta \ll \epsilon$ *there exists* $n_0 \in \mathbb{N}$ *such that* $w_n \ll \epsilon$ *for all* $n \geq n_0$.

*A nondecreasing mapping* $\psi : P \to P$ *is called a* $\psi$*-map if*

$(\psi_1)$ $\psi(w) = \theta$ if and only if $w = \theta$,

$(\psi_2)$ for every $w_n \in P$, $w_n \to \theta$ if and only if $\psi(w_n) \to \theta$,

$(\psi_3)$ for every $w_1, w_2 \in P$, $\psi(w_1 + w_2) \preceq \psi(w_1) + \psi(w_2)$.

**Definition 8.** *[5] Let* $X$ *be a nonempty set. Suppose the mapping* $d : X \times X \to E$ *satisfies*

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

*Then* $d$ *is called a cone metric on* $X$*, and* $(X, d)$ *is called a cone metric space.*

**Definition 9.** *[5] Let* $(X, d)$ *be a cone metric space. Let* $(x_n)$ *be a sequence in* $X$ *and* $x \in X$*. If for every* $c \in E$ *with* $\theta \ll c$ *there is a natural number* $n_0$ *such that for all* $n > n_0$*,* $d(x_n, x) \ll c$*, then* $(x_n)$ *is said to be convergent and* $(x_n)$ *converges to* $x$*, and* $x$ *is the limit of* $(x_n)$*. We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ $(n \to \infty)$.

**Definition 10.** *[5] Let* $(X, d)$ *be a cone metric space,* $(x_n)$ *be a sequence in* $X$*. If for any* $c \in E$ *with* $\theta \ll c$*, there is a natural number* $n_0$ *such that for all* $n, m > n_0$*,* $d(x_n, x_m) \ll c$*, then* $(x_n)$ *is called a Cauchy sequence in* $X$*.*

**Definition 11.** *[5] Let* $(X, d)$ *be a cone metric space, if every Cauchy sequence is convergent in* $X$*, then* $X$ *is called a complete cone metric space.*

**Lemma 1.** *[19] Every cone metric space* $(X, d)$ *is a topological space. For* $c \gg \theta$*,* $c \in E$*,* $x \in X$ *let* $B(x, c) = \{y \in X : d(y, x) \ll c\}$ *and* $\beta = \{B(x, c) : x \in X, c \gg \theta\}$*. Then* $\tau_c = \{U \subseteq X : \forall x \in U, \exists B \in \beta, x \in B \subseteq U\}$ *is a topology on* $X$*.*

**Definition 12.** *[19] Let* $(X, d)$ *be a cone metric space. A map* $T : (X, d) \to (X, d)$ *is called sequentially continuous if* $x_n \in X, x_n \to x$ *implies* $Tx_n \to Tx$*.*

**Lemma 2.** *[19] Let* $(X, d)$ *be a cone metric space, and* $T : (X, d) \to (X, d)$ *be any map. Then,* $T$ *is continuous if and only if* $T$ *is sequentially continuous.*

**Lemma 3.** *[14] Let* $E$ *be a real Banach space with a cone* $P$*. Then*
(i) *If* $a \ll b$ *and* $b \ll c$*, then* $a \ll c$*.*
(ii) *If* $a \preceq b$ *and* $b \ll c$*, then* $a \ll c$*.*

**Lemma 4.** *[5] Let* $E$ *be a real Banach space with cone* $P$*. Then one has the following.*
(i) *If* $\theta \ll c$*, then there exists* $\delta > 0$ *such that* $\|b\| < \delta$ *implies* $b \ll c$*.*
(ii) *If* $a_n$*,* $b_n$ *are sequences in* $E$ *such that* $a_n \to a$*,* $b_n \to b$ *and* $a_n \preceq b_n$ *for all* $n \geq 1$*, then* $a \preceq b$*.*

**Proposition 1.** *[6] If* $E$ *is a real Banach space with cone* $P$ *and if* $a \preceq \lambda a$ *where* $a \in P$ *and* $0 \leq \lambda < 1$ *then* $a = \theta$*.*

## 3 Main Results

In this section we always suppose that $E$ is a real Banach space, $P$ is a cone in $E$ with $\mathrm{int}(P) \neq \emptyset$ and $\preceq$ is the partial ordering on $E$ with respect to $P$. Also, we mean by $\varphi$ the $\varphi$-map and by $\psi$ the $\psi$-map, unless otherwise stated. Now, we state and prove our main results.

