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Spacetime singularity, singular bounds and compactness for solutions of the Poisson's equation

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ABSTRACT

A black hole is a spacetime region in whose interior lies a structure known as a spacetime singularity whose scientific description is profoundly elusive, and which depends upon the still missing theory of quantum gravity. Using the classical weak comparison principle we are able to obtain new bounds, compactness results and concentration phenomena in the theory of Newtonian potentials of distributions with compact support which gives a suitable mathematical theory of spacetime singularity. We derive a rigorous renormalization of the Newtonian gravity law using nonlinear functional analysis and we have a solid set of astronomical observations supporting our new equation. This general setting introduces a new kind of ill posed problem with a very simple physical interpretation.

RESUMEN

Un hoyo negro es una región espacio-temporal en cuyo interior hay una estructura llamada singularidad espacio-temporal cuya descripción científica es difícil de encontrar, y que depende de la aún inexistente teoría de la gravedad cuántica. Usando el clásico principio de comparación débil, aquí probamos nuevas cotas, resultados de compacidad y fenómenos de concentración en la teoría de potenciales Newtonianos de distribuciones de soporte compacto, que dan una teoría matemática adecuada de la singularidad espacio-temporal. Derivamos una rigurosa renormalización de la ley de gravitación Newtoniana usando análisis funcional no lineal y tenemos un contundente conjunto de datos de observaciones astronómicas que apoyan nuestra nueva ecuación. Este marco general introduce una nueva forma de problema mal-puesto con una interpretación física muy simple.

Keywords and Phrases: Black hole, spacetime singularity, quantum field theory, Newtonian potentials, elliptic equations, compact imbedding, Sobolev's spaces.

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1 Introduction.

In [2, 3], the authors introduced a new concentration phenomena for the Poisson's equation using techniques from nonlinear functional analysis. In this article we are concerned with several simple consequences of this new concentration of compactness results. Using the classical theory of Newtonian potentials of distributions with compact support we are able to derive concentration of compactness for Newtonian potentials with singular behaviour. For a review of this topic see [10].

Newtonian potentials are useful in the description of gravity fields of celestial bodies [5, 20, 21]. Today black holes in gravity theory and astronomy plays a central role [8, 12, 15, 27, 30, 31, 33, 40, 46]. The interior of a black hole is usually called spacetime singularity [7, 31]. In [3] we obtain the existence of a sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty} \in C^2(\Omega)$ for any Ω bounded domain in $\mathbb{R}^N$ such that $-\lim_{j\to\infty} \Delta\mathfrak{P}_j = \infty$ uniformly on Ω and $0 < \mathfrak{P}_j(x) \leq \mathfrak{P}_{j+1}(x) \leq Cte$ for all $x \in \Omega$. This sequence proof that it is possible to do rigorous treatment of divergence to infinite in the frame of Newtonian potentials extending rigorous quantum field theories on large scale [37, 38]. Black holes are complicated real objects, and our Newtonian equation is a mathematical object but we have astronomical observations of supermassive black holes given a concrete physical significance to this new theoretical frame [27]. This is a remarkable fact in gravity theory [13, 41].

Lemma 1.1 (Lemma 1 page 277 [11]). *Let Ω be an open set of $\mathbb{R}^N$, $f \in \mathfrak{D}'(\Omega)$ and u a solution (in the sense of distribution) of Poisson's equation $\Delta u = f$ on Ω. Then for every bounded open set Ω_1 with $\overline{\Omega}_1 \subset \Omega$ there exists $f_1 \in \mathscr{E}'$ the space of distributions on $\mathbb{R}^N$ with compact support, such that*

$$f_1 = f \textit{ on } \Omega, \; u = \textit{ the Newtonian potential of } f_1 \textit{ on } \Omega_1. \tag{1}$$

Therefore for $\Omega_1 \subset \Omega$ and our sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ there exists a sequence $\{f_{1,j}\}_{j=1}^{\infty} \in \mathscr{E}'$ the space of distributions on $\mathbb{R}^N$ with compact support, such that

$$f_{1,j} = \Delta\mathfrak{P}_j \text{ on } \Omega, \; \mathfrak{P}_j = \text{ the Newtonian potential of } f_{1,j} \text{ on } \Omega_1. \tag{2}$$

We have associated to each pair $\{\mathfrak{P}_j, \; \Omega_1\}$ a gravitational Newtonian potential defined on all $\mathbb{R}^N$. Using this lemma we have a very simple Newtonian interpretation of the interior of a black hole. We have a set of solid astronomic observations supporting our new equation.

Black hole name	Solar mass (sun=1)	References
Holmberg 15A	170.000.000.000	[23]
S5 0014+813	40.000.000.000	[16]
SDSS J085543.40-001517.7	25.000.000.000	[47]
APM 08279+5255	23.000.000.000	[35]
NGC 4889	21.000.000.000	[25]
Central black hole of Phoenix cluster	20.000.000.000	[26]
SDSS J07451.78+734336.1	19.500.000.000	[47]
OJ 285 primary	18.000.000.000	[39]
SDSS J08019.69+373047.3	15.140.000.000	[47]
SDSS J115954.33+20192.1	14.120.000.000	[47]
SDSS J075303.34+423130.8	13.500.000.000	[47]
SDSS J081855.77+095848.0	12.000.000.000	[47]
SDSS J0100+2802	12.000.000.000	[44]
SDSS J082535.19+512706.3	11.220.000.000	[47]
SDSS J013127.34-0321000.1	11.000.000.000	$\diamondsuit$
Central black hole of RX J1532.9+3021	10.000.000.000	$\diamondsuit$
QSO B2126-158	10.000.000.000	[16]
SDSS J015741.57-010629.6	9.800.000.00	[47]
NGC 3842	9.700.000.000	[25]
SDSS J2330301.45-093930.7	9.120.000.000	[47]
SDSS J075819.70+202300.9	7.800.000.000	[47]
SDSS J080956.02+50200.9	6.450.000.000	[47]
SDSS J0142114.75+0023224.2	6.310.000.000	[47]
Messier 87	6.300.000.000	$\diamondsuit$
QSO B0746+254	5.000.000.000	[16]
QSO B2149-306	5.000.000.000	[16]
NGC 1277	5.000.000.000	$\diamondsuit$
SDSS J090033.50+421547.0	4.700.000.000	[47]
Messier 60	4.500.000.000	$\diamondsuit$
SDSS J011521.20+152453.3	4.100.000.000	[47]
QSO B0222+185	4.000.000.000	[16]
Hercules A (3C 348)	4.000.000.000	$\diamondsuit$
SDSS J213023.61+122252.0	3.500.000.000	[47]

J173352.23+540030.4	3.400.000.000	[47]
SDSS J025021.76-075749.9	3.100.000.000	[47]
SDSS J030341.04-002321.9	3.000.000.000	[47]
QSO B0836+710	3.000.000.000	[16]
SDSS J224956.08+000218.0	2.630.000.000	[47]
SDSS J030449.85-000813.4	2.400.000.000	[47]
SDSS J234625.66-001600.4	2.240.000.000	[47]
ULAS J1120+0641	2.000.000.000	♢
QSO 0537-286	2.000.000.000	[16]
NGC 3115	2.000.000.000	♢
Q0906+6930	2.000.000.000	♢
QSO B0805+614	1.500.000.000	[16]
Messier 84	1.500.000.000	♢
QSO B225155+2217	1.000.000.000	[16]
QSO B1210+330	1.000.000.000	[16]
NGC 6166	1.000.000.000	♢
Cygnus A	1.000.000.000	♢
Sombrero Galaxy	1.000.000.000	♢
Markarian 501	900.000.000-3.400.000.000	♢
PG 1426+015	467.740.000	[28]
3C 273	550.000.000	[28]
Messier 49	560.000.000	♢
PG 0804+761	190.550.000	[28]
PG 1617+175	275.420.000	[28]
PG 1700 + 518	60.260.000	[28]
NGC 4261	400.000.000	♢
NGC 1275	340.000.000	♢
3C 390.3	338.840.000	[28]
II Zwicky 136	144.540.000	[28]
PG 0052+251	218.780.000	[28]
Messier 59	270.000.000	♢
PG 1411+442	79.430.000	[28]
Markarian 876	240.000.000	[28]
Andromeda Galaxy	230.000.000	♢
PG 0953+414	182.000.000	[28]
PG 0026+129	53.700.000	[28]
Fairall 9	79.430.000	[28]

Markarian 1095	182.000.000	[28]
Messier 105	200.000.000	♢
Markarian 509	57.550.000	[28]
OJ 287 secondary	100.000.000	[39]
RX J124236.9-1111935	100.000.000	♢
Messier 85	100.000.000	♢
NGC 5548	123.000.000	♢
PG 1221+143	40.740.000	[28]
Messier 88	80.000.000	♢
Messier 81	70.000.000	♢
Markarian 771	75.860.000	[28]
Messier 58	70.000.000	♢
PG 0844+349	21.380.000	[28]
Centaurus A	55.000.000	♢
Markarian 79	52.500.000	[28]
Messier 96	48.000.000	♢
Markarian 817	43.650.000	[28]
NGC 3227	38.900.000	[28]
NGC 4151 primary	40.000.000	♢
3C 120	22.900.000	♢
Markarian 279	41.700.000	♢
NGC 3516	23.000.000	♢
NGC 863	17.700.00	♢
Messier 82	30.000.000	♢
Messier 108	24.000.000	♢
NGC 3783	9.300.000	♢
Markarian 110	5.620.000	♢
Markarian 335	6.310.000	♢
NGC 4151 secondary	10.000.000	♢
NGC 7469	6.460.000	♢
IC 4329 A	5.010.000	♢
NGC 4593	8.130.000	♢
Messier 61	5.000.000	♢
Messier 32	1.500.000-5.000.000	♢
Sagittarius A*	4.100.000	♢
NGC 4051	1.300.000	♢

♢ Only main

references are provided.

