Gronwall-Bellman type integral inequalities and applications to global uniform asymptotic stability

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ABSTRACT

In this paper, some new nonlinear generalized Gronwall-Bellman-Type integral inequalities are established. These inequalities can be used as handy tools to research stability problems of perturbed dynamic systems. As applications, based on these new established inequalities, some new results of practical uniform stability are also given. A numerical example is presented to illustrate the validity of the main results.

RESUMEN

En este artículo, establecemos algunas desigualdades integrales nolineales nuevas de tipo Gronwall-Bellman. Estas desigualdades pueden ser usadas como herramientas utiles para estudiar problemas de estabilidad de sistemas dinámicos perturbados. Como aplicaciones, basados en las nuevas desigualdades establecidas, también damos algunos resultados nuevos de estabilidad uniforme prácticos. Un ejemplo numérico es presentado para ilustrar la validez de los resultados principales.

1 Introduction

In 1919, T.H. Gronwall [6] proved a remarkable inequality which has attracted and continues to attract considerable attention in the literature. In the qualitative theory of differential, the Gronwall type inequalities of one variable for the real functions play a very important role. The first use of the Gronwall inequality to establish boundedness and uniqueness is due to R. Bellman [1] . Gronwall-Bellman inequality, which is usually proved in elementary differential equations using continuity arguments, is an important tool in the study of of qualitative behavior of solutions of differential and stability.

The problem of stability analysis of nonlinear time-varying systems has attracted the attention of several researchers and has produced a vast body of important results (see [2]-[15] and the references therein). In this paper, we present a new generalization of the Gronwall- Bellman lemma. This new generalization can develop a simple command to exponentially stabilize a large class of nonlinear systems. In this paper, some new nonlinear generalized Gronwall-Bellman-Type integral inequalities are given. As applications, we give some new classes of time-varying perturbed systems which are globally uniformly practically asymptotically stable. Moreover, we give an example to illustrate the applicability of the results.

2 Definitions and notations

We consider the following system :

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where $t \in \mathbb{R}_+$ is the time and $x \in \mathbb{R}^n$ is the state.

Definition 1. *(uniform boundedness).*
A solution of (1) *is said to be globally uniformly bounded if for every* $\alpha > 0$ *there exists* $c = c(\alpha)$ *such that, for all* $t_0 \geq 0$,

$$\parallel x_0 \parallel \leq \alpha \Rightarrow \parallel x(t) \parallel \leq c, \quad \forall t \geq t_0.$$

Let $r \geq 0$ *and* $B_r = \{x \in \mathbb{R}^n / \parallel x \parallel \leq r\}$. *First, we give the definition of uniform stability and uniform attractivity of* B_r.

Definition 2. *(uniform stability of* B_r).

i. B_r *is uniformly stable if for all* $\varepsilon > r$, *there exists* $\delta = \delta(\varepsilon) > 0$ *such that for all* $t_0 \geq 0$,

$$\parallel x_0 \parallel \leq \delta \Rightarrow \parallel x(t) \parallel \leq \varepsilon, \quad \forall t \geq t_0.$$

ii. B_r *is globally uniformly stable if it is uniformly stable and the solutions of system* (1) *are globally uniformly bounded.*

Definition 3. *(uniform attractivity of* B_r*).*
B_r *is globally uniformly attractive if for all* $\varepsilon > r$ *and* c, *there exists* $T(\varepsilon, c) > 0$ *such that for all* $t_0 \geq 0$,

$$\| x(t) \| \leq \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon, c), \| x_0 \| \leq c$$

The system (1) is globally uniformly practically asymptotically stable if there exists $r \geq 0$ such that B_r is globally uniformly stable and globally uniformly attractive.

Definition 4. *A continuous function* $\alpha : [0, a) \to [0, +\infty)$ *is said to belong to class* K *if it is strictly increasing and* $\alpha(0) = 0$. *It is said to belong to class* K_∞ *if* $a = +\infty$ *and* $\alpha(r) \to +\infty$ *as* $r \to +\infty$.

Definition 5. *A continuous function* $\beta : [0, a) \times [0, +\infty) \to [0, +\infty)$ *is said to belong to class* KL *if, for each fixed* s, *the mapping* $\beta(r, s)$ *belongs to class* K *with respect to* r *and, for each* r, *the mapping* $\beta(r, s)$ *is decreasing with respect to* s *and* $\beta(r, s) \to 0$ *as* $s \to +\infty$.