**Theorem 1.** *Let* $(X, \sqsubseteq)$ *be a partially ordered set and* $(X, d)$ *be a complete cone metric space. Suppose* $F : X \times X \to X$ *and* $g : X \to X$ *be two continuous and commuting functions with* $F(X \times X) \subseteq g(X)$*. Let* $F$ *satisfy mixed* $g$*-monotone property and*

$$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \preceq \varphi(\psi(d(gx, gu) + d(gy, gv))) \tag{1}$$

*for all* $x, y, u, v \in X$ *with* $(gx \sqsubseteq gu)$ *and* $(gy \sqsupseteq gv)$ *or* $(gx \sqsupseteq gu)$ *and* $(gy \sqsubseteq gv)$*. If there exist* $x_0,\ y_0 \in X$ *satisfying* $gx_0 \sqsubseteq F(x_0, y_0)$ *and* $F(y_0, x_0) \sqsubseteq gy_0$*, then* $F$ *and* $g$ *have a coupled coincidence point.*

*Proof.* Let $x_0,\ y_0$ be such that $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Continuing this process one can construct sequences $(x_n)$ and $(y_n)$ in $X$ such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for all $n \geq 0$. We shall show that

$$gx_n \sqsubseteq gx_{n+1} \text{ and } gy_n \sqsupseteq gy_{n+1} \tag{2}$$

for all $n \geq 0$.
We shall use the mathematical induction. For $n = 0$, (2) follows by the choice of $x_0$ and $y_0$. Suppose now (2) holds for $n = k,\ k \geq 0$. Then $gx_k \sqsubseteq gx_{k+1}$ and $gy_k \sqsupseteq gy_{k+1}$. Mixed $g$-monotonicity of $F$ now implies that

$$gx_{k+1} = F(x_k, y_k) \sqsubseteq F(x_{k+1}, y_k) \sqsubseteq F(x_{k+1}, y_{k+1}) = gx_{k+2}.$$

Similarly, we have $gy_{k+1} \sqsupseteq gy_{k+2}$. Thus (2) follows for $k + 1$. Hence, by the mathematical induction we conclude that (2) holds for $n \geq 0$.

Now for all $n \in \mathbb{N}$,

$$\begin{aligned}
\psi(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})) &= \psi\begin{pmatrix} d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ \\ +d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \end{pmatrix} \\
&\preceq \varphi(\psi(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n))) \\
&\preceq \varphi^2(\psi(d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1}))) \\
&\ . \\
&\ . \\
&\ . \\
&\preceq \varphi^n(\psi(d(gx_0, gx_1) + d(gy_0, gy_1))).
\end{aligned}$$

Let $\epsilon \in int(P)$, then by $(\varphi_2)$, $\ \epsilon_0 = \epsilon - \varphi(\epsilon) \in int(P)$. By $(\varphi_3)$,

$$\lim_{n\to\infty} \varphi^n(\psi(d(gx_0, gx_1) + d(gy_0, gy_1))) = \theta.$$

So, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n_0$,

$$\psi(d(gx_m, gx_{m+1}) + d(gy_m, gy_{m+1})) \ll \epsilon - \varphi(\epsilon).$$

We show that

$$\psi(d(gx_m, gx_{n+1}) + d(gy_m, gy_{n+1})) \ll \epsilon, \tag{3}$$

for a fixed $m \geq n_0$ and $n \geq m$.
Clearly, this holds for $n = m$. We now suppose that (3) holds for some $n \geq m$. Then by using $(\psi_3)$ and condition (1), we obtain

$$\begin{aligned}
\psi(d(gx_m, gx_{n+2}) + d(gy_m, gy_{n+2})) &\preceq \psi\left(\begin{array}{c} d(gx_m, gx_{m+1}) + d(gx_{m+1}, gx_{n+2}) \\ \\ +d(gy_m, gy_{m+1}) + d(gy_{m+1}, gy_{n+2}) \end{array}\right) \\
&\preceq \psi(d(gx_m, gx_{m+1}) + d(gy_m, gy_{m+1})) \\
&\quad +\psi(d(gx_{m+1}, gx_{n+2}) + d(gy_{m+1}, gy_{n+2})) \\
&\preceq \psi(d(gx_m, gx_{m+1}) + d(gy_m, gy_{m+1})) \\
&\quad +\varphi(\psi(d(gx_m, gx_{n+1}) + d(gy_m, gy_{n+1}))) \\
&\ll \epsilon - \varphi(\epsilon) + \varphi(\epsilon) = \epsilon.
\end{aligned}$$