Our function $\mathfrak{P} = \lim_{j\to\infty} \mathfrak{P}_j$ is very 'rough' from the point of view of regularity on Sobolev's spaces in opposition of the Poisson's equation $\Delta u = 0$ on Ω.

Theorem 1.2 (Weyl page 118[10])**.** *Let $\Omega \subset \mathbb{R}^N$ be open and $u \in L^1_{loc}(\Omega)$ satisfy $\int_\Omega u\Delta v = 0$ for all $v \in C_0^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$.*

A simpler use of Green's identity allow us to imply the discontinuity at infinitum of the functional $\mathfrak{G} : C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \to \mathbb{R}$,

$$\mathfrak{G}(u,v) = \int_\Omega (v\Delta u - u\Delta v)\, dx - \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\gamma, \tag{3}$$

where Ω is a bounded domain with C^1 boundary $\partial\Omega$. The functional $\mathfrak{G}$ for a fixed pair $(u,v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ satisfies the Green's identity $\mathfrak{G}(u,v) = 0$ (for a proof of Green's formula see page 20 [29]). This equality is a consequence of the divergence Theorem, (for a proof of the divergence Theorem stated by E. Heinz see page 46 [36], also page 570 [11]). Let us to remember several classical results:

Theorem 1.3. *[Theorem 6.6 [18]] Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^N$ and let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of the equation*

$$Lu = \sum_{i,j=1}^N a_{ij} u_{x_i,x_j} + \sum_{i=1}^N b_i u_{x_i} + cu = f, \tag{4}$$

where $f \in C^\alpha(\overline{\Omega})$ and the coefficients of L satisfy, for positive constants λ, Λ

$$\sum_{i,j=1}^N a_{ij}\xi_i\xi_j \geq \lambda \mid \xi \mid^2, \text{ for all } x \in \Omega,\ \xi \in \mathbb{R}^N, \tag{5}$$

$$\mid a_{i,j} \mid_{0,\alpha;\Omega},\ \mid b_i \mid_{0,\alpha;\Omega}, \mid c_i \mid_{0,\alpha;\Omega} \leq \Lambda. \tag{6}$$

Let $\varphi \in C^{2,\alpha}(\overline{\Omega})$ and suppose $u = \varphi$ on $\partial\Omega$. Then

$$\mid u \mid_{2,\alpha;\Omega} \leq C\{\mid u \mid_{0,\Omega} + \mid \varphi \mid_{2,\alpha;\Omega} + \mid f \mid_{0,\alpha;\Omega}\}, \tag{7}$$

where $C = C(N, \alpha, \lambda, \Lambda, \Omega)$.

$$Lu = D_i\left(a^{ij}(x)D_j u + b^i(x)u\right) + c^i(x)D_i u + d(x)u, \tag{8}$$

$$L^*u = D_i\left(a^{ij}D_j u - c^i u\right) - b^i D_i u + du, \tag{9}$$

$$a^{ij}(x)\xi_i\xi_j \geq \lambda \mid \xi \mid^2, \text{ for all } x \in \Omega,\ \xi \in \mathbb{R}^N, \tag{10}$$

$$\sum \mid a^{ij}(x) \mid^2 \leq \Lambda^2, \tag{11}$$

$$\lambda^{-2}\sum\left(\mid b^i(x) \mid^2 + \mid c^i(x) \mid^2\right) + \lambda^{-1} \mid d(x) \mid \leq \mathfrak{v}^2, \tag{12}$$

Theorem 1.4. *[Theorem 9.11 [18]] Let Ω be an open set in $\mathbb{R}^N$ and $W^{2,p}(\Omega)\cap L^p(\Omega)$, $1 < p < \infty$, a strong solution of the equation $Lu = f$ in Ω where the coefficients of L satisfy for positive constants $\lambda,\ \Lambda$*

$$a^{ij} \in C^0(\Omega),\ \ b^i, c \in L^\infty(\Omega),\ \ f \in L^p(\Omega); \tag{13}$$

$$a^{ij}\xi_i\xi_j \geq \lambda \mid \xi \mid^2 \ \ \textit{for all } \xi \in \mathbb{R}^N, \tag{14}$$

$$\mid a^{i,j} \mid,\ \mid b^i \mid,\ \mid c \mid \leq \Lambda, \tag{15}$$

where $i, j = 1, \ldots, N$. Then for any domain $\Omega' \subset\subset \Omega$,

$$\| u \|_{2,p;\Omega'} \leq C\left(\| u \|_{p;\Omega} + \| f \|_{p;\Omega}\right), \tag{16}$$

where C depends on $N, p, \lambda, \Lambda, \Omega', \Omega$ and the moduli of continuity of the coefficients a^{ij} on Ω'.

Now the existence of our concentrating sequence of non negative, bounded functions $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ is a compactness property not established and noncontradictory with the statements in Theorems 1.3 and 1.4. This is a singular property that using very elementary techniques, allow us to obtain new bounds and several interesting results related to the Newtonian potential in the theory of distributions [11]. Moreover we built a sequence $\mathfrak{F}_j : [a, b] \to [0, \infty)$ satisfying $\lim_{j\to\infty} \mathfrak{F}_j''(x) = -\infty$ in measure, $Cte \geq \mathfrak{F}_{j+1} \geq \mathfrak{F}_j \geq 0$ and each $\mathfrak{F}_j$ non increasing on $[a, b]$. By considering $\mathfrak{F}_j(x_1, x_2, \ldots, x_N) = \mathfrak{F}_j(x_1)$ we obtain a sequence for bounded smooth domains in $\mathbb{R}^N$, $N \geq 2$ with similar properties.

2 Preliminaries.

The results of this section are contained in [3]. For the sake of the readability, we stated and prove it here again.

Lemma 2.1. *Let $B(0, R)$ be a ball of radius $R > 0$ in $\mathbb{R}^N$, $N > 2$. Consider the singular nonlinear elliptic equation*

$$\begin{aligned} -\Delta\mathfrak{P}_j &= \mathfrak{H}_j(\mathfrak{P}_j) && \textit{in } B(0, R) \\ \mathfrak{P}_j &= 0 && \textit{on } \partial B(0, R). \end{aligned} \tag{17}$$

where $\mathfrak{H}_j : (0, \infty) \to (0, \infty)$ is locally Hölder continuous function

$$\mathfrak{H}_j(s) = \begin{cases} s^{-j} & \textit{if } 0 < s < 1, \\ s^{-1} & \textit{if } s \geq 1. \end{cases}$$

Then the next properties holds:
(i) The sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty} \in C^2(B(0, R)) \cap C(\overline{B(0, R)})$ are radial functions with $\frac{\partial\mathfrak{P}}{\partial r} < 0$.
(ii) The sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ satisfies $\mathfrak{P}_j \leq \mathfrak{P}_{j+1}$.
(iii) The sequence $\{\mathfrak{H}_j(\mathfrak{P}_j)\}_{j=1}^{\infty}$ satisfies $\mathfrak{H}_j(\mathfrak{P}_j) \leq \mathfrak{H}_{j+1}(\mathfrak{P}_{j+1})$.
(iv) The sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ satisfies $w \leq \mathfrak{P}_j \leq e$, where $-\Delta\nu = \nu^{-1}$ in $B(0, R)$, $\nu = 0$ on $\partial B(0, R)$,

$-\Delta e = e^{-1}$ *in* $B(0,R)$, $e = 1$ *on* $\partial B(0,R)$ *and* $-\Delta w = e^{-1}$ *in* $B(0,R)$, $w = 0$ *on* $\partial B(0,R)$.

Proof. The enunciate (i) is a consequence of classical results on radial symmetry. The points (ii) and (iii) have been stated at [2, 3] and the point (iv) is proved in [1]. $\square$

Theorem 2.2 ([3]). *Let* $B(0,R) \subset \mathbb{R}^N$, *a ball of radius* R, *with* $N \geq 3$. *Then there exists a sequence of radial, nonnegative and bounded functions* $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ *and* $0 \leq R_0 < R$ *such that*

$$-\lim_{j\to\infty} \Delta\mathfrak{P}_j = \infty \text{ on } A(R_0,R), \tag{18}$$

where $A(R_0,R)$ *is the annulus of external radius* R *and internal radius* R_0. *Moreover* $\mathfrak{P}_j \in C^{\infty}(A(R_0,R))$ *and* $\mathfrak{P}_j \leq \mathfrak{P}_{j+1}$.

