CUBO A Mathematical Journal
Vol.17, Nº02, (73–87). June 2015

On an anisotropic Allen-Cahn system

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ABSTRACT

Our aim in this paper is to prove the existence and uniqueness of solutions for an Allen-Cahn type system based on a modification of the Ginzburg-Landau free energy proposed in [11]. In particular, the free energy contains an additional term called Willmore regularization and takes into account anisotropy effects.

RESUMEN

Nuestro propósito en este trabajo es probar la existencia y unicidad de soluciones para un Sistema de tipo Allen-Cahn basados en una modificación de la energía libre Ginzburg-Landau propuesta en [11]. En particular, la energía libre contiene un término adicional llamado regularización de Willmore y considera efectos de anisotropía.

Keywords and Phrases: Allen-Cahn equation, Willmore regularization, anisotropy effects, well-posedness

2010 AMS Mathematics Subject Classification: 35B45, 35K55

1 Introduction

The Allen-Cahn equation,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = 0, \tag{1.1}$$

where u is the order parameter and $f(s) = s^3 - s$, describes important processes related with phase separation in binary alloys, namely, the ordering of atoms in a lattice (see [1]). This equation is obtained by considering the Ginzburg-Landau free energy,

$$\Psi_{\rm GL} = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u))\, dx, \tag{1.2}$$

where Ω is the domain occupied by the material and, typically, $F(s) = \frac{1}{4}(s^2-1)^2$. Assuming a relaxation dynamics, i.e., writing

$$\frac{\partial u}{\partial t} = -\frac{D\Psi_{\rm GL}}{Du}, \tag{1.3}$$

where $\frac{D}{Du}$ denotes a variational derivative, we obtain (1.1).

In [11] (see also [2]), the authors introduced the following modification of the Ginzburg-Landau free energy:

$$\Psi_{\rm AGL} = \int_\Omega (\delta(\frac{\nabla u}{|\nabla u|})(\frac{1}{2}|\nabla u|^2 + F(u)) + \frac{\beta}{2}\omega^2)\, dx,\ \ \beta > 0, \tag{1.4}$$

$$\omega = -\Delta u + f(u), \tag{1.5}$$

where $G(u) = \frac{1}{2}\omega^2$ is called nonlinear Willmore regularization, β is a small regularization parameter and the function δ accounts for anisotropy effects. The Willmore regularization is relevant, e.g., in determining the equilibrium shape of a crystal in its own liquid matrix, when anisotropy effects are strong. Indeed, in that case, the equilibrium interface may not be a smooth curve, but may present facets and corners with slope discontinuities (see, e.g., [9]), which can lead to an ill-posed problem and requires regularization.

The Allen-Cahn equation associated with (1.4) has been studied in [6] in the particular cases $\delta \equiv 1$ (isotropic case) and $\delta \equiv -1$ (in that case, $\Psi_{\rm AGL}$ is also called functionalized Cahn-Hilliard energy in [8]). In particular, well-posedness results have been obtained. The Cahn-Hilliard equation associated with (1.4) (obtained by writing $\frac{\partial u}{\partial t} = \Delta \frac{D\Psi_{\rm AGL}}{Du}$) has been studied in [4], again, in the isotropic case $\delta \equiv 1$; we also refer the reader to [2] and [12] for numerical studies.

In this paper, we actually consider the following regularization of $\Psi_{\rm AGL}$:

$$\Psi_{\mathrm{RAGL}} = \int_\Omega (\delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})(\frac{1}{2}|\nabla u|^2 + F(u)) + \frac{\beta}{2}\omega^2)\,dx,\ \ \epsilon > 0. \tag{1.6}$$