Therefore, by induction (3) holds.
Since $\psi$ is nondecreasing, it follows from (3) that

$$\psi(d(gx_m, gx_{n+1})) \preceq \psi(d(gx_m, gx_{n+1}) + d(gy_m, gy_{n+1})) \ll \epsilon$$

for a fixed $m \geq n_0$ and $n \geq m$.
Similarly,

$$\psi(d(gy_m, gy_{n+1})) \ll \epsilon$$

for a fixed $m \geq n_0$ and $n \geq m$.
Therefore, by using $(\psi_2)$ we deduce that $(gx_n)$ and $(gy_n)$ are Cauchy sequences in $X$. Since $X$ is complete, there exist $x^*$, $y^* \in X$ such that $gx_n \to x^*$ and $gy_n \to y^*$ as $n \to \infty$. By continuity of $g$ we get $\lim_{n\to\infty} ggx_n = gx^*$ and $\lim_{n\to\infty} ggy_n = gy^*$. Commutativity of $F$ and $g$ now implies that

$$ggx_n = g(F(x_{n-1}, y_{n-1})) = F(gx_{n-1}, gy_{n-1})$$

for all $n \in \mathbb{N}$ and

$$ggy_n = g(F(y_{n-1}, x_{n-1})) = F(gy_{n-1}, gx_{n-1})$$

for all $n \in \mathbb{N}$. Since $F$ is continuous,

$$\begin{aligned}
gx^* = \lim_{n\to\infty} ggx_n &= \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1}) \\
&= F(\lim_{n\to\infty} gx_{n-1}, \lim_{n\to\infty} gy_{n-1}) \\
&= F(x^*, y^*)
\end{aligned}$$

and

$$\begin{aligned} gy^* = \lim_{n\to\infty} ggy_n &= \lim_{n\to\infty} F(gy_{n-1}, gx_{n-1}) \\ &= F(\lim_{n\to\infty} gy_{n-1}, \lim_{n\to\infty} gx_{n-1}) \\ &= F(y^*, x^*). \end{aligned}$$

Thus, $F$ and $g$ have a coupled coincidence point. □

If we let $\psi$ be the identity map in Theorem 1, then we have the following Corollary.

**Corolary 1.** *Let* $(X, \sqsubseteq)$ *be a partially ordered set and* $(X, d)$ *be a complete cone metric space. Suppose* $F : X \times X \to X$ *and* $g : X \to X$ *be two continuous and commuting functions with* $F(X \times X) \subseteq g(X)$. *Let* $F$ *satisfy mixed* $g$*-monotone property and*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \preceq \varphi(d(gx, gu) + d(gy, gv))$$

*for all* $x, y, u, v \in X$ *with* $(gx \sqsubseteq gu)$ *and* $(gy \sqsupseteq gv)$ *or* $(gx \sqsupseteq gu)$ *and* $(gy \sqsubseteq gv)$. *If there exist* $x_0$, $y_0 \in X$ *satisfying* $gx_0 \sqsubseteq F(x_0, y_0)$ *and* $F(y_0, x_0) \sqsubseteq gy_0$, *then* $F$ *and* $g$ *have a coupled coincidence point.*

**Corolary 2.** *Let* $(X, \sqsubseteq)$ *be a partially ordered set and* $(X, d)$ *be a complete cone metric space. Suppose* $F : X \times X \to X$ *and* $g : X \to X$ *be two continuous and commuting functions with* $F(X \times X) \subseteq g(X)$. *Let* $F$ *satisfy mixed* $g$*-monotone property and*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \preceq k(d(gx, gu) + d(gy, gv))$$

*for some* $k \in [0, 1)$ *and all* $x, y, u, v \in X$ *with* $(gx \sqsubseteq gu)$ *and* $(gy \sqsupseteq gv)$ *or* $(gx \sqsupseteq gu)$ *and* $(gy \sqsubseteq gv)$. *If there exist* $x_0$, $y_0 \in X$ *satisfying* $gx_0 \sqsubseteq F(x_0, y_0)$ *and* $F(y_0, x_0) \sqsubseteq gy_0$, *then* $F$ *and* $g$ *have a coupled coincidence point.*

*Proof.* The proof can be obtained from Theorem 1 by taking $\psi = I$, the identity map and $\varphi(x) = kx$, where $k \in [0, 1)$ is a constant. □

The following Corollary is a generalization of the result [[3], Theorem 2.1].