Proposition 1. *If there exists a class* K-*function* α, *and a constant* $r > 0$ *such that, given any initial state* x_0, *the solution satisfies*

$$\| x(t) \| \leq \alpha(\| x_0 \|) + r \quad \forall t \geq t_0,$$

then the system (1) *is globally uniformly practically stable.*

Proposition 2. *If there exist a class* KL-*function* β, *a constant* $r > 0$ *such that, given any initial state* x_0, *the solution satisfies*

$$\| x(t) \| \leq \beta(\| x_0 \|, t - t_0) + r \quad \forall t \geq t_0,$$

then the system (1) *is globally uniformly practically asymptotically stable.*

The next definition concerns a special case of practical global uniform asymptotic stability, namely, if the class KL in the above proposition consists of functions $\beta(r, s) = kr e^{-\gamma s}$.

Definition 6. B_r *is globally uniformly exponentially stable if there exist* $\gamma > 0$ *and* $k \geq 0$ *such that for all* $t_0 \in \mathbb{R}_+$ *and* $x_0 \in \mathbb{R}^n$,

$$\| x(t) \| \leq k \| x_0 \| \exp(-\gamma(t - t_0)) + r \quad \forall t \geq t_0.$$

System (1) is globally practically uniformly exponentially stable if there exist $r > 0$ such that B_r is globally uniformly exponentially stable.

3 Basic results

Lemma 1. *Let* u, v *and* w *nonnegative piecewise continuous functions on* $[0, +\infty)$ *for which the inequality*

$$u(t) \leq c + \int_a^t (uv + w) \quad \forall t \geq a \tag{2}$$

holds, where a *and* c *are nonnegative constants. Then,*

$$u(t) \leq ce^{\int_a^t v} + re^{\int_a^t (v + \frac{w}{r})} \quad \forall t \geq a, \forall r > 0. \tag{3}$$

Proof

It follows from (2) and the classic inequality

$$e^x > x + 1 \quad \forall x > 0$$

that for all $r > 0$ and $t \geq a$

$$0 \leq u(t) < (c + re^{\int_a^t \frac{w}{r}}) + \int_a^t uv \tag{4}$$

which implies that

$$\frac{u(t)}{c + re^{\int_a^t \frac{w}{r}} + \int_a^t uv} \leq 1.$$

Since $v \geq 0$, we obtain

$$\frac{u(t)v(t) + w(t)e^{\int_a^t \frac{w}{r}}}{c + re^{\int_a^t \frac{w}{r}} + \int_a^t uv} \leq v(t) + \frac{w(t)e^{\int_a^t \frac{w}{r}}}{c + re^{\int_a^t \frac{w}{r}}} \tag{5}$$

then we take

$$f(t) = \int_a^t v + \log(c + re^{\int_a^t \frac{w}{r}}) - \log(c + re^{\int_a^t \frac{w}{r}} + \int_a^t uv) \quad \forall t \geq a.$$

It is clear that f is defined, continuous and piecewise continuously differentiable on $[a, +\infty)$. Consequently, we get for all $b > a$, a sequence $\{a_0, ..., a_n\}$ of $[a, b]$ verifying

$$f'(t) = v(t) + \frac{w(t)e^{\int_a^t \frac{w}{r}}}{c + re^{\int_a^t \frac{w}{r}}} - \frac{w(t)e^{\int_a^t \frac{w}{r}} + u(t)v(t)}{c + re^{\int_a^t \frac{w}{r}} + \int_a^t uv} \quad \forall t \in [a, b] - \{a_0, ..., a_n\}.$$

By using the inequality (5), we obtain

$$f'(t) \geq 0.$$

Thus, f is increasing on the intervals $[a, a_0), ...(a_n, b]$. Since f is continuous on $[a, b]$, then f is increasing on $[a, b]$. Consequently, we get

$$f(b) \geq f(a)$$

however, $f(a) = 0$ which implies that

$$f(b) \geq 0 \quad \forall b \geq a.$$

Consequently

$$\log(c + re^{\int_a^t \frac{w}{r}} + \int_a^t uv) \leq \int_a^t v + \log(c + re^{\int_a^t \frac{w}{r}}) \quad \forall t \geq a$$

hence

$$c + re^{\int_a^t \frac{w}{r}} + \int_a^t uv \leq (c + re^{\int_a^t \frac{w}{r}})e^{\int_a^t v}$$

by using the inequality (4), we have finally

$$u(t) \leq ce^{\int_a^t v} + re^{\int_a^t (v + \frac{w}{r})}.$$

Lemma 2. *Let $\phi \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ where $p \in [1, +\infty]$. We denote by $\| \phi \|_p$ the p-norm of ϕ. Then, for all $t_0 \geq 0$, $s \geq 0$ and $t \geq t_0$*

$$\int_{t_0}^t \phi \leq N + L(t - t_0) \tag{6}$$

where $N = \int_0^s \phi + \frac{M_s}{p}$ and $L = \frac{p-1}{p}M_s$ with $M_s = \| \phi_{|[s, +\infty)} \|_p$.