Proof. Our proof is a reductio ad absurdum procedure. Let us to consider

$$\mathfrak{P} = \lim_{j\to\infty} \mathfrak{P}_j. \tag{19}$$

Now if there exists no sequence such that it is stated in our Theorem 2.2, then

$$\lim_{r\nearrow R} \mathfrak{P}(r) \geq 1, \tag{20}$$

because if $\lim_{r\nearrow R} \mathfrak{P}(r) < 1$ for a all nonnegative and small enough ϵ, there exist $\delta > 0$ such that $\mathfrak{P}(r) \leq 1-\epsilon$ for all $r \in (R-\delta,R)$. Therefore $\mathfrak{P}_j(r) \leq \mathfrak{P}(r) \leq 1-\epsilon$ for all $r \in (R-\delta,R)$ and

$$\begin{aligned} -\lim_{j\to\infty} \Delta\mathfrak{P}_j &= \lim_{j\to\infty} \mathfrak{H}_j(\mathfrak{P}_j) \\ &= \lim_{j\to}(\mathfrak{P}_j)^{-j} \\ &\geq \lim_{j\to\infty}(1-\epsilon)^{-j} \\ &= \infty \text{ on } A(R-\delta,R). \end{aligned} \tag{21}$$

Similarly if there exists no sequence satisfying our Theorem 2.2 then

$$\lim_{j\to\infty} \mathfrak{H}_j(\mathfrak{P}_j(r)) < \infty \text{ for all } r \in [0,R), \tag{22}$$

because if

$$\lim_{j\to\infty} \mathfrak{H}_j(\mathfrak{P}_j(r)) = \infty \text{ for } r_0 \in [0,R), \tag{23}$$

from

$$\mathfrak{H}_j(\mathfrak{P}_j(r)) \geq \mathfrak{H}_j(\mathfrak{P}_j(r_0)) \text{ for all } \mid x \mid = r \in [r_0,R), \tag{24}$$

we deduce

$$\begin{aligned} -\lim_{j\to\infty} \Delta\mathfrak{P}_j &= \lim_{j\to\infty} \mathfrak{H}_j(\mathfrak{P}_j) \\ &\geq \lim_{j\to\infty} \mathfrak{H}_j(\mathfrak{P}_j(r_0)) = \infty \text{ on } A(r_0,R). \end{aligned} \tag{25}$$

Contradiction so 22 holds and we obtain that $\mathfrak{P} \in C^1(B(0,R))$, because using Theorem 9.11 page 235 in [18] we derive

$$\begin{aligned} \| \mathfrak{P}_j \|_{H^{2,p}(\Omega'')} &\leq C(N,p,\Omega',\Omega'') \left\{ \| \mathfrak{P}_j \|_{L^p(\Omega')} + \| \mathfrak{H}_j(\mathfrak{P}_j) \|_{L^p(\Omega')} \right\} \\ &\leq C(N,p,\Omega',\Omega'') \left\{ \| e \|_{L^p(\Omega')} + \| \lim_{j\to\infty} \mathfrak{H}_j(\mathfrak{P}_j(r_3)) \|_{L^p(\Omega')} \right\} \end{aligned} \tag{26}$$

where $\overline{\Omega'} \subset \Omega''$, $\overline{\Omega''} \subset B(0,R)$, $p > N$ and $r_3 = \sup_{x\in\Omega'} | x |$. According to Theorem 7.26 page 171 [18], we have the bounds

$$\| \mathfrak{P}_j \|_{C^{1,1-\frac{N}{p}}\overline{(\Omega'')}} \leq \| \mathfrak{P}_j \|_{H^{2,p}(\Omega'')} . \tag{27}$$

Therefore $\mathfrak{P} \in C^1(B(0,R))$ and $\mathfrak{P}_j \to \mathfrak{P}$ in $C^{1,\alpha}_{loc}(B(0,R)$ for $0 < \alpha < 1 - \frac{N}{p}$.
One more time if there exist no sequence satisfying the conclusions of Theorem 2.2 we imply $\mathfrak{P} : [0,R) \to [0,\infty)$ is a strictly nonincreasing radial function because if there exist $0 \leq r_1 < r_2 < R$ with $\mathfrak{P}(r_1) = \mathfrak{P}(r_2)$ and the fact of being $\mathfrak{P}$ nonincreasing implies $-\Delta\mathfrak{P} = 0$ on the annulus $A(r_1,r_2)$. Using $\| \mathfrak{P}_j \|_{C^{1,\alpha}(A(r_1,r_2)} \leq C$ and a nonnegative test function φ with support contained in $A(r_1,r_2)$:

$$\begin{aligned} 0 &= \int_{A(r_1,r_2)} \nabla\mathfrak{P} \cdot \nabla\varphi dx \\ &= \lim_{j\to\infty} \int_{A(r_1,r_2)} \nabla\mathfrak{P}_j \cdot \nabla\varphi dx \\ &= \int_{A(r_1,r_2)} \mathfrak{H}_j(\mathfrak{P}_1)\varphi dx \\ &\geq \int_{A(r_1,r_2)} \mathfrak{H}_1(\mathfrak{P}_1)\varphi dx > 0. \end{aligned} \tag{28}$$

Contradiction.
So from the negation of the conclusion of Theorem 2.2 we derive that the function $\mathfrak{P} : [0,R) \to \mathbb{R}$ satisfies $\mathfrak{P} \in C^1(0,R)$, $\mathfrak{P}$ is strictly nonincreasing in $(0,R)$ and $\lim_{r\nearrow R} \mathfrak{P} \geq 1$ and therefore

$$\mathfrak{P}(r) > 1 \text{ for all } r \in [0,R). \tag{29}$$

Finally we are ready to finish the proof of Theorem 2.2. Independently of the hypothesis in the reductio ad absurdum, there exists $0 < r_0 < R$ such that $\mathfrak{H}_1(\mathfrak{P}_1(r_0)) > \mathfrak{H}_1(1) = 1$ and therefore using (iii) of Lemma 2.1

$$1 < \mathfrak{H}_1(\mathfrak{P}_1(r_0)) \leq \mathfrak{H}_j(\mathfrak{P}_j(r_0)) \text{ for all } j \geq 1. \tag{30}$$

But $\lim_{j\to\infty} \mathfrak{P}_j(r_0) = \mathfrak{P}(r_0) > 1$ and therefore for j big enough $\mathfrak{P}_j(r_0) > 1$ and

$$\mathfrak{H}_j(\mathfrak{P}_j(r_0)) = \mathfrak{H}_1(\mathfrak{P}_j(r_0)) < \mathfrak{H}_1(1) = 1. \tag{31}$$

Therefore

$$1 < \mathfrak{H}_1(\mathfrak{P}_1(r_0)) \leq \mathfrak{H}_j(\mathfrak{P}_j(r_0)) < 1, \tag{32}$$

for j big enough. In page 291 [11] it is stated that for every function $f \in C^{m,\alpha}(\Omega)$, $0 < \alpha < 1$, $m \geq 1$ the solutions of Poisson's equation $\Delta u = f$ on Ω are of class $C^{m+2,\alpha}$ on Ω, and so $\mathfrak{P}_j$ are of class $C^\infty(A(r,R))$. This end the proof. $\square$

The next lemma is new

Lemma 2.3. *The sequence* $\{\mathfrak{H}_j(\mathfrak{P}_j)\}_{j=1}^{\infty}$ *is unbounded in* $C^{\alpha}_{loc}(A(r,R))$.

Proof. In Theorem 4.6 page 60 [18] it is stated that: let Ω be a domain in $\mathbb{R}^N$ and let $u \in C^2(\Omega)$, $f \in C^{\alpha}(\Omega)$ satisfy Poisson's equation $\Delta u = f$. Then $u \in C^{2,\alpha}(\Omega)$ and for any two concentric balls $B_1 = B(x_0, R)$, $B_2 = B(x_0, 2R) \subset\subset \Omega$ we have

$$| u |'_{2,\alpha;B_1} \leq C(| u |_{0;B_2} + R^2 | f |'_{0,\alpha;B_2}). \tag{33}$$

Therefore $\lim_{j\to\infty} | \mathfrak{H}_j(\mathfrak{P}_j) |'_{0,\alpha;B_2} = \infty$. □

3 The construction of the sequence of functions $\mathfrak{F}_j$.

In [19] it is stated that

> [page XVIII, [19]] When we refer to a set F as a fractal, therefore, we will typically have the following in mind.
> (i) F has a fine structure, i. e. detail on arbitray small scales.
> (ii) F is too irregular to be described in traditional geometrical language, both locally and globally.
> (iii) Often F has some form of self-similarity, perhaps approximate or statistical.
> (iv) Usually, the 'fractal dimension' of F (defined in some way) is greater than its topological dimension.
> (v) In most cases of interest F is defined in a very simple way, perhaps recursively.

Moreover

> [Page XXII, [19]] The highly intricate structure of the Julia set illustrated in figure 0.6 stems from the single quadratic function $f(z) = z^2 + c$ for a suitable constant c. Although the set is not strictly self-similar in the sense that the Cantor's set and von Koch curve are, it is 'quasi-self-similar' in that arbitrarily small portions of the set can be magnified an distorted smoothly to coincide with a large part of the set.