**Corolary 3.** *Let* $(X, \sqsubseteq)$ *be a partially ordered set and* $(X, d)$ *be a complete cone metric space. Suppose* $F : X \times X \to X$ *and* $g : X \to X$ *be two continuous and commuting functions with* $F(X \times X) \subseteq g(X)$. *Let* $F$ *satisfy mixed* $g$*-monotone property and*

$$d(F(x, y), F(u, v)) \preceq a d(gx, gu) + b d(gy, gv) \tag{4}$$

*for some* $a$, $b \in [0, 1)$ *with* $a + b < 1$ *and all* $x, y, u, v \in X$ *with* $(gx \sqsubseteq gu)$ *and* $(gy \sqsupseteq gv)$ *or* $(gx \sqsupseteq gu)$ *and* $(gy \sqsubseteq gv)$. *If there exist* $x_0$, $y_0 \in X$ *satisfying* $gx_0 \sqsubseteq F(x_0, y_0)$ *and* $F(y_0, x_0) \sqsubseteq gy_0$, *then* $F$ *and* $g$ *have a coupled coincidence point.*

*Proof.* Let $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. Using (4), we have

$$d(F(x,y), F(u,v)) \preceq ad(gx, gu) + bd(gy, gv)$$

and

$$d(F(y,x), F(v,u)) \preceq ad(gy, gv) + bd(gx, gu).$$

Therefore,

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \preceq (a+b)(d(gx,gu) + d(gy,gv)).$$

The result follows from Corollary 2. □

**Theorem 2.** *Let $(X, \sqsubseteq)$ be a partially ordered set and $(X, d)$ be a cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is a complete subspace of $X$. Let $F$ satisfy mixed $g$-monotone property and*

$$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \preceq \varphi(\psi(d(gx,gu) + d(gy,gv)))$$

*for all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. Suppose $X$ has the following property:*
*(i) if a nondecreasing sequence $(x_n) \to x$, then $x_n \sqsubseteq x$ for all $n$.*
*(ii) if a nonincreasing sequence $(y_n) \to y$, then $y \sqsubseteq y_n$ for all $n$.*
*If there exist $x_0,\ y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then $F$ and $g$ have a coupled coincidence point.*

*Proof.* Consider Cauchy sequences $(gx_n)$ and $(gy_n)$ as in the proof of Theorem 1. Since $(g(X), d)$ is complete, there exist $x^*, y^* \in X$ such that $gx_n \to gx^*$ and $gy_n \to gy^*$. It is to be noted that the sequence $(gx_n)$ is nondecreasing and converges to $gx^*$. By given condition $(i)$ we have, therefore, $gx_n \sqsubseteq gx^*$ for all $n \geq 0$ and similarly $gy_n \sqsupseteq gy^*$ for all $n \geq 0$.
By $(\psi_2)$, for $\theta \ll c$, one can choose a natural number $n_0$ such that $\psi(d(gx_n, gx^*)) \ll \frac{c}{4}$ and $\psi(d(gy_n, gy^*)) \ll \frac{c}{4}$ for all $n \geq n_0$.

Then,

$$
\begin{aligned}
\psi\begin{pmatrix} d(F(x^*,y^*),gx^*) \\ \\ +d(F(y^*,x^*),gy^*) \end{pmatrix} &\preceq \psi\begin{pmatrix} d(F(x^*,y^*),gx_{n+1})+d(gx_{n+1},gx^*) \\ \\ +d(F(y^*,x^*),gy_{n+1})+d(gy_{n+1},gy^*) \end{pmatrix} \\
&\preceq \psi(d(gx_{n+1},gx^*)+d(gy_{n+1},gy^*)) \\
&\quad +\psi\begin{pmatrix} d(F(x^*,y^*),F(x_n,y_n)) \\ \\ +d(F(y^*,x^*),F(y_n,x_n)) \end{pmatrix} \\
&\preceq \psi(d(gx_{n+1},gx^*))+\psi(d(gy_{n+1},gy^*)) \\
&\quad +\varphi(\psi(d(gx_n,gx^*)+d(gy_n,gy^*))) \\
&\prec \psi(d(gx_{n+1},gx^*))+\psi(d(gy_{n+1},gy^*)) \\
&\quad +\psi(d(gx_n,gx^*)+d(gy_n,gy^*)) \\
&\preceq \psi(d(gx_{n+1},gx^*))+\psi(d(gy_{n+1},gy^*)) \\
&\quad +\psi(d(gx_n,gx^*))+\psi(d(gy_n,gy^*)) \\
&\ll \frac{c}{4}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4}=c.
\end{aligned}
$$