Proof

We first consider the case $p \in (1, +\infty)$. By using Hölder inequality to the function ϕ, we obtain for all $t \geq t_0$:

$$\int_{t_0}^t \phi \leq (\int_{t_0}^t \phi^p)^{\frac{1}{p}}(\int_{t_0}^t 1)^{\frac{p-1}{p}}$$

$$\leq (t - t_0)^{\frac{p-1}{p}}(\int_{t_0}^{+\infty} \phi^p)^{\frac{1}{p}}.$$

We put

$$f(x) = \frac{1}{p} + \frac{p-1}{p}x - x^{\frac{p-1}{p}} \quad \forall x > 0$$

then, f is differentiable on $(0, +\infty)$ and verifying

$$f'(x) = \frac{p-1}{p}(1 - x^{-\frac{1}{p}}).$$

Hence, f is decreasing on $(0, 1]$ and increasing on $[1, +\infty)$. Since $f(1) = 0$, we conclude that f is positive on $(0, +\infty)$ which means that

$$x^{\frac{p-1}{p}} \leq \frac{1}{p} + \frac{p-1}{p}x \quad \forall x > 0$$

consequently, we have

$$(t - t_0)^{\frac{p-1}{p}} \leq \frac{1}{p} + \frac{p-1}{p}(t - t_0) \quad \forall t \geq t_0$$

then

$$0 \leq \int_{t_0}^t \phi \leq M_{t_0}[\frac{1}{p} + \frac{p-1}{p}(t - t_0)] \tag{7}$$

where $M_{t_0} = \| \phi_{|[t_0, +\infty)} \|_p$. This inequality holds also for $p \in \{1, +\infty\}$.

Now, for all $t_0 \geq 0$, $s \geq 0$ and $t \geq t_0$, we distingue three cases :

- $s \leq t_0 \leq t$

In view of (7), we obtain

$$\int_{t_0}^{t} \varphi \leq M_{t_0}[\frac{1}{p} + \frac{p-1}{p}(t - t_0)]$$

$$\leq \frac{M_s}{p} + \frac{p-1}{p}(t - t_0)M_s.$$

Now, since $\int_0^s \varphi \geq 0$, we obtain

$$\int_{t_0}^{t} \varphi \leq (\int_0^s \varphi + \frac{M_s}{p}) + \frac{p-1}{p}(t - t_0)M_s.$$

- $t_0 < s \leq t$

We can write by using (7)

$$\int_{t_0}^{t} \varphi \leq \int_{t_0}^{s} \varphi + \int_{s}^{t} \varphi$$

$$\leq \int_0^s \varphi + [\frac{1}{p} + \frac{p-1}{p}(t - s)]M_s.$$

then

$$\int_{t_0}^{t} \varphi \leq \int_0^s \varphi + \frac{M_s}{p} + \frac{p-1}{p}(t - s)M_s$$

however, $s \in (t_0, t]$ then

$$\int_{t_0}^{t} \varphi \leq (\int_0^s \varphi + \frac{M_s}{p}) + \frac{p-1}{p}(t - t_0)M_s.$$

- $t_0 \leq t < s$

It is clear that

$$\int_{t_0}^{t} \varphi \leq \int_0^s \varphi$$

$$\leq (\int_0^s \varphi + \frac{M_s}{p}) + \frac{p-1}{p}(t - t_0)M_s.$$

then the lemma is proved.

Lemma 3. *Consider the differential system*

$$\dot{x}(t) = A(t)x(t) + h(t, x(t)) \tag{8}$$

where :

i. A *is an* $n \times n$ *matrix whose entries are all real-valued piecewise-continuous functions of* $t \in \mathbb{R}_+$.

ii. *The function* h *is defined on* $\mathbb{R}_+ \times \mathbb{R}^n$, *piecewise continuous in t, and locally Lipshitz in x.*

iii. *There exist ϕ and ε piecewise continuous functions, positives and verifying*

$$\| h(t,x) \|\le \phi(t) \| x \| +\varepsilon(t) \quad \forall t \in \mathbb{R}_+. \tag{9}$$

Then, for all $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exist a unique maximal solution x of (8) such that $x(t_0) = x_0$. Moreover, x is defined on $[t_0,+\infty)$.