Let $[0, a]$ be a bounded interval in $\mathbb{R}$. Let us to consider, the infinite sequence of linear functions

$$\begin{aligned}
\mathfrak{h}(x) &= a - x,\\
\mathfrak{h}_0(x) &= a + 1 - x,\\
\mathfrak{h}_1(x) &= a + 1 + \frac{1}{2} - x,\\
&\dots\\
\mathfrak{h}_j(x) &= a + \sum_{n=0}^{j} \frac{1}{2^n} - x.
\end{aligned}$$

We introduce a infinite sequence of functions defined on $[0, a]$,

$$\begin{aligned}
\mathfrak{p}(x) = \mathfrak{p}(x; s; k) &= -t(x - \frac{s}{k}a)^2 + \mathfrak{c}_s \text{ for all } x \in [a\frac{s}{k}, a\frac{s+1}{k}],\ s = 0, 1, 2, \dots k - 1,\\
\mathfrak{p}_0(x) = \mathfrak{p}_0(x; s; k_0) &= -t_0(x - \frac{s}{k_0}a)^2 + \mathfrak{c}_{0,s} \text{ for all } x \in [a\frac{s}{k_0}, a\frac{s+1}{k_0}],\ s = 0, 1, 2, \dots k_0 - 1,\\
\mathfrak{p}_1(x) = \mathfrak{p}_1(x; s; k_1) &= -t_1(x - \frac{s}{k_1}a)^2 + \mathfrak{c}_{1,s} \text{ for all } x \in [a\frac{s}{k_1}, a\frac{s+1}{k_1}],\ s = 0, 1, 2, \dots k_1 - 1,\\
&\dots,\\
\mathfrak{p}_j(x) = \mathfrak{p}_j(x; s; k_j) &= -t_j(x - \frac{s}{k_j}a)^2 + \mathfrak{c}_{j,s} \text{ for all } x \in [a\frac{s}{k_j}, a\frac{s+1}{k_j}],\ s = 0, 1, 2, \dots k_j - 1.
\end{aligned}$$

Where the sequence

$$\begin{aligned}
\mathfrak{c}_0 &= 1,\\
\mathfrak{c}_{0,0} &= 1 + 1,\\
\mathfrak{c}_{1,0} &= 1 + 1 + \frac{1}{2},\\
&\dots,\\
\mathfrak{c}_{j,0} &= 1 + \sum_{s=0}^{j} \frac{1}{2^j}.
\end{aligned}$$

satisfies

$$\mathfrak{c}_s = \mathfrak{p}\left(\frac{s}{k}; s; k\right) \text{ for all } s = 1, \dots, k - 1, \tag{34}$$

$$\mathfrak{c}_{0,s} = \mathfrak{p}_0\left(\frac{s}{k}; s; k_0\right) \text{ for all } s = 1, \dots, k_0 - 1, \tag{35}$$

$$\mathfrak{c}_{1,s} = \mathfrak{p}_1\left(\frac{s}{k}; s; k_1\right) \text{ for all } s = 1, \dots, k_1 - 1, \tag{36}$$

$$\dots, \tag{37}$$

$$\mathfrak{c}_{j,s} = \mathfrak{p}_j\left(\frac{s}{k_j}; s; k_j\right) \text{ for all } s = 1, \dots, k_j - 1. \tag{38}$$

We have the association:

$$\begin{aligned}
\{\mathfrak{c}_s\}_{s=0}^{\infty} &\longleftrightarrow \mathfrak{h} \longleftrightarrow \mathfrak{p},\\
\{\mathfrak{c}_{0,s}\}_{s=0}^{\infty} &\longleftrightarrow \mathfrak{h}_0 \longleftrightarrow \mathfrak{p}_0,\\
\{\mathfrak{c}_{1,s}\}_{s=0}^{\infty} &\longleftrightarrow \mathfrak{h}_1 \longleftrightarrow \mathfrak{p}_1,\\
&\ldots,\\
\{\mathfrak{c}_{j,s}\}_{s=0}^{\infty} &\longleftrightarrow \mathfrak{h}_j \longleftrightarrow \mathfrak{p}_j.
\end{aligned}$$

The choice of the sequence of non negative numbers $k \cup \{k_j\}_{j=1}^{\infty}$ determines $t \cup \{t_j\}_{j=1}^{\infty}$. We keep k such that the equation

$$\mathfrak{p}(x; 0, k) = \mathfrak{h}_0(x), \tag{39}$$

has no solutions. Similarly we divide this first interval $[0, \frac{a}{k}]$ in $\mathfrak{k}_0$ intervals and setting $k_0 = k\mathfrak{k}_0$ such that the equation

$$\mathfrak{p}_0(x; 0, k) = \mathfrak{h}_1(x), \tag{40}$$

has no solution. Now we complete the procedure by induction. Therefore

$$\lim_{j\to\infty} t_j = \infty. \tag{41}$$

It is follow that the non decreasing, bounded, sequence of functions $\{\mathfrak{p}\}_{j=0}^{\infty}$ defined on $[0, a]$ has second derivative defined almost everywhere and $\mathfrak{p}_j''(x) = -2t_j$.
Using the Rolle's Theorem for the functions $\mathfrak{p}_j(\cdot; s; k_j)$, $\mathfrak{h}_j(\cdot)$ in the interval $[\frac{sa}{k_j}, \frac{(s+1)a}{k_j}]$, we deduce the existence of $x_{j,s} \in (\frac{sa}{k_j}, \frac{(s+1)a}{k_j})$ such that

$$\mathfrak{p}_j'(x_{j,s}; s; k_j) = \mathfrak{h}_j'(x_{j,s}) = -1, \tag{42}$$

where

$$x_{j,s} = \frac{1}{2t_j} + \frac{sa}{k_j}. \tag{43}$$

Similarly for

$$\tilde{x}_{j,s} = \frac{1}{20t_j} + \frac{sa}{k_j}, \tag{44}$$

we have

$$\mathfrak{p}_j'(\tilde{x}_{j,s}; s; k_j) = -\frac{1}{10}. \tag{45}$$

For j big enough, let us to consider the sequence of intervals $(\frac{sa}{k_j} - \delta_j, \frac{sa}{k_j} + \delta_j) \subset [0, a]$ for $s = 1, 2, \ldots, k_j - 1$ where

$$\delta_j = \min\{\frac{1}{200t_j}, \frac{a}{k_j j^2}, \frac{\frac{a}{k_j} - \frac{1}{2t_j}}{10}\}. \tag{46}$$

By smoothing each $\mathfrak{p}_j$ on $(\frac{sa}{k_j}-\delta_j, \frac{sa}{k_j}+\delta_j)$, we obtain the desired sequence $\{\mathfrak{F}_j\}_{j=1}^{\infty} \in C^{\infty}[a,b]$. We point that C^{∞} extension is a nontrivial task, see for example page 136 [18] or the Nikolskii's extension method page 69 [29] and the Calderon's extension method page 72 [29]. Therefore we give a full description of our procedure. For the smoothing method we use the functions

$$S(x;\mathfrak{a};\mathfrak{b}) = x - \int_0^x g(t)dt, \tag{47}$$

where $g(x)=0$ for $x<\mathfrak{a}-\delta$, $g(x)=1$ for $\mathfrak{a}<x<\mathfrak{b}$, $g(x)=0$ for $x>\mathfrak{b}+\delta$, $0<\delta\in\mathbb{R}$, and $g\in C^{\infty}(\mathbb{R})$ (see the C^{∞} Urysohn Lemma page 245, [14]). It it follows that $S(x;\mathfrak{a};\mathfrak{b})=x$ for $x<\mathfrak{a}-\delta$, $S(x;\mathfrak{a};\mathfrak{b})=x-c$ for $x>\mathfrak{b}+\delta$ and $S'(x;\mathfrak{a};\mathfrak{b})\geq 0$. Taking a suitable composition with functions $S(\cdot;\mathfrak{a};\mathfrak{b})$ we accomplish with the smoothing procedure and moreover $\mathfrak{F}_j'(x)\leq 0$. The sequence $\{\Delta\mathfrak{F}_j\}_{j=1}^{\infty}$ converges in measure to $-\infty$.