So, $\frac{c}{i}-\psi(d(F(x^*,y^*),gx^*)+d(F(y^*,x^*),gy^*))\in P$, for all $i\geq 1$. Since $\frac{c}{i}\to\theta$ as $i\to\infty$ and $P$ is closed, $-\psi(d(F(x^*,y^*),gx^*)+d(F(y^*,x^*),gy^*))\in P$. But $P\cap(-P)=\theta$ gives that

$$\psi(d(F(x^*,y^*),gx^*)+d(F(y^*,x^*),gy^*))=\theta.$$

By $(\psi_1)$, we get

$$d(F(x^*,y^*),gx^*)+d(F(y^*,x^*),gy^*)=\theta.$$

This shows that $d(F(x^*,y^*),gx^*)=d(F(y^*,x^*),gy^*)=\theta$ and so $F(x^*,y^*)=gx^*$, $F(y^*,x^*)=gy^*$. Thus, $F$ and $g$ have a coupled coincidence point. □

If we let $\psi$ be the identity map in Theorem 2, then we have the following Corollary.

**Corolary 4.** *Let $(X,\sqsubseteq)$ be a partially ordered set and $(X,d)$ be a cone metric space. Suppose $F:X\times X\to X$ and $g:X\to X$ be two functions such that $F(X\times X)\subseteq g(X)$ and $(g(X),d)$ is a complete subspace of $X$. Let $F$ satisfy mixed $g$-monotone property and*

$$d(F(x,y),F(u,v))+d(F(y,x),F(v,u))\preceq\varphi(d(gx,gu)+d(gy,gv))$$

*for all $x,y,u,v\in X$ with $(gx\sqsubseteq gu)$ and $(gy\sqsupseteq gv)$ or $(gx\sqsupseteq gu)$ and $(gy\sqsubseteq gv)$. Suppose $X$ has the following property:*
*(i) if a nondecreasing sequence $(x_n)\to x$, then $x_n\sqsubseteq x$ for all $n$.*
*(ii) if a nonincreasing sequence $(y_n)\to y$, then $y\sqsubseteq y_n$ for all $n$.*
*If there exist $x_0$, $y_0\in X$ satisfying $gx_0\sqsubseteq F(x_0,y_0)$ and $F(y_0,x_0)\sqsubseteq gy_0$, then $F$ and $g$ have a coupled coincidence point.*

**Corolary 5.** *Let* $(X, \sqsubseteq)$ *be a partially ordered set and* $(X, d)$ *be a cone metric space. Suppose* $F : X \times X \to X$ *and* $g : X \to X$ *be two functions such that* $F(X \times X) \subseteq g(X)$ *and* $(g(X), d)$ *is a complete subspace of* $X$. *Let* $F$ *satisfy mixed* $g$*-monotone property and*

$$d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) \preceq k(d(gx,gu) + d(gy,gv))$$

*for some* $k \in [0,1)$ *and all* $x, y, u, v \in X$ *with* $(gx \sqsubseteq gu)$ *and* $(gy \sqsupseteq gv)$ *or* $(gx \sqsupseteq gu)$ *and* $(gy \sqsubseteq gv)$. *Suppose* $X$ *has the following property:*
*(i) if a nondecreasing sequence* $(x_n) \to x$, *then* $x_n \sqsubseteq x$ *for all* $n$.
*(ii) if a nonincreasing sequence* $(y_n) \to y$, *then* $y \sqsubseteq y_n$ *for all* $n$.
*If there exist* $x_0$, $y_0 \in X$ *satisfying* $gx_0 \sqsubseteq F(x_0, y_0)$ *and* $F(y_0, x_0) \sqsubseteq gy_0$, *then* $F$ *and* $g$ *have a coupled coincidence point.*

*Proof.* The proof can be obtained from Theorem 2 by taking $\psi = I$, the identity map and $\varphi(x) = kx$, where $k \in [0,1)$ is a constant. □

The following Corollary is a generalization of the result [[3], Theorem 2.2].