Proof
It is clear that the system (8) can be written

$$\dot{x}(t) = f(t,x(t))$$

where

$$f(t,x) = A(t)x + h(t,x).$$

The function f is piecewise continuous in t and locally Lipshitz in x, then we have : For all $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exist a unique maximal solution x of (8) such that $x(t_0) = x_0$.
We will prove that x is defined on $[t_0,+\infty)$. Supposed that is not true, then there exist $T_{max} \in (t_0,+\infty)$ such that x is defined on $[t_0,T_{max})$. Then, for all $t \in [t_0,T_{max})$

$$\| \dot{x}(t) \|\le (M_1 + M_2) \| x(t) \| +M_3$$

where

$$M_1 = \sup_{[t_0,T_{max}]} \| A(t) \|,$$

$$M_2 = \sup_{[t_0,T_{max}]} \| \phi(t) \|$$

and

$$M_3 = \sup_{[t_0,T_{max}]} \| \varepsilon(t) \| .$$

It is clear that M_1, M_2 and $M_3 \in \mathbb{R}_+$, therefore

$$\begin{aligned} \| \int_{t_0}^{t} \dot{x}(s)ds \| &\le \int_{t_0}^{t} \| \dot{x}(s) \| ds \\ &\le \int_{t_0}^{t} [(M_1 + M_2) \| x(s) \| +M_3]ds \end{aligned}$$

then

$$\| x(t) \|\le\| x(t_0) \| + \int_{t_0}^{t} [(M_1 + M_2) \| x(s) \| +M_3]ds$$

By using the lemma 1, we obtain for all $t \in [t_0,T_{max})$

$$\begin{aligned} \| x(t) \| &\le \| x(t_0) \| e^{\int_{t_0}^{t} (M_1+M_2)ds} + e^{\int_{t_0}^{t} (M_1+M_2+M_3)ds} \\ &\le M_4 \end{aligned}$$

where
$$M_4 = \| x(t_0) \| e^{(M_1+M_2)T_{max}} + e^{(M_1+M_2+M_3)T_{max}}.$$

Consequently, x remains within the compact B_{M_4}, which is impossible. So, we conclude that

$$T_{max} = +\infty.$$

Theorem 1. *Consider the following time-varying :*

$$\dot{x}(t) = A(t)x(t) + h(t, x(t)) \tag{10}$$

where :

(1) *A is an $n \times n$ matrix whose entries are all real-valued piecewise-continuous functions of $t \in \mathbb{R}_+$.*

(2) *The transition matrix for the system*

$$\dot{x} = A(t)x$$

satisfies :

$$\| R(t,s) \| \le k e^{-\gamma(t-s)} \quad \forall (t,s) \in \mathbb{R}_+^2 \tag{11}$$

for some $k > 0$ and $\gamma > 0$.

(3) *The function h is defined on $\mathbb{R}_+ \times \mathbb{R}^n$, piecewise continuous in t, and locally Lipshitz in x.*

(4) *There exist ϕ and ε piecewise continuous functions, positives and verifying*

$$\| h(t,x) \| \le \phi(t) \| x \| + \varepsilon(t) \quad \forall t \in \mathbb{R}_+. \tag{12}$$

(5) *$\phi \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ for some $p \in [1, +\infty)$.*

(6) *There exist a constant $M > 0$ such that*

$$\varepsilon(t) \le M e^{-\gamma t}. \tag{13}$$

Then for all $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the maximal solution x of (10) such that $x(t_0) = x_0$, is verifying :

i. *The function x is defined on $[t_0, +\infty)$.*

ii. *For all $t \ge t_0$*

$$\| x(t) \| \le L \| x_0 \| e^{-\delta(t-t_0)} + N e^{-\theta t}$$

where $N, L > 0$ and $\delta, \theta \in (0, \gamma]$.