Note that the set of numbers $\{x_{j,s},\tilde{x}_{j,s}\}$ are numerable and dense in $[0,a]$. Therefore by 42 and 45 we imply if $\mathfrak{F}_j(x_1,x_2,\ldots,x_N)=\mathfrak{F}_j(x_1)$ then

$$\lim_{j\to\infty} \parallel \mathfrak{F}_j \parallel_{C^{1,\alpha}(\overline{B(x,r)})} = \infty, \tag{48}$$

for any ball $B(x,r)$ with center at x with radius r and $\overline{B(x,r)}\subset\Omega$. It follows that

$$\lim_{j\to\infty} \parallel \mathfrak{F}_j \parallel_{C^{2,\alpha}(\overline{B(x,r)})} = \infty. \tag{49}$$

Let us to consider the function

$$I_{A_j}(x) = \begin{cases} 1 & \text{if } x\in A_j, \\ 0 & \text{if } x\notin A_j, \end{cases} \tag{50}$$

where $A_j = \{x\in(\frac{sa}{k_j}+\delta_j, \frac{(s+1)a}{k_j}-\delta_j) \mid \ s=0,1,2,\ldots k_j-1\}$. Therefore

$$\begin{aligned}\int_0^a \mathfrak{F}_j'' I_{A_j}\,dx &= t_j\sum_{s=0}^{k_j-1}\left(\frac{a}{k_j}-2\delta_j\right)\\ &= t_j a - 2\sum_{s=0}^{k_j-1}\delta_j\\ &\geq t_j a - 2\frac{a}{j^2}.\end{aligned}$$

So $\lim_{j\to\infty}\int_0^a \mathfrak{F}_j'' I_{A_j}\,dx=\infty$ and

$$\lim_{j\to\infty} \parallel \mathfrak{F}_j'' \parallel_{L^p[0,a]} = \infty \text{ for all } 1\leq p\leq\infty. \tag{51}$$

4 A primer analysis.

4.1 A new kind of ill-posed problem.

The concept of a well-posed problem of mathematical physics was introduced by J. Hadamard. The solution of any quantitative problem usually ends in a equation $z=R(u)$ where u is the

initial data and z is the solution, $R : U \to Z$, U and Z are metric spaces with distances ρ_U and ρ_Z respectively. The problem of determining the solution z in the space Z from the initial data u in the space U is said to be well-posed on the pair of metric space (Z, U) if the following three conditions are satisfied:
(i) For every element $u \in U$ there exists a solution z in the space Z.
(ii) The solution is unique.
(iii) For every positive number $\epsilon > 0$ there exists a positive number δ such that $\rho_U(u_1, u_2) \leq \delta$ implies $\rho_Z(z_1, z_2) \leq \epsilon$, where $z_1 = S(u_1)$, $z_2 = S(u_2)$.
Problems that do not satisfy them are said ill-posed. The sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ is a new kind of ill-posed problem related to Sobolev's spaces or even for the Laplacian operator in the the context of distributions.

Given Ω a bounded domain in $\mathbb{R}^N$ and $u \in W^{1,1}(\Omega)$ whose laplacian is a bounded measure μ on Ω, we call normal derivative in the sense of distributions of u on $\partial\Omega$ the distribution $\mathfrak{v}_1$ defined on $\mathbb{R}^N$ by

$$\langle \mathfrak{v}_1, \varphi \rangle = \int_\Omega \varphi d\mu + \int_\Omega \nabla\varphi \cdot \nabla u dx, \ \ \varphi \in \mathscr{D}(\mathbb{R}^N) \tag{52}$$

The distribution $\mathfrak{v}_1$ defined by 52 is of compact support in $\partial\Omega$, if $\operatorname{supp}\varphi \cap \partial\Omega = \emptyset$ then φI_Ω and $\nabla(\varphi I_\Omega) = (\nabla\varphi) I_\Omega$. Then by the definition of Laplacian in the sense of distributions

$$\begin{array}{rcl} \int_\Omega \varphi d\mu & = & \langle \Delta u, \varphi I_\Omega \rangle \\ & = & -\int \nabla(\varphi I_\Omega) \cdot \nabla u dx \\ & = & -\int_\Omega \nabla\varphi \cdot \nabla u dx. \end{array} \tag{53}$$

Proposition 4.1 (page 500 [11])**.** *Let Ω be a regular bounded open set with boundary of class $W^{2,\infty}$, μ a bounded Radon measure and $\mathfrak{v}_1$ a Radon measure on $\partial\Omega$. Let us to consider the Neumann problem*

$$u \in W^{1,1}(\Omega), \ \ \Delta u = \mu \ \textit{in} \ \mathscr{D}'(\Omega), \tag{54}$$

$$\mathfrak{v}_1 \ \textit{is the normal derivative on} \ \partial\Omega \ \textit{of} \ u. \tag{55}$$

(i) There exists a solution of 54,55 if and only if $\int_{\partial\Omega} d\mathfrak{v}_1 = \int_\Omega d\mu$.
(ii) If $\int_{\partial\Omega} d\mathfrak{v}_1 = \int_\Omega d\mu$ then the solution of 54, 55 is defined to whitin an additive constant and $u \in W^{1,p}(\Omega)$ for all $1 \leq p < \frac{N}{N-1}$.

Our sequence $\{\Delta\mathfrak{P}_j\}_{j=1}^{\infty}$ is non bounded in $L^\infty(\Omega)$ therefore the limit $\mathfrak{P}$ is outside of the scope of application of Proposition 4.1.

4.1.1 Green's identities.

We show the nature of the discontinuity of the the sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ in Sobolev's spaces and in the context of distribution theory. From (page 17 [18] or [6])

$$\int_{\Omega} (\Delta u)\, v dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\gamma - \int_{\Omega} \nabla u \cdot \nabla v dx \text{ for all } u \in C^2(\overline{\Omega}), \text{ for all } v \in C^1(\overline{\Omega}), \tag{56}$$

we calculate

$$\lim_{j\to\infty} \int_{\Omega} \nabla \mathfrak{P}_j \cdot \nabla v dx = \infty \text{ for all non negative } v \in C^1(\overline{\Omega}) \text{ with } v = 0 \text{ on } \partial\Omega. \tag{57}$$

Taking $v \equiv 1$ in 56, we derive

$$\lim_{j\to\infty} \int_{\partial\Omega} \frac{\partial \mathfrak{P}_j}{\partial n} d\gamma = -\infty. \tag{58}$$

Now we use the second Green's identity

$$\int_{\Omega} (v\Delta u - u\Delta v)\, dx = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\gamma \text{ for all } u, v \in C^2(\overline{\Omega}), \tag{59}$$

it follows

$$\lim_{j\to\infty} \int_{\partial\Omega} \left(v\frac{\partial \mathfrak{P}_j}{\partial n} - \mathfrak{P}_j\frac{\partial v}{\partial n} \right) d\gamma = -\infty \text{ for all nonnegative } v \in C^2(\overline{\Omega}). \tag{60}$$

Therefore for non negative $v \in C^2(\overline{\Omega})$ with $v = 0$ on $\partial\Omega$, we deduce

$$\int_{\Omega} ((v+\epsilon)\Delta\mathfrak{P}_j - \mathfrak{P}_j\Delta v)\, dx = \epsilon \int_{\partial\Omega} \frac{\partial \mathfrak{P}_j}{\partial n} d\gamma - \int_{\partial\Omega} \mathfrak{P}_j \frac{\partial v}{\partial n} d\gamma, \tag{61}$$

$$\int_{\Omega} (v\Delta\mathfrak{P}_j - \mathfrak{P}_j\Delta v)\, dx = -\int_{\partial\Omega} \mathfrak{P}_j \frac{\partial v}{\partial n} d\gamma, \tag{62}$$

and letting $j \to \infty$ we demonstrate the discontinuity at infinitum of the functional 3.

Theorem 4.2 (Theorem 4.11 page 85 [29]). *Let $\Omega \in \mathbb{R}^N$ a bounded domain with smooth boundary; if $1 \le p < N$ put $\frac{1}{q} = \frac{1}{p} - \frac{1}{N-1}\frac{p-1}{p}$; if $p = N$, put $q \ge 1$. There exists a unique mapping $Z \in [W^{2,p}(\Omega) \to W^{1,q}(\partial\Omega)]$ such that $u \in C^{\infty}(\overline{\Omega}) \Longrightarrow Zu = u$.*

The second Green's identity 59 is valid on $W^{2,2}(\Omega)$, it follows from 58 and Theorem 4.2 that our sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ is unbounded in $W^{1,2\frac{N-1}{N-2}}(\partial\Omega)$. From 57 we imply

$$\lim_{j\to\infty} \| \mathfrak{P}_j \|_{W^{1,p}(\Omega)} = \infty \text{ for all } 1 \le p \le \infty, \tag{63}$$

and

$$\lim_{j\to\infty} \| \mathfrak{P}_j \|_{W^{2,p}(\Omega)} = \infty \text{ for all } 1 \le p \le \infty. \tag{64}$$

Therefore the sequence is unbounded in the domain of definition of the trace operator stated in Theorem 4.2. Similar considerations are implied easily from

Theorem 4.3 (page 5 [29]). *Let Ω be a bounded domain with lipschitzian boundary. Then there exists a uniquely defined, linear an continuous mapping $T : W^{k,2}(\Omega) \to L^2(\partial\Omega)$ such that for $x \in \partial\Omega$ and $\mathfrak{v} \in C^\infty(\Omega)$, it is defined by $T(\mathfrak{v})(x) = \mathfrak{v}(x)$*

Theorem 4.4 (page 135 [36]). *We consider a solution $u = u(x) \in C^2(\Omega)$ of Poisson's differential equation $\Delta u(x) = f(x)$, $x \in \Omega$ in the domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$. For each ball $B_R(a) \subset\subset \Omega$ then we have the identity*

$$u(a) = \frac{1}{R^{N-1}\omega_N}\int_{|x-a|=R} u(x)d\sigma - \frac{1}{(N-2)\omega_N}\int_{|x-a|\leq R}\left(|\, x-a\,|^{2-N} - R^{2-N}\right) f(x)dx \tag{65}$$

The same discontinuity at infinitum appears in the context of singular phenomena in nonlinear elliptic problems [1, 2, 9, 17, 34].

$$\int_\Omega ((\nu+\epsilon)\Delta\mathfrak{U}_j - \mathfrak{U}_j\Delta\nu)\, dx = \epsilon\int_{\partial\Omega}\frac{\partial\mathfrak{U}_j}{\partial n}d\gamma - \int_{\partial\Omega}\mathfrak{U}_j\frac{\partial\nu}{\partial n}d\gamma, \tag{66}$$

$$\int_\Omega (\nu\Delta\mathfrak{U}_j - \mathfrak{U}_j\Delta\nu)\, dx = -\int_{\partial\Omega}\mathfrak{U}_j\frac{\partial\nu}{\partial n}d\gamma, \tag{67}$$

where $\mathfrak{U}_j$ solves the problem

$$-\Delta\mathfrak{U}_j = \mathfrak{U}_j^{-\gamma} \text{ in } \Omega, \tag{68}$$

$$\mathfrak{U}_j = \frac{1}{j} \text{ on } \partial\Omega, \tag{69}$$

with $\gamma > 1$. Moreover $\lim_{j\to\infty}\mathfrak{U}_j = \mathfrak{U} \in C^2(\Omega)\cap C^0(\overline{\Omega})$ and $\int_\Omega \mathfrak{U}^{-\gamma}dx = \infty$, showing the same kind of discontinuity at infinitum. This discontinuity property is an interesting example in the Friedrichs method of extension of semibounded operators to self-adjoint operators (page 228 [4], see also page 205 [24]).