We have, in that case and formally,

$$D\Psi_{\mathrm{RAGL}} = \int_\Omega (\delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})(\nabla u \cdot \nabla Du + f(u)Du) + \beta\omega f'(u)Du - \beta\omega\Delta Du)\,dx$$

$$+ \int_\Omega \delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \frac{\nabla Du}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}(\frac{1}{2}|\nabla u|^2 + F(u))\,dx$$

$$- \int_\Omega \delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u \frac{\nabla u . \nabla Du}{(\epsilon + |\nabla u|^2)^{\frac{3}{2}}}(\frac{1}{2}|\nabla u|^2 + F(u))\,dx,$$

where δ' denotes the differential (gradient) of δ. Therefore, assuming proper boundary conditions,

$$\frac{D\Psi_{\mathrm{RAGL}}}{Du} = -\mathrm{div}(\delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})\nabla u) + \delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})f(u) \tag{1.7}$$

$$-\frac{1}{2}\mathrm{div}(\frac{|\nabla u|^2}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})) - \mathrm{div}(\frac{F(u)}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}))$$

$$+\frac{1}{2}\mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u \frac{|\nabla u|^2 \nabla u}{(\epsilon + |\nabla u|^2)^{\frac{3}{2}}}) + \mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u \frac{F(u)\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{3}{2}}})$$

$$+\beta\omega f'(u) - \beta\Delta\omega.$$

Assuming again a relaxation dynamics, we finally obtain the following regularized anisotropic Allen-Cahn system:

$$\frac{\partial u}{\partial t} - \mathrm{div}(\delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})\nabla u) + \delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})f(u) \tag{1.8}$$

$$-\frac{1}{2}\mathrm{div}(\frac{|\nabla u|^2}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})) - \mathrm{div}(\frac{F(u)}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}))$$

$$+\frac{1}{2}\mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u \frac{|\nabla u|^2 \nabla u}{(\epsilon + |\nabla u|^2)^{\frac{3}{2}}}) + \mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u \frac{F(u)\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{3}{2}}})$$

$$+\beta\omega f'(u) - \beta\Delta\omega = 0,$$

$$\omega = -\Delta u + f(u). \tag{1.9}$$

We proved in [5] the existence and uniqueness of solutions to (1.8)-(1.9), but only in one space dimension, due to a lack of regularity on $\frac{\partial u}{\partial t}$. Thus, in order to handle the problem in higher space dimensions, we consider in this paper the following further regularized Allen-Cahn system:

$$\frac{\partial u}{\partial t} - \alpha\frac{\partial \Delta u}{\partial t} - \mathrm{div}(\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\nabla u) + \delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})f(u) \tag{1.10}$$

$$-\frac{1}{2}\mathrm{div}(\frac{|\nabla u|^2}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})) - \mathrm{div}(\frac{F(u)}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}))$$

$$+\frac{1}{2}\mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\cdot\nabla u\frac{|\nabla u|^2\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{3}{2}}}) + \mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\cdot\nabla u\frac{F(u)\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{3}{2}}})$$

$$+\beta\omega f'(u) - \beta\Delta\omega = 0,\ \ \alpha > 0,$$

$$\omega = -\Delta u + f(u). \tag{1.11}$$

A term of the form $-\alpha\frac{\partial\Delta u}{\partial t}$ appears in generalizations of the Allen-Cahn equation proposed in [3], based on a separate balance law for internal microforces, i.e., for interactions at a microscopic level (we can note that the derivation proposed in [3] is strongly based on the usual Ginzburg-Landau free energy; it would thus be interesting to go back to the arguments in [3] and see whether/how they can be adapted to a more general free energy). Such a regularization is also similar to the viscous Cahn-Hilliard equation proposed in [7]. Actually, the approach in [3], applied to the Cahn-Hilliard equation, allows to recover the viscous Cahn-Hilliard equation.

We prove, in the next sections, the existence and uniqueness of solutions to (1.10)-(1.11). It is important to note however that our estimates are not independent of ϵ, so that we cannot pass to the limit as ϵ goes to 0. This is not surprising, as the problem formally obtained by taking $\epsilon = 0$ cannot correspond to the (Allen-Cahn) problem associated with the free energy (1.4) (see also [2] and [11]). Actually, this is related with a proper functional setting for the limit problem and, more precisely, for the Allen-Cahn system associated with (1.4). This will be studied elsewhere.