Proof of theorem 1

i. By using the lemma 3, we proved that the system (10) has a unique maximal solution x such that $x(t_0) = x_0$. Moreover, x is defined on $[t_0, +\infty)$.

ii. We can write the solution x of (10) as

$$x(t) = R(t, t_0)x(t_0) + \int_{t_0}^t R(t, s)h(s, x(s))ds$$

where $R(t, t_0)$ is the transition matrix of the system

$$\dot{x} = A(t)x.$$

Further, we have :

$$\begin{aligned} \| x(t) \| &\leq \| R(t, t_0) \| \| x(t_0) \| + \int_{t_0}^t \| R(t, s) \| \| h(s, x(s)) \| ds \\ &\leq ke^{-\gamma(t-t_0)} \| x_0 \| + \int_{t_0}^t ke^{-\gamma(t-s)} \| h(s, x(s)) \| ds. \end{aligned}$$

From the inequalities (11) and (12), we deduce that

$$u(t) \leq ku(t_0) + \int_{t_0}^t [k\phi(s)u(s) + ke^{\gamma s}\varepsilon(s)]ds$$

where

$$u(t) = e^{\gamma t} \| x(t) \| .$$

Now by the lemma 1, we get

$$u(t) \leq ku(t_0)e^{\int_{t_0}^t k\phi} + re^{\int_{t_0}^t [k\phi(s) + \frac{ke^{\gamma s}\varepsilon(s)}{r}]ds} \qquad \forall t \geq t_0, \forall r > 0$$

since

$$\| x(t) \| = e^{-\gamma t}u(t)$$

we obtain the estimation

$$\| x(t) \| \leq k \| x_0 \| e^{\int_{t_0}^t k\phi - \gamma(t-t_0)} + re^{\int_{t_0}^t [k\phi(s) + \frac{ke^{\gamma s}\varepsilon(s)}{r}]ds - \gamma t}. \tag{14}$$

Let us denote

$$M = \sup_{t \geq 0}[e^{\gamma t}\varepsilon(t)] \quad \text{and} \quad M_s = (\int_s^{+\infty} \phi^p)^{\frac{1}{p}}$$

we deduce from the assumptions 5 and 6, that

$$M, M_s \in \mathbb{R}_+$$

it follows that

$$\int_{t_0}^{t} \frac{ke^{\gamma s}\varepsilon(s)}{r}ds \leq \frac{kM}{r}t \quad \forall t \geq t_0. \tag{15}$$

Moreover $\phi \in L^p(\mathbb{R}_+, \mathbb{R}_+)$, then

$$\int_{t}^{+\infty} \phi^p \xrightarrow[t\to+\infty]{} 0$$

and so there exist $s \geq 0$ such that

$$M_s < \frac{\gamma}{k}\frac{p}{p-1}.$$

By using the lemma 2, we find for all $t \geq t_0$

$$\int_{t_0}^{t} \phi \leq \int_{0}^{s} \phi + \frac{M_s}{p} + M_s\frac{p-1}{p}(t-t_0) \tag{16}$$

from (15) and (16), we get :

$$\int_{t_0}^{t} k\phi - \gamma(t-t_0) \leq k(\int_{0}^{s} \phi + \frac{M_s}{p}) + [kM_s\frac{p-1}{p} - \gamma](t-t_0)$$

and

$$\int_{t_0}^{t} [k\phi(s) + \frac{ke^{\gamma s}\varepsilon(s)}{r}]ds - \gamma t \leq [-\gamma + kM_s\frac{p-1}{p} + \frac{kM}{r}]t + k(\int_{0}^{s} \phi + \frac{M_s}{p}).$$

Thus, (14) yields

$$\| x(t) \| \leq ke^{k(\int_{0}^{s} \phi + \frac{M_s}{p})} \| x_0 \| e^{-[\gamma - kM_s\frac{p-1}{p}](t-t_0)} + re^{-[\gamma - kM_s\frac{p-1}{p} - \frac{kM}{r}]t + k(\int_{0}^{s} \phi + \frac{M_s}{p})}.$$

Taking

$$r > \frac{M}{\frac{\gamma}{k} - \frac{p-1}{p}M_s}$$

$$L = ke^{k(\frac{M_s}{p} + \int_{0}^{s} \phi)}$$

$$N = re^{k(\frac{M_s}{p} + \int_{0}^{s} \phi)} = \frac{r}{k}L$$

$$\delta = \gamma - k\frac{p-1}{p}M_s \in (0, \gamma]$$

$$\theta = \gamma - k\frac{p-1}{p}M_s - \frac{kM}{r} \in (0, \delta).$$

Finally, we obtain

$$\| x(t) \| \leq L \| x_0 \| e^{-\delta(t-t_0)} + Ne^{-\theta t} \quad \forall t \geq t_0.$$

Corollary 1. *Under the same assumptions of theorem* 1*, we get*
$\forall r > 0, \forall t \geq t_0, \forall x_0 \in \mathbb{R}^n \setminus B_r$:

$$\| x(t) \| \leq P \| x_0 \| e^{-\theta(t-t_0)}$$

where $P > 0$ *and* $\theta \in (0, \gamma)$.