4.1.2 Integration by parts.

In the one dimension if $u, \nu \in W^{1,p}(I)$ with $1 \leq p \leq \infty$, then $\int_x^y u'\nu = u(x)\nu(x) - u(y)\nu(x) - \int_y^x u\nu'$ for all $x, y \in \overline{I}$ but even if a distribution has second distributional derivative the integration by parts is not true. For example for $f_h = \int_0^x \frac{1}{h} I_{[-h,0]}(t)dt$, we have:

$$\begin{array}{rcll} \int_x^y [f_h']\varphi' & = & [f_h'(y)]\varphi(y) - [f_h'(x)]\varphi(x) - \int_x^y [f_h'']\varphi = 0 & \text{for all } \varphi \in C_0^\infty(x,y) \\ \int_x^y [f_h']\varphi' & = & [f_h(y)]\varphi'(y) - [f_h(x)]\varphi'(x) - \int_x^y [f_h]\varphi'' & \\ & = & \frac{\varphi'(-h)-\varphi'(0)}{-h} & \text{for all } \varphi \in C_0^\infty(x,y). \end{array} \tag{70}$$

Taking a radial $\varphi \in \mathscr{D}(A(r,R))$ we get

$$\begin{array}{rcl} \int_{A(r,R)} \nabla\mathfrak{P}_j\cdot\nabla\varphi dx & = & \int_{A(r,R)} \frac{\partial\mathfrak{P}_j}{\partial\mathfrak{r}}\frac{x_i}{\mathfrak{r}}\frac{\varphi}{\partial\mathfrak{r}}\frac{x_i}{\mathfrak{r}}dx \\ & = & \int_{A(r,R)} \frac{\partial\mathfrak{P}_j}{\partial\mathfrak{r}}\frac{\partial\varphi}{\partial\mathfrak{r}}\mathfrak{r}dx \\ & = & \int_r^R \mathfrak{r}^{N-1}d\mathfrak{r}\int_{S^{N-1}}\frac{\partial\mathfrak{P}_j}{\partial\mathfrak{r}}\frac{\partial\varphi}{\partial\mathfrak{r}}\mathfrak{r}d\omega \\ & = & \int_r^R \frac{\partial\mathfrak{P}_j}{\partial\mathfrak{r}}\frac{\partial\varphi}{\partial\mathfrak{r}}\mathfrak{r}^N d\mathfrak{r}\int_{S^{N-1}} d\omega. \end{array} \tag{71}$$

So

$$\lim_{j\to\infty}\int_r^R \frac{\partial \mathfrak{P}_j}{\partial \mathfrak{r}}\frac{\partial \varphi}{\partial \mathfrak{r}} d\mathfrak{r} = \infty \tag{72}$$

and the integration by parts rule not hold in the limit because

$$\int_r^R \frac{\partial \mathfrak{P}_j}{\partial \mathfrak{r}}\frac{\partial \varphi}{\partial \mathfrak{r}} d\mathfrak{r} = \mathfrak{P}_j(R)\frac{\partial \varphi}{\partial \mathfrak{r}}(R) - \mathfrak{P}_j(r)\frac{\partial \varphi}{\partial \mathfrak{r}}(r) - \int_r^R \mathfrak{P}_j \frac{\partial^2 \varphi}{\partial \mathfrak{r}^2} d\mathfrak{r}. \tag{73}$$

Also the Cantor's function not satisfies the integration by part rule (this function is monotone and it has zero derivative almost everywhere).

4.1.3 A detour with monotone functions.

Let Ω be a domain in $\mathbb{R}^N$, P a line verifying $P \cap \Omega$ is a nonempty set. A function defined almost everywhere in Ω is said absolutely continuous on the line P if it is continuous on each closed interval of $P \cap \Omega$.

Theorem 4.5 (page 55 [29]). *Suppose $u \in L^1_{loc}(\Omega)$ and $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$, $p \geq 1$. This function changed on a set of measure zero is absolutely continuous on almost all lines parallel to axis x_i. Let us denote by $\left[\frac{\partial u}{\partial x_i}\right]$ the usual derivative and by $\frac{\partial u}{\partial x_i}$ the distribution derivative. Then we have almost everywhere $\left[\frac{\partial u}{\partial x_i}\right] = \frac{\partial u}{\partial x_i}$. Conversely, if $u \in L^1_{loc}(\Omega)$ is absolutely continuous on almost all lines parallel to the axis x_i with $\left[\frac{\partial u}{\partial x_i}\right] \in L^p(\Omega)$, then we have $\frac{\partial u}{\partial x_i} = \left[\frac{\partial u}{\partial x_i}\right]$.*

By the Lebesgue's Differentiation Theorem the function $\mathfrak{P}$ has derivative almost everywhere with respect the radius.

$$\begin{array}{rcll} \mathrm{Var}\mathfrak{P}_j & = & \sup_{r\leq r_0<r_1<\cdots<r_n\leq R}\left\{\sum_{i=1}^n \mid \mathfrak{P}_j(r_i) - \mathfrak{P}_j(r_{j-1}) \mid\right\} & \text{(page 39 [22])} \\ & = & \int_r^R \mid [\frac{d\mathfrak{P}_j}{d\mathfrak{r}}]) \mid d\mathfrak{r} & \text{(page 41 [22])} \\ & = & \sup_{[r,R]}\mathfrak{P}_j - \inf_{[r,R]}\mathfrak{P}_j & \text{(page 43 [22])} \\ & \leq & C & \text{(Lemma 2.1 iv)}. \end{array} \tag{74}$$

Moreover from page 62 [22]

$$\mathrm{Var}\mathfrak{P} \leq \liminf_{n\to\infty} \mathrm{Var}\mathfrak{P}_j. \tag{75}$$

We infer

$$\| [\frac{d\mathfrak{P}}{d\mathfrak{r}}] \|_{L^1[r,R]} \leq \| [\frac{d\mathfrak{P}_j}{d\mathfrak{r}}] \|_{L^1[r,R]} \leq Cte. \tag{76}$$

From

$$\mathfrak{P}_j''(\mathfrak{r}) + \frac{N-1}{\mathfrak{r}}\mathfrak{P}_j'(\mathfrak{r}) \leq C_j \text{ on } B(r,R) \text{ with } \lim_{j\to\infty} C_j = -\infty, \tag{77}$$

we derive

$$\lim_{j\to\infty} \| \frac{\partial^2 \mathfrak{P}_j}{\partial \mathfrak{r}^2} \|_{L^p(B(r,R))} = \infty, \tag{78}$$

$1 \leq p \leq \infty$. Similarly

$$\lim_{j\to\infty} \| \mathfrak{F}'' \|_{L^p[0,a]} = \infty \tag{79}$$

for all $1 \leq p \leq \infty$ and

$$\| [\frac{d\mathfrak{F}}{dx}] \|_{L^1[0,a]} \leq \| [\frac{d\mathfrak{F}_j}{dx}] \|_{L^1[0,a]} \leq Cte. \tag{80}$$

It follows that both sequences $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ and $\{\mathfrak{F}_j\}_{j=1}^{\infty}$ are ill-posed in Sobolev's spaces. This anomalous behaviour is also present in the sequence $f_h(x) = \int_0^x \frac{1}{h} I_{[-h,0]}(t)dt$, where $\int_{-1}^{1} | [\frac{df_h}{dt}] | dt = 1$ for all $h > 0$ and $\lim_{h\to 0^+} \| [\frac{df_h}{dt}] \|_{L^p[-1,1]} = \infty$ for all $1 < p \leq \infty$. Moreover for all nonnegative test function $\varphi \in C_0^{\infty}(-1,1)$, we have

$$\lim_{h\to 0^+} \int_{-1}^{1} [f_h'(x)]\, \varphi'(x)dx = \lim_{h\to 0^+} \int_{-h}^{0} \frac{\varphi'(x)}{h} dx = \lim_{h\to 0^+} \frac{\varphi(-h) - \varphi(0)}{-h} = (-1)(D\delta_0)\varphi. \tag{81}$$

Therefore if we define the distribution $\Lambda_h(\varphi) = \int_{-1}^{1} [f_h'(x)]\, \varphi'(x)dx$ then $\lim_{h\to 0^+} \Lambda_h = -(D\delta_0)$ where $D\delta_0$ is the distributional derivative of Dirac's δ distribution and it is well known that distribution has not weak derivative. The space of functions of pointwise bounded variation admits discontinuous functions and therefore both topologies on the same set $C^{\infty}(r,R)$ produce completely different objects in metrics and associated functionals. Our functions The sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ is bounded in $W^{1,1}(A(r,R))$ but is it not ensured the strong or weak convergence. The function $\mathfrak{P}$ has derivative $\left[\frac{\partial \mathfrak{P}}{\partial r}\right]$ almost everywhere on (r,R) because is a monotone function and moreover we have $\mathfrak{P} = \mathfrak{P}_{AC} + \mathfrak{P}_C + \mathfrak{P}_j$, where $\mathfrak{P}_{AC}$ is an absolutely continuous function, $\mathfrak{P}_j$ is continuous and singular, and $\mathfrak{P}_J$ is the jump function of $\mathfrak{P}$.