2 Setting of the problem

We consider the following initial and boundary value problem (for simplicity, we take $\alpha = \beta = 1$):

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} - \mathrm{div}(\delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})\nabla u) + \delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})f(u) \tag{2.1}$$

$$-\frac{1}{2}\mathrm{div}(\frac{|\nabla u|^2}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})) - \mathrm{div}(\frac{F(u)}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}))$$

$$+\frac{1}{2}\mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u \frac{|\nabla u|^2 \nabla u}{(\epsilon + |\nabla u|^2)^{\frac{3}{2}}}) + \mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u \frac{F(u)\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{3}{2}}})$$

$$+\omega f'(u) - \Delta\omega = 0,$$

$$\omega = -\Delta u + f(u), \tag{2.2}$$

$$u = \omega = 0 \text{ on } \Gamma, \tag{2.3}$$

$$u|_{t=0} = u_0, \tag{2.4}$$

in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n = 1$, 2 or 3, with boundary Γ.

As far as the nonlinear terms δ and f are concerned, we assume that

$$\delta \text{ is of class } \mathcal{C}^2, \tag{2.5}$$

$$f \text{ is of class } \mathcal{C}^2,\ f(0) = 0,\ f' \geq -c_0,\ c_0 \geq 0, \tag{2.6}$$

$$sf'(s)f(s) - f(s)^2 \geq -c_1,\ c_1 \geq 0,\ s \in \mathbb{R}, \tag{2.7}$$

$$sf''(s) \geq -c_2,\ c_2 \geq 0,\ s \in \mathbb{R}, \tag{2.8}$$

$$|F(s)| \le \sigma f(s)^2 + c_\sigma,\ \forall \sigma > 0,\ s \in \mathbb{R}, \tag{2.9}$$

$$|F(s)| \le c_3(|s|^p + 1),\ c_3 > 0,\ p \ge 0 \text{ if } n = 1 \text{ or } 2,\ p \in [0,7] \text{ if } n = 3, \tag{2.10}$$

where F is an antiderivative of f.

These assumptions are satisfied by polynomials of the form $f(s) = \sum_{i=1}^{q} a_i s^i$, $q \ge 3$ odd ($q \le 5$ when $n = 3$), $a_q > 0$, and, in particular, by the usual cubic nonlinear term $f(s) = s^3 - s$.

We denote by $((\cdot,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$, and we denote by $\|\cdot\|_X$ the norm in the Banach space X.

Throughout the paper, the same letter c (and, sometimes, c') denotes constants which may vary from line to line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may vary from line to line.

3 A priori estimates

We multiply (2.1) by u and have, integrating over Ω and by parts and owing to (2.2),

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|\nabla u\|^2) + ((\delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})\nabla u, \nabla u)) + ((\delta(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})f(u), u)) \tag{3.1}$$

$$+\frac{1}{2}((\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}), \frac{|\nabla u|^2 \nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}})) + ((\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}), \frac{F(u)\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}))$$

$$-\frac{1}{2}((\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u, \frac{|\nabla u|^4}{(\epsilon + |\nabla u|^2)^{\frac{3}{2}}})) - ((\delta'(\frac{\nabla u}{(\epsilon + |\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u, \frac{F(u)|\nabla u|^2}{(\epsilon + |\nabla u|^2)^{\frac{3}{2}}}))$$

$$+\|\omega\|^2 + \int_\Omega (uf'(u)f(u) - f(u)^2)\, dx + ((uf''(u)\nabla u, \nabla u)) = 0.$$

We note that

$$\frac{|s|}{(\epsilon + |s|^2)^{\frac{1}{2}}} \le 1,\ s \in \mathbb{R}^n,$$

so that

$$|\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})| \le c, \; |\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})| \le c'.$$