**Corolary 6.** *Let* $(X, \sqsubseteq)$ *be a partially ordered set and* $(X, d)$ *be a cone metric space. Suppose* $F : X \times X \to X$ *and* $g : X \to X$ *be two functions such that* $F(X \times X) \subseteq g(X)$ *and* $(g(X), d)$ *is a complete subspace of* $X$. *Let* $F$ *satisfy mixed* $g$*-monotone property and*

$$d(F(x,y),F(u,v)) \preceq a d(gx,gu) + b d(gy,gv)$$

*for some* $a$, $b \in [0,1)$ *with* $a + b < 1$ *and all* $x, y, u, v \in X$ *with* $(gx \sqsubseteq gu)$ *and* $(gy \sqsupseteq gv)$ *or* $(gx \sqsupseteq gu)$ *and* $(gy \sqsubseteq gv)$. *Suppose* $X$ *has the following property:*
*(i) if a nondecreasing sequence* $(x_n) \to x$, *then* $x_n \sqsubseteq x$ *for all* $n$.
*(ii) if a nonincreasing sequence* $(y_n) \to y$, *then* $y \sqsubseteq y_n$ *for all* $n$.
*If there exist* $x_0$, $y_0 \in X$ *satisfying* $gx_0 \sqsubseteq F(x_0, y_0)$ *and* $F(y_0, x_0) \sqsubseteq gy_0$, *then* $F$ *and* $g$ *have a coupled coincidence point.*

*Proof.* The proof follows from Theorem 2 by an argument similar to that used in Corollary 3. □

**Theorem 3.** *In addition to hypothesis of either Theorem 1 or Theorem 2, suppose that any two elements of* $g(X)$ *are comparable and* $g$ *is one-one. Then* $F$ *and* $g$ *have a coupled coincidence point of the form* $(x^*, x^*)$ *for some* $x^* \in X$.

*Proof.* We first note that the set of coupled coincidence points of $F$ and $g$ is nonempty. We will show that if $(x^*, y^*)$ is a coupled coincidence point of $F$ and $g$, then $x^* = y^*$. Since the elements of $g(X)$ are comparable, we may assume that $gx^* \sqsubseteq gy^*$. Suppose that $d(gx^*, gy^*) \neq \theta$. Then, by using $(\varphi_1)$ we have

$$\begin{aligned}
\psi(d(gx^*,gy^*) + d(gy^*,gx^*)) &= \psi(d(F(x^*,y^*),F(y^*,x^*)) + d(F(y^*,x^*),F(x^*,y^*))) \\
&\preceq \varphi(\psi(d(gx^*,gy^*) + d(gy^*,gx^*))) \\
&\prec \psi(d(gx^*,gy^*) + d(gy^*,gx^*)),
\end{aligned}$$

a contradiction. Therefore, $d(gx^*, gy^*) = \theta$ which gives that $gx^* = gy^*$. Since $g$ is one-one, it follows that $x^* = y^*$. □

We conclude with an example.

**Example 1.** *Let* $E = \mathbb{R}^2$*, the Euclidean plane and* $P = \{(x, x) \in \mathbb{R}^2 : x \geq 0\}$ *a cone in* $E$*. Let* $X = [0, \infty)$ *with the usual ordering and define* $d : X \times X \to E$ *by*

$$d(x, y) = (|\, x - y\,|, |\, x - y\,|)$$

*for all* $x, y \in X$*. Then* $(X, d)$ *is a partially ordered complete cone metric space. Define* $F : X \times X \to X$ *as follows:*

$$F(x, y) = \begin{cases} \frac{x-y}{6}, & \text{if } x \geq y \\ \\ 0, & \text{if } x < y, \end{cases}$$