Theorem 4.6 (page 3 [22])**.** *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be a monotone function. Then u has as most countable many discontinuity points. Conversely, given a countable set $E \subset \mathbb{R}$, there exists a monotone function $u : \mathbb{R} \to \mathbb{R}$ whose set of discontinuity points is exactly E.*

So by Theorem 4.5 the function u has derivative $[u']$ but no weak derivative if for example E is dense on I.

4.2 The solid mean value.

Despite the difficulty posed by the discontinuity of Green's identities on the sequence $\{\mathfrak{P}_j\}_{j=1}^{\infty}$ we can obtain several properties.

If a function u is absolutely continuous on the interval (a,b) page 225 [22], then

$$u(x) - \frac{1}{b-a}\int_a^b u(t)dt = \frac{1}{b-a}\left[\int_a^x (t-a)u'(t)dt - \int_x^b (b-t)u'(t)dt\right]. \tag{82}$$

Using the Lebesgue's Dominated Convergence Theorem, we obtain a one dimensional solid mean average identity

$$\mathfrak{F}(x) - \frac{1}{b-a}\int_a^b \mathfrak{F}(t)dt = \lim_{j\to\infty}\left\{\frac{1}{b-a}\left[\int_a^x (t-a)\mathfrak{F}_j'(t)dt - \int_x^b (b-t)\mathfrak{F}_j'(t)dt\right]\right\}. \tag{83}$$

Now we recall the extension to Poisson's equation of the solid mean value for Laplace's equation in [18] Chapter 4: Let $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $-\Delta v = f$ then for any ball $B = B_R(y)$, we have

$$v(y) = \frac{1}{|B|}\int_B v dx + \frac{1}{N\omega_N}\int_B f(x)\Theta(r,R)dx, \ r = |x-y|, \tag{84}$$

where

$$\Theta(r,R) = \frac{1}{N-2}\left(r^{2-N} - R^{2-N}\right) - \frac{1}{2R^N}\left(R^2 - r^2\right), \tag{85}$$

for $N > 2$ and

$$\Theta(r,R) = \log\left(\frac{R}{r}\right) - \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right), \tag{86}$$

for $N = 2$, where ω_N is the volume of the unit ball in $\mathbb{R}^N$. We deduce that if $\mathfrak{P}(y) = lim_{j\to\infty}\mathfrak{P}_j(y)$, using the Lebesgue's Dominated Convergence Theorem then

$$\mathfrak{P}(y) - \frac{1}{|B|}\int_B \mathfrak{P} dx = \lim_{j\to\infty}\frac{1}{N\omega_N}\int_B \mathfrak{H}_j(\mathfrak{P}_j(x))\Theta(r,R)dx, \tag{87}$$

$$r = |x-y|. \tag{88}$$

This elementary result involves several strong indeterminations.

Lemma 4.7 (Lemma 3.1.1 page 113 [45]). *Let* $u \in W^{1,p}[B(x_0,r)]$, $p \geq 1$, *where* $x_0 \in \mathbb{R}^N$ *and* $r > 1$. *Let* $0 < \delta < r$. *Then*

$$\begin{aligned}\frac{\int_{B(x_0,r)} u(y)dy}{r^N} - \frac{\int_{B(x_0,\delta)} u(y)dy}{\delta^N} &= \frac{\int_{B(x_0,r)}[\nabla u(y)\cdot(y-x_0)]dy}{Nr^N} \\ &\quad - \frac{\int_{B(x_0,\delta)}[\nabla u(y)\cdot(y-x_0)]dy}{N\delta^N}.\end{aligned} \tag{89}$$

The Lebesgue's Dominated Convergence Theorem implies

$$\begin{aligned}\frac{\int_{B(x_0,r)} \mathfrak{P}(y)dy}{r^N} - \frac{\int_{B(x_0,\delta)} \mathfrak{P}(y)dy}{\delta^N} &= \lim_{j\to\infty}\left\{\frac{\int_{B(x_0,r)}[\nabla \mathfrak{P}_j(y)\cdot(y-x_0)]dy}{Nr^N}\right. \\ &\quad \left. - \frac{\int_{B(x_0,\delta)}[\nabla \mathfrak{P}_j(y)\cdot(y-x_0)]dy}{N\delta^N}\right\}.\end{aligned} \tag{90}$$

4.3 Newtonian potentials.

The theory of Newtonian potentials for distributions with compact support are defined on $\Omega \subset \mathbb{R}^N$, $N \geq 1$. We shall make use of the following results:

Proposition 4.8 (Proposition 5 page 281 [11]). *Let* Ω *be a regular bounded open set and let* $u \in C^2(\Omega) \cap C^1_n(\overline{\Omega})$ *with* $\Delta u \in L^1(\Omega)$. *Then*

$$u = u_0 + u_1 + u_2 \ \textit{on}\ \Omega, \tag{91}$$

where $u_0,\ u_1,\ u_2$ *are the Newtonian potentials of the distributions* $f_0,\ f_1,\ f_2$ *on* $\mathbb{R}^N$ *defined by*

$$\langle f_0, \zeta\rangle = \int_\Omega \zeta \Delta u dx, \tag{92}$$

$$\langle f_1, \zeta\rangle = \int_{\partial\Omega} \zeta \left(-\frac{\partial u}{\partial n}\right) d\gamma, \tag{93}$$

$$\langle f_2, \zeta\rangle = \int_{\partial\Omega} \frac{\partial \zeta}{\partial n} u d\gamma. \tag{94}$$

We note that $f_0, f_1, f_2 \in \mathscr{E}'$*:*
f_0 *is an integrable function on* $\mathbb{R}^N$ *with support contained in* $\overline{\Omega}$*,*
f_1 *is a measure on* $\mathbb{R}^N$ *with support contained in* $\partial\Omega$*,*
f_2 *is a distribution of order* 1 *on* $\mathbb{R}^N$ *with support contained in* $\partial\Omega$*.*
We say that u_1 *is the simple layer (respectively double layer) potential defined by the function* $-\frac{\partial u}{\partial n}$ *(respectively* u*) continuous on* $\partial\Omega$*.*

We apply this results to our sequence $\{\mathfrak{P}_j\}_{j=1}^\infty$. Using Proposition 4.8, we have

$$\mathfrak{P}_j = \mathfrak{P}_{0,j} + \mathfrak{P}_{1,j} + \mathfrak{P}_{2,j} \text{ on } \Omega, \tag{95}$$

where $\mathfrak{P}_{0,j}, \mathfrak{P}_{1,j}, \mathfrak{P}_{2,j}$ are the Newtonian potentials of the distributions $\mathfrak{f}_{0,j}, \mathfrak{f}_{1,j}, \mathfrak{f}_{2,j}$. Therefore

$$\Delta\mathfrak{P}_{0,j} = \mathfrak{f}_{0,j} \text{ on } \mathbb{R}^N, \tag{96}$$

$$\Delta\mathfrak{P}_{1,j} = \mathfrak{f}_{1,j} \text{ on } \mathbb{R}^N, \tag{97}$$

$$\Delta\mathfrak{P}_{2,j} = \mathfrak{f}_{2,j} \text{ on } \mathbb{R}^N. \tag{98}$$

We obtain

$$\lim_{j\to\infty} \langle \mathfrak{f}_{0,j}, \zeta\rangle = -\infty, \tag{99}$$

$$\lim_{j\to\infty} \langle \mathfrak{f}_{1,j}, 1\rangle = \int_{\partial\Omega} 1 \left(-\frac{\partial \mathfrak{P}_j}{\partial n}\right) d\gamma = \infty, \text{ simple layer potentials,} \tag{100}$$

$$\lim_{j\to\infty} \langle \mathfrak{f}_{2,j}, \zeta\rangle = \int_{\partial\Omega} \frac{\partial \zeta}{\partial n} \left(\lim_{j\to\infty} \mathfrak{P}_j\right) d\gamma, \text{ double layer potentials.} \tag{101}$$

Proposition 4.9 (Proposition 2 page 278 [11]). *Let* $f \in \mathscr{E}'$ *and let* u *be the Newtonian potential of* f*. Then for every multi-index* $\alpha \in \mathbb{N}$

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x) = \langle f, 1\rangle + O\left(\frac{1}{|\, x\,|^{n+|\alpha|-1}}\right) \ \textit{when} \ |\, x\,| \to \infty. \tag{102}$$

In particular

$$\textit{if } N \geq 3,\ \lim_{|x|\to\infty} u(x) = 0, \tag{103}$$

$$\textit{if } N \geq 2,\ \lim_{|x|\to\infty} \nabla u(x) = 0. \tag{104}$$

The last proposition is useful in the description of the sequence of Newtonian potentials $\{f_{1,j}\}_{j=1}^\infty$, $\{\mathfrak{f}_{0,j}\}_{j=1}^\infty$, $\{\mathfrak{f}_{1,j}\}_{j=1}^\infty$ and $\{\mathfrak{f}_{2,j}\}_{j=1}^\infty$.