Therefore,

$$|((\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\nabla u, \nabla u))| \le c\|\nabla u\|^2, \tag{3.2}$$

$$|((\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})f(u), u))| \le \sigma\|f(u)\|^2 + c_\sigma\|\nabla u\|^2, \; \forall \sigma > 0, \tag{3.3}$$

$$|((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}), \frac{|\nabla u|^2\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}))| \le c\|\nabla u\|^2 \tag{3.4}$$

$$((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}), \frac{F(u)\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})) \le c\int_\Omega |F(u)|\,dx \tag{3.5}$$

$$\le \text{(owing to (2.9))}$$

$$\le \sigma\|f(u)\|^2 + c_\sigma, \; \forall \sigma > 0,$$

$$|((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u, \frac{|\nabla u|^4}{(\epsilon+|\nabla u|^2)^{\frac{3}{2}}}))| \le c\|\nabla u\|^2 \tag{3.6}$$

and, as above,

$$|((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}) \cdot \nabla u, \frac{F(u)|\nabla u|^2}{(\epsilon+|\nabla u|^2)^{\frac{3}{2}}}))| \le c\int_\Omega |F(u)|\,dx \le \sigma\|f(u)\|^2 + c_\sigma, \; \forall \sigma > 0. \tag{3.7}$$

It thus follows from (2.7)-(2.8) and (3.1)-(3.7) that

$$\frac{d}{dt}(\|u\|^2 + \|\nabla u\|^2) + 2\|\omega\|^2 \le c\|\nabla u\|^2 + \sigma\|f(u)\|^2 + c_\sigma, \; \forall \sigma > 0. \tag{3.8}$$

We then note that, owing to (2.6),

$$\|\omega\|^2 \ge \|\Delta u\|^2 + \|f(u)\|^2 - 2c_0\|\nabla u\|^2 \tag{3.9}$$

and it follows from (3.8)-(3.9) that, taking $\sigma = 1$,

$$\frac{d}{dt}(\|u\|^2+\|\nabla u\|^2)+c(\|u\|^2_{H^2(\Omega)}+\|f(u)\|^2)\le c'(\|u\|^2_{H^1(\Omega)}+1),\ c>0. \tag{3.10}$$

We then multiply (2.1) by $\frac{\partial u}{\partial t}$ and obtain, owing to (2.2),

$$\|\frac{\partial u}{\partial t}\|^2+\|\nabla\frac{\partial u}{\partial t}\|^2+((\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}})\nabla u,\nabla\frac{\partial u}{\partial t}))+((\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}})f(u),\frac{\partial u}{\partial t})) \tag{3.11}$$

$$+\frac12((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}}),\frac{|\nabla u|^2\nabla\frac{\partial u}{\partial t}}{(\epsilon+|\nabla u|^2)^{\frac12}}))+((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}}),\frac{F(u)\nabla\frac{\partial u}{\partial t}}{(\epsilon+|\nabla u|^2)^{\frac12}}))$$

$$-\frac12((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}})\cdot\nabla u,\frac{|\nabla u|^2\nabla u\cdot\nabla\frac{\partial u}{\partial t}}{(\epsilon+|\nabla u|^2)^{\frac32}}))-((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}})\cdot\nabla u,\frac{F(u)\nabla u\cdot\nabla\frac{\partial u}{\partial t}}{(\epsilon+|\nabla u|^2)^{\frac32}}))$$

$$+\frac12\frac{d}{dt}\|\omega\|^2=0.$$

We have, proceeding as above,

$$|((\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}})\nabla u,\nabla\frac{\partial u}{\partial t}))|+|((\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}})f(u),\frac{\partial u}{\partial t}))| \tag{3.12}$$

$$+\frac12|((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}}),\frac{|\nabla u|^2\nabla\frac{\partial u}{\partial t}}{(\epsilon+|\nabla u|^2)^{\frac12}}))|+\frac12|((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}})\cdot\nabla u,\frac{|\nabla u|^2\nabla u\cdot\nabla\frac{\partial u}{\partial t}}{(\epsilon+|\nabla u|^2)^{\frac32}}))|$$

$$\le c(\|\nabla u\|+\|f(u)\|)\|\nabla\frac{\partial u}{\partial t}\|$$

$$\le\frac12\|\nabla\frac{\partial u}{\partial t}\|^2+c(\|\nabla u\|^2+\|f(u)\|^2).$$