*for all* $x, y \in X$ *and* $g : X \to X$ *with* $gx = \frac{x}{3}$ *for all* $x \in X$*. Then* $F(X \times X) \subseteq g(X) = X$ *and* $F$ *satisfy mixed* $g$*-monotone property. Also* $F$ *and* $g$ *are continuous and commuting,* $g(0) \leq F(0, 1)$ *and* $g(1) \geq F(1, 0)$*.*
*Let* $\psi, \varphi : P \to P$ *be defined by* $\psi(x, x) = (\frac{x}{2}, \frac{x}{2})$ *and* $\varphi(x, x) = (\frac{3x}{4}, \frac{3x}{4})$*.*
*Let* $x, y, u, v \in X$ *be such that* $gx \leq gu$ *and* $gy \geq gv$*. Now, we have*
**Case-I** *(*$y > x$ *and* $v > u$*). Then*

$$\begin{aligned} &\psi(d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))) \\ &\quad= \psi\left(d(0, 0) + d\left(\frac{y-x}{6}, \frac{v-u}{6}\right)\right) \\ &\quad= \psi\left(\frac{|\, y - x - v + u\,|}{6}, \frac{|\, y - x - v + u\,|}{6}\right) \\ &\quad= \left(\frac{|\, y - x - v + u\,|}{12}, \frac{|\, y - x - v + u\,|}{12}\right) \\ &\quad\preceq \left(\frac{|\, x - u\,|}{12} + \frac{|\, y - v\,|}{12}, \frac{|\, x - u\,|}{12} + \frac{|\, y - v\,|}{12}\right). \end{aligned} \tag{5}$$

*Again,*

$$\begin{aligned} &\varphi(\psi(d(gx, gu) + d(gy, gv))) = \varphi\left(\psi\left(d\left(\tfrac{x}{3}, \tfrac{u}{3}\right) + d\left(\tfrac{y}{3}, \tfrac{v}{3}\right)\right)\right) \\ &\quad= \varphi\left(\psi\left(\left(\frac{|\, x - u\,|}{3}, \frac{|\, x - u\,|}{3}\right) + \left(\frac{|\, y - v\,|}{3}, \frac{|\, y - v\,|}{3}\right)\right)\right) \\ &\quad= \left(3\frac{|\, x - u\,|}{24} + 3\frac{|\, y - v\,|}{24}, 3\frac{|\, x - u\,|}{24} + 3\frac{|\, y - v\,|}{24}\right). \end{aligned} \tag{6}$$

*It follows from conditions (5) and (6) that*

$$\psi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u)))\preceq\varphi(\psi(d(gx,gu)+d(gy,gv))).$$

**Case-II** *($y>x$ and $u\geq v$). Then*

$$\begin{aligned}
&\psi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u)))\\
&= \psi\left(d\left(0,\frac{u-v}{6}\right)+d\left(\frac{y-x}{6},0\right)\right)\\
&= \psi\left(\left(\frac{u-v}{6},\frac{u-v}{6}\right)+\left(\frac{y-x}{6},\frac{y-x}{6}\right)\right)\\
&= \left(\frac{u-v+y-x}{12},\frac{u-v+y-x}{12}\right)\\
&\preceq \left(\frac{|x-u|}{12}+\frac{|y-v|}{12},\frac{|x-u|}{12}+\frac{|y-v|}{12}\right)\\
&\prec \left(3\frac{|x-u|}{24}+3\frac{|y-v|}{24},3\frac{|x-u|}{24}+3\frac{|y-v|}{24}\right)\\
&= \varphi(\psi(d(gx,gu)+d(gy,gv))).
\end{aligned}$$

**Case-III** *($x\geq y$ and $u\geq v$). Then*

$$\begin{aligned}
&\psi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u)))\\
&= \psi\left(d\left(\frac{x-y}{6},\frac{u-v}{6}\right)+d(0,0)\right)\\
&= \left(\frac{|x-y-u+v|}{12},\frac{|x-y-u+v|}{12}\right)\\
&\preceq \left(\frac{|x-u|}{12}+\frac{|y-v|}{12},\frac{|x-u|}{12}+\frac{|y-v|}{12}\right)\\
&\preceq \left(3\frac{|x-u|}{24}+3\frac{|y-v|}{24},3\frac{|x-u|}{24}+3\frac{|y-v|}{24}\right)\\
&= \varphi(\psi(d(gx,gu)+d(gy,gv))).
\end{aligned}$$

*The case $x\geq y$ and $v>u$ is not possible. As $gx\leq gu$ and $gy\geq gv$, it follows that $x\leq u$ and $y\geq v$. So, $v\leq y\leq x\leq u$ when $x\geq y$. Thus, we have all the conditions of Theorem 1. Moreover, $(0,0)$ is the coupled coincidence point of $F$ and $g$.*

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