4.4 The spherical average.

For u in 56 with $\{0\} \in \Omega \subset \mathbb{R}^N,\ N \geq 2$, let $\overline{u}$ be the spherical average of u, i.e.,

$$\overline{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} u(x) d\gamma_x. \tag{105}$$

With the change of variable $x \to y$, we have

$$\overline{u}(r) = \frac{1}{\omega_N} \int_{|y|=1} u(ry) d\gamma_y, \tag{106}$$

and

$$\frac{d\overline{u}}{dr} = \frac{1}{\omega_N} \int_{|y|=1} \nabla u(ry) \cdot y d\gamma_y. \tag{107}$$

Hence

$$\frac{d\overline{u}}{dr} = \frac{1}{\omega_N} \int_{|y|=1} \frac{\partial u}{\partial r}(ry) d\gamma_y = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \frac{\partial u}{\partial r}(x) d\gamma_x, \tag{108}$$

that is

$$\frac{d\overline{u}}{dr} = \frac{1}{\omega_N r^{N-1}} \int_{B(0,r)} \Delta u(x) dx. \tag{109}$$

Therefore from Theorem 2.2 it follows that

$$\lim_{j\to\infty} \frac{d\overline{\mathfrak{P}}_j}{dr} = -\infty. \tag{110}$$

5 Statement and proof of the main results.

Our main Theorem states the concentration of compactness. It can be regarded as a classical counterpart of Helly's Selection Theorem in the space of functions of bounded point variations (Theorem 2.35 page 59 [22]).

Theorem 5.1. *Let Ω be a bounded domain in $\mathbb{R}^N,\ N \geq 3$ and any sequence of functions $\{u_j\}_{j=1}^{\infty}$ in $C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying*

$$-\Delta \mathfrak{P}_j \geq -\Delta u_j \ \text{in } \Omega, \tag{111}$$

$$\mathfrak{P}_j \geq u_j \ \text{on } \partial\Omega. \tag{112}$$

Then there exist a constant C depending only on the sequence $\{\mathfrak{P}\}_{j=1}^{\infty}$, such that $u_j \leq C$ for all $j = 1, \ldots, \infty$.

Proof. It is a simple consequence of Theorem 3.3 page 33 in [18]. □

Theorem 5.2 (Strong concentration of compactness for Newtonian potentials)**.** *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N > 2$. Then there exist a sequence $\{f_{1,j}\}_{j=1}^{\infty} \in \mathscr{E}'$ the space of distributions on $\mathbb{R}^N$ with compact support such that:*
(i) $f_{1,j} = \Delta \mathfrak{P}_j$ *on Ω in the sense of the distributions.*
(ii) The sequence of functions $\{\mathfrak{P}_j\}_{j=1}^{\infty} \in C^{\infty}(\overline{\Omega})$ is non decreasing and bounded.
(iii) $\lim_{j\to\infty} \Delta\mathfrak{P}_j = -\infty$ *uniformly on Ω.*
(iv) $\mathfrak{P}_j$ is the Newtonian potential of $f_{1,j}$ on Ω.
(v) The simple layer potential a $\mathfrak{f}_{1,j} \in \mathscr{E}'$ associated to $\mathfrak{P}_j$ satisfies
$\lim_{j\to\infty}\langle \mathfrak{f}_{1,j}, 1\rangle = \int_{\partial\Omega} 1 \left(-\frac{\partial \mathfrak{P}_j}{\partial n}\right) d\gamma = \infty.$
(vi) $\lim_{j\to\infty} \parallel \Delta\mathfrak{P}_j \parallel'_{C^{\alpha}(\overline{B(x_0,R)})} = \infty$ *for all $B(x_0, R) \subset\subset \Omega$.*
(vii) Solid mean value property.

$$\mathfrak{P}(y) - \frac{1}{\mid B \mid}\int_B \mathfrak{P} dx = \lim_{j\to\infty} \frac{(-1)}{N\omega_N}\int_B \Delta\mathfrak{P}_j(x)\Theta(r, R)dx, \tag{113}$$

$$r = \mid x - y \mid . \tag{114}$$

where

$$\Theta(r, R) = \frac{1}{N-2}\left(r^{2-N} - R^{2-N}\right) - \frac{1}{2R^N}\left(R^2 - r^2\right), \tag{115}$$

for $N > 2$, where ω_N is the volume of the unit ball in $\mathbb{R}^N$.
(viii) For $N > 2$ we have

$$\begin{aligned} \frac{\int_{B(x_0,r)} \mathfrak{P}(y)dy}{r^N} - \frac{\int_{B(x_0,\delta)} \mathfrak{P}(y)dy}{\delta^N} \quad = \quad & \lim_{j\to\infty}\left\{ \frac{\int_{B(x_0,r)}[\nabla\mathfrak{P}_j(y)\cdot(y-x_0)]dy}{Nr^N} \right. \\ & \left. - \frac{\int_{B(x_0,\delta)}[\nabla\mathfrak{P}_j(y)\cdot(y-x_0)]dy}{N\delta^N} \right\}. \end{aligned} \tag{116}$$

(ix) The spherical average $\overline{\mathfrak{P}}_j$ satisfy

$$\lim_{j\to\infty} \frac{d\overline{\mathfrak{P}}_j}{dr} = -\infty. \tag{117}$$

Proof. This theorem is a collection of results stated in the *A primer analysis* section. ☐

Theorem 5.3 (Weak concentration of compactness for Newtonian potentials)**.** *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$. Then there exist a sequence $\{\mathbf{f}_{1,j}\}_{j=1}^{\infty} \in \mathscr{E}'$ the space of distributions on $\mathbb{R}^N$ with compact support such that:*
(i) $\mathbf{f}_{1,j} = \Delta\mathfrak{F}_j$ *on Ω in the sense of the distributions.*
(ii) The sequence of functions $\{\mathfrak{F}_j\}_{j=1}^{\infty} \in C^{\infty}(\overline{\Omega})$ is non decreasing and bounded.
(iii) $\lim_{j\to\infty} \Delta\mathfrak{F}_j = -\infty$ *in measure on Ω.*
(iv) $\mathfrak{F}_j$ is the Newtonian potential of $\mathbf{f}_{1,j}$ on Ω.
(v) $\lim_{j\to\infty} \parallel \mathfrak{F} \parallel_{C^{1,\alpha}(\overline{B(x_0,R)})} = \infty$ *for all $B(x_0, R) \subset\subset \Omega$.*

(vi) One dimensional mean value property.

$$\mathfrak{F}(x) - \frac{1}{b-a}\int_a^b \mathfrak{F}(t)dt = \lim_{j\to\infty}\left\{\frac{1}{b-a}\left[\int_a^x (t-a)\mathfrak{F}_j'(t)dt - \int_x^b (b-t)\mathfrak{F}_j'(t)dt\right]\right\}. \tag{118}$$

(vii) Solid mean value property.

$$\mathfrak{F}(y) - \frac{1}{|B|}\int_B \mathfrak{F}dx = \lim_{j\to\infty}\frac{(-1)}{N\omega_N}\int_B \Delta\mathfrak{F}_j(x)\Theta(r,R)dx, \tag{119}$$

$$r = |x-y|. \tag{120}$$

where

$$\Theta(r,R) = \frac{1}{N-2}\left(r^{2-N} - R^{2-N}\right) - \frac{1}{2R^N}\left(R^2 - r^2\right), \tag{121}$$

for $N > 2$ *and*

$$\Theta(r,R) = \log\left(\frac{R}{r}\right) - \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right), \tag{122}$$

for $N = 2$*, where* ω_N *is the volume of the unit ball in* $\mathbb{R}^N$*.*
(viii) For $N > 2$

$$\begin{aligned}\frac{\int_{B(x_0,r)}\mathfrak{F}(y)dy}{r^N} - \frac{\int_{B(x_0,\delta)}\mathfrak{F}(y)dy}{\delta^N} &= \lim_{j\to\infty}\left\{\frac{\int_{B(x_0,r)}[\nabla\mathfrak{F}_j(y)\cdot(y-x_0)]dy}{Nr^N}\right. \\ &\left. - \frac{\int_{B(x_0,\delta)}[\nabla\mathfrak{F}_j(y)\cdot(y-x_0)]dy}{N\delta^N}\right\}.\end{aligned} \tag{123}$$

Proof. This theorem is a collection of results stated in the *A primer analysis* section. □

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