Furthermore, for the most difficult case $n=3$ and $p=7$ and owing to (2.10) and Agmon's inequality (see, e.g., [10]),

$$|((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}}),\frac{F(u)\nabla\frac{\partial u}{\partial t}}{(\epsilon+|\nabla u|^2)^{\frac12}}))|+|((\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac12}})\cdot\nabla u,\frac{F(u)\nabla u\cdot\nabla\frac{\partial u}{\partial t}}{(\epsilon+|\nabla u|^2)^{\frac32}}))| \tag{3.13}$$

$$\le c\epsilon^{-\frac12}\int_\Omega|F(u)||\nabla\frac{\partial u}{\partial t}|\,dx\le\frac12\|\nabla\frac{\partial u}{\partial t}\|^2+c'\epsilon^{-1}\int_\Omega(|u|^{14}+1)\,dx$$

$$\leq \frac{1}{2}\|\nabla \frac{\partial u}{\partial t}\|^2 + c\epsilon^{-1}(\|u\|^8_{L^\infty(\Omega)} + 1)(\|u\|^6_{L^6(\Omega)} + 1)$$

$$\leq \frac{1}{2}\|\nabla \frac{\partial u}{\partial t}\|^2 + c\epsilon^{-1}(\|u\|^4_{H^1(\Omega)}\|u\|^4_{H^2(\Omega)} + 1)(\|u\|^6_{H^1(\Omega)} + 1)$$

$$\leq \frac{1}{2}\|\nabla \frac{\partial u}{\partial t}\|^2 + c\epsilon^{-1}(\|u\|^{10}_{H^1(\Omega)} + 1)(\|u\|^4_{H^2(\Omega)} + 1).$$

We thus deduce from (3.11)-(3.13) that

$$\frac{d}{dt}\|\omega\|^2 + \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} \leq c\epsilon^{-1}(\|u\|^{10}_{H^1(\Omega)} + 1)(\|u\|^4_{H^2(\Omega)} + 1). \tag{3.14}$$

Recalling that

$$\|\omega\|^2 \geq \|\Delta u\|^2 + \|f(u)\|^2 - 2c_0\|\nabla u\|^2,$$

we finally deduce that

$$\frac{d}{dt}\|\omega\|^2 + \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} \leq c\epsilon^{-1}(\|u\|^{10}_{H^1(\Omega)} + 1)(\|u\|^2_{H^2(\Omega)} + 1)(\|\omega\|^2 + \|u\|^2_{H^1(\Omega)} + 1). \tag{3.15}$$

We now multiply (2.1) by $-\Delta u$ and have, owing to (2.2),

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2 + \|\Delta u\|^2) + ((\mathrm{div}\varphi_1(\nabla u), \Delta u)) - ((\varphi_2(\nabla u)f(u), \Delta u)) \tag{3.16}$$

$$+((\mathrm{div}\varphi_3(\nabla u), \Delta u)) + ((\mathrm{div}(F(u)\varphi_4(\nabla u)), \Delta u))$$

$$-((\mathrm{div}\varphi_5(\nabla u), \Delta u)) - ((\mathrm{div}(F(u)\varphi_6(\nabla u)), \Delta u))$$

$$-((\omega f'(u), \Delta u)) + ((\Delta f(u), \Delta u)) + \|\nabla\Delta u\|^2 = 0,$$

where

$$\varphi_1(s) = \delta(\frac{s}{(\epsilon + |s|^2)^{\frac{1}{2}}})s,\ \ \varphi_2(s) = \delta(\frac{s}{(\epsilon + |s|^2)^{\frac{1}{2}}}),\ \ \varphi_3(s) = \frac{1}{2}\frac{|s|^2}{(\epsilon + |s|^2)^{\frac{1}{2}}}\delta'(\frac{s}{(\epsilon + |s|^2)^{\frac{1}{2}}}),$$

$$\varphi_4(s)=\frac{1}{(\epsilon+|s|^2)^{\frac{1}{2}}}\delta'(\frac{s}{(\epsilon+|s|^2)^{\frac{1}{2}}}),\ \varphi_5(s)=\frac{1}{2}\delta'(\frac{s}{(\epsilon+|s|^2)^{\frac{1}{2}}})\cdot s\frac{|s|^2 s}{(\epsilon+|s|^2)^{\frac{3}{2}}},$$

$$\varphi_6(s)=\delta'(\frac{s}{(\epsilon+|s|^2)^{\frac{1}{2}}})\cdot s\frac{s}{(\epsilon+|s|^2)^{\frac{3}{2}}}.$$

Noting that

$$\mathrm{div}\varphi_i(\nabla u)=\varphi_i'(\nabla u)\cdot\nabla\nabla u,\ i=1,\ 3,\ 5,$$

$$\mathrm{div}(F(u)\varphi_i(\nabla u))=F(u)\varphi_i'(\nabla u)\cdot\nabla\nabla u+f(u)\varphi_i(\nabla u)\cdot\nabla u,\ i=4,\ 6,$$

it follows from the continuous embedding $H^2(\Omega)\subset\mathcal{C}(\overline{\Omega})$ and (2.2) that

$$|((\mathrm{div}\varphi_1(\nabla u),\Delta u))|+|((\varphi_2(\nabla u)f(u),\Delta u))|$$

$$+|((\mathrm{div}\varphi_3(\nabla u),\Delta u))|+|((\mathrm{div}(F(u)\varphi_4(\nabla u)),\Delta u))|$$

$$+|((\mathrm{div}\varphi_5(\nabla u),\Delta u))|+|((\mathrm{div}(F(u)\varphi_6(\nabla u)),\Delta u))|$$

$$+|((\omega f'(u),\Delta u))+((\Delta f(u),\Delta u))|\le Q(\epsilon^{-1},\|u\|_{H^2(\Omega)})$$

(here, we have used the facts that the φ_i''s are bounded and that $\frac{1}{\epsilon+|s|^2}\le\epsilon^{-1}$ and $\frac{|s|^2}{\epsilon+|s|^2}\le 1$), hence,

$$\frac{d}{dt}(\|\nabla u\|^2+\|\Delta u\|^2)+c\|u\|^2_{H^3(\Omega)}\le Q(\epsilon^{-1},\|u\|_{H^2(\Omega)}),\ c>0. \qquad (3.17)$$

We finally multiply (2.1) by $-\Delta\frac{\partial u}{\partial t}$ and find, owing to (2.2),

$$\|\nabla\frac{\partial u}{\partial t}\|^2+\|\Delta\frac{\partial u}{\partial t}\|^2+((\mathrm{div}\varphi_1(\nabla u),\Delta\frac{\partial u}{\partial t}))-((\varphi_2(\nabla u)f(u),\Delta\frac{\partial u}{\partial t}))$$

$$+((\mathrm{div}\varphi_3(\nabla u),\Delta\frac{\partial u}{\partial t}))+((\mathrm{div}(F(u)\varphi_4(\nabla u)),\Delta\frac{\partial u}{\partial t}))$$

$$-((\mathrm{div}\varphi_5(\nabla u),\Delta\frac{\partial u}{\partial t}))-((\mathrm{div}(F(u)\varphi_6(\nabla u)),\Delta\frac{\partial u}{\partial t}))$$

$$-((\omega f'(u), \Delta\frac{\partial u}{\partial t})) + ((\Delta f(u), \Delta\frac{\partial u}{\partial t})) + \frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2 = 0,$$

which yields, proceeding as above,

$$\frac{d}{dt}\|\nabla\Delta u\|^2 + c\|\frac{\partial u}{\partial t}\|^2_{H^2(\Omega)} \le Q(\epsilon^{-1}, \|u\|_{H^2(\Omega)}),\ c > 0. \tag{3.18}$$

4 Existence and uniqueness of solutions

We have the

Theorem 1. We assume that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. Then, (2.1)-(2.4) possesses a unique solution u such that $u \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap L^2(0, T; H^3(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0, T; H^1_0(\Omega))$, $\forall T > 0$. Furthermore, if $u_0 \in H^3(\Omega) \cap H^1_0(\Omega)$, then $u \in L^\infty(0, T; H^3(\Omega) \cap H^1_0(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$, $\forall T > 0$.

Proof. **a) Existence:**

The proof of existence is based on the a priori estimates derived in the previous section and, e.g., a classical Galerkin scheme.

In particular, we first deduce from (3.10) that $u \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^2(\Omega))$, $\forall T > 0$. Having this, it follows from (3.15) that $u \in L^\infty(0, T; H^2(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0, T; H^1_0(\Omega))$.

The only difficulty here is to pass to the limit in the nonlinear terms when considering Galerkin approximations. More precisely, we have, owing to classical Aubin-Lions compactness results, a sequence u_m of solutions to approximated problems such that

$$u_m \to u \text{ in } L^\infty(0, T; H^2(\Omega)) \text{ weak star},\ L^2(0, T; H^1(\Omega)) \text{ and a.e..}$$

We consider, for instance, the passage to the limit in the term $F(u_m)\varphi_4(\nabla u_m)$ (the other terms can be handled similarly or are simpler to treat). We have

$$|F(u_m)\varphi_4(\nabla u_m) - F(u)\varphi_4(\nabla u)|$$

$$\le |F(u_m)(\varphi_4(\nabla u_m) - \varphi_4(\nabla u))| + |(F(u_m) - F(u))\varphi_4(\nabla u)|,$$

so that, proceeding as in the previous section (using, in particular, the fact that φ_4' is bounded),

$$\|F(u_m)\varphi_4(\nabla u_m) - F(u)\varphi_4(\nabla u)\| \le Q(\epsilon^{-1}, \|u_m\|_{H^2(\Omega)}, \|u\|_{H^2(\Omega)})\|u_m - u\|_{H^1(\Omega)},$$

hence the convergence in $L^2(0,T;L^2(\Omega))$. Furthermore, noting that ω_m (defined as in (2.2)) converges to ω in $L^\infty(0,T;L^2(\Omega))$ weak star and $f'(u_m)$ converges to $f'(u)$ in $L^2(0,T;H^1(\Omega))$, we easily see that $\omega_m f'(u_m)$ converges to $\omega f'(u)$ in $L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(\Omega))$ weak.

b) Uniqueness:

Let u_1 and u_2 be two solutions to (2.1)-(2.3) (ω_1 and ω_2 being defined as in (2.2)) with initial data $u_{0,1}$ and $u_{0,2}$, respectively. We set $u = u_1 - u_2$, $\omega = \omega_1 - \omega_2$, $u_0 = u_{0,1} - u_{0,2}$ and have

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} - \mathrm{div}(\varphi_1(\nabla u_1) - \varphi_1(\nabla u_2)) + \varphi_2(\nabla u_1)f(u_1) - \varphi_2(\nabla u_2)f(u_2) \tag{4.1}$$

$$-\mathrm{div}(\varphi_3(\nabla u_1) - \varphi_3(\nabla u_2)) - \mathrm{div}(F(u_1)\varphi_4(\nabla u_1) - F(u_2)\varphi_4(\nabla u_2))$$

$$+\mathrm{div}(\varphi_5(\nabla u_1) - \varphi_5(\nabla u_2)) + \mathrm{div}(F(u_1)\varphi_6(\nabla u_1) - F(u_2)\varphi_6(\nabla u_2))$$

$$+\omega_1 f'(u_1) - \omega_2 f'(u_2) - \Delta\omega = 0,$$

$$\omega = -\Delta u + f(u_1) - f(u_2), \tag{4.2}$$

$$u = \omega = 0 \text{ on } \Gamma, \tag{4.3}$$

$$u|_{t=0} = u_0. \tag{4.4}$$

We multiply (4.1) by u and obtain, owing to (4.2),

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|\nabla u\|^2) + ((\varphi_1(\nabla u_1) - \varphi_1(\nabla u_2), \nabla u)) \tag{4.5}$$

$$+((\varphi_2(\nabla u_1)f(u_1) - \varphi_2(\nabla u_2)f(u_2), u)) + ((\varphi_3(\nabla u_1) - \varphi_3(\nabla u_2), \nabla u))$$

$$+((F(u_1)\varphi_4(\nabla u_1) - F(u_2)\varphi_4(\nabla u_2)), \nabla u)) - ((\varphi_5(\nabla u_1) - \varphi_5(\nabla u_2), \nabla u))$$

$$-((F(u_1)\varphi_6(\nabla u_1) - F(u_2)\varphi_6(\nabla u_2)), \nabla u)) + ((\omega_1 f'(u_1) - \omega_2 f'(u_2), u))$$

$$+((f(u_1)-f(u_2),u))+\|\Delta u\|^2=0.$$

We have, for instance (again, the other terms can be handled similarly or are easier to treat) and proceeding as in the previous section,

$$|((F(u_1)\varphi_4(\nabla u_1)-F(u_2)\varphi_4(\nabla u_2)),\nabla u))|\leq|((F(u_1)(\varphi_4(\nabla u_1)-\varphi_4(\nabla u_2)),\nabla u))|$$

$$+|(((F(u_1)-F(u_2))\varphi_4(\nabla u_2),\nabla u))|\leq Q(\epsilon^{-1},T,\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})\|\nabla u\|^2.$$

Furthermore, owing to (4.2) and a proper interpolation inequality,

$$|((\omega_1 f'(u_1)-\omega_2 f'(u_2),u))|\leq|((\omega f'(u_1),u))|+|((\omega_2(f'u_1)-f'(u_2)),u))|$$

$$\leq Q(T,\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})\|\Delta u\|\|u\|.$$

We thus find an inequality of the form

$$\frac{d}{dt}(\|u\|^2+\|\nabla u\|^2)\leq Q(\epsilon^{-1},T,\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})\|\nabla u\|^2,$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the H^1-norm.

□

Remark 4.1. The viscous Cahn-Hilliard system associated with the free energy (1.6) reads

$$\frac{\partial u}{\partial t}-\alpha\frac{\partial\Delta u}{\partial t}-\Delta[-\mathrm{div}(\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\nabla u)+\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})f(u) \tag{4.6}$$

$$-\frac{1}{2}\mathrm{div}(\frac{|\nabla u|^2}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}))-\mathrm{div}(\frac{F(u)}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}))$$

$$+\frac{1}{2}\mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\cdot\nabla u\frac{|\nabla u|^2\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{3}{2}}})+\mathrm{div}(\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\cdot\nabla u\frac{F(u)\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{3}{2}}})$$

$$+\beta\omega f'(u)-\beta\Delta\omega]=0,\ \ \alpha>0,$$

$$\omega = -\Delta u + f(u). \tag{4.7}$$

Taking, for simplicity, Dirichlet boundary conditions,

$$u = \Delta u = \omega = \Delta\omega = 0 \text{ on } \Gamma,$$

we can rewrite (4.6) equivalently as

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial t} - \operatorname{div}(\delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\nabla u) + \delta(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})f(u) \tag{4.8}$$

$$-\frac{1}{2}\operatorname{div}(\frac{|\nabla u|^2}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})) - \operatorname{div}(\frac{F(u)}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}}))$$

$$+\frac{1}{2}\operatorname{div}(\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\cdot\nabla u\frac{|\nabla u|^2\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{3}{2}}}) + \operatorname{div}(\delta'(\frac{\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{1}{2}}})\cdot\nabla u\frac{F(u)\nabla u}{(\epsilon+|\nabla u|^2)^{\frac{3}{2}}})$$

$$+\beta\omega f'(u) - \beta\Delta\omega = 0.$$

Even though (4.8) bears some resemblance with (2.1), we have less regularity on $\frac{\partial u}{\partial t}$ and thus cannot proceed as above to prove the existence and uniqueness of solutions. We can however prove the well-posedness in one space dimension (see [5]).

Received: May 2013. Accepted: May 2013.

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