Proof

Due to theorem 1, we have

$$\| x(t) \| \le L \| x_0 \| e^{-\delta(t-t_0)} + N e^{-\theta t} \quad \forall t \ge t_0.$$

Let $r > 0$, then for all $x_0 \in \mathbb{R}^n \setminus B_r$

$$\begin{aligned} \| x(t) \| &\le& L \| x_0 \| e^{-\delta(t-t_0)} + \frac{N}{r} r e^{-\theta(t-t_0)} \\ &\le& (L + \frac{N}{r}) \| x_0 \| e^{-\theta(t-t_0)}. \end{aligned}$$

Taking

$$P = L + \frac{N}{r} > 0,$$

we obtain

$$\| x(t) \| \le P \| x_0 \| e^{-\theta(t-t_0)}.$$

Remark 1. *Take limit as* $r \to \dfrac{M}{\frac{\gamma}{k} - \frac{p-1}{p} M_s}$ *in theorem 1, we obtain*

$$\| x(t) \| \le L \| x_0 \| e^{-\delta(t-t_0)} + N \quad \forall t \ge t_0 \tag{17}$$

with

$$N = \frac{M}{\frac{\gamma}{k} - \frac{p-1}{p} M_s} e^{k(\frac{M_s}{p} + \int_0^s \phi)}.$$

In particular, if we choose $p = 1$*, we find*

$$\| x(t) \| \le L \| x_0 \| e^{-\delta(t-t_0)} + N \quad \forall t \ge t_0 \tag{18}$$

with

$$L = k e^{k \| \phi \|_1}$$

and

$$N = \frac{kM}{\gamma} e^{k \| \phi \|_1}.$$

The estimation (17) *and* (18) *implies that the system* (10) *is globally uniformly practically asymptotically stable in the sense that the ball* B_N *is globally uniformly asymptotically stable.*

Theorem 2. *Consider the following time-varying :*

$$\dot{x}(t) = A(t)x(t) + h(t, x(t)) \tag{19}$$

where :

(1) A *is an* $n \times n$ *matrix whose entries are all real-valued piecewise-continuous functions of* $t \in \mathbb{R}_+$.

(2) *The transition matrix for the system*

$$\dot{x} = A(t)x$$

satisfies :

$$\parallel R(t,s) \parallel \le ke^{-\gamma(t-s)} \quad \forall (t,s) \in \mathbb{R}_+^2 \tag{20}$$

for some $k > 0$ *and* $\gamma > 0$.

(3) *The function* h *is defined on* $\mathbb{R}_+ \times \mathbb{R}^n$, *piecewise continuous in t, and locally Lipshitz in x.*

(4) *There exist* ϕ *and* ε *piecewise continuous functions, positives and verifying*

$$\parallel h(t,x) \parallel \le \phi(t) \parallel x \parallel + \varepsilon(t) \quad \forall t \in \mathbb{R}_+. \tag{21}$$

(5) $\sup_{[s,+\infty)} \phi < \dfrac{\gamma}{k}$ *for some* $s \in [0,+\infty)$.

(6) *There exist a constant* $M > 0$ *such that*

$$\varepsilon(t) \le Me^{-\gamma t}. \tag{22}$$

Then for all $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, *the maximal solution* x *of* (10) *such that* $x(t_0) = x_0$, *is verifying :*

i. *The function* x *is defined on* $[t_0, +\infty)$.

ii. *For all* $t \ge t_0$

$$\parallel x(t) \parallel \le L \parallel x_0 \parallel e^{-\delta(t-t_0)} + Ne^{-\theta t}$$

where $N, L > 0$ *and* $\delta, \theta \in (0, \gamma]$.

Proof of theorem 2
i. By using the lemma 3, we proved that the system (19) has a unique maximal solution x such that $x(t_0) = x_0$. Moreover, x is defined on $[t_0, +\infty)$.
ii. Similar to the proof of theorem 1, it can be shown that :

$$\parallel x(t) \parallel \le k \parallel x_0 \parallel e^{\int_{t_0}^{t} k\phi - \gamma(t - t_0)} + re^{\int_{t_0}^{t} [k\phi(s) + \frac{ke^{\gamma s}\varepsilon(s)}{r}]ds - \gamma t} \quad \forall t \ge t_0, \forall r > 0.$$

Let us denote

$$M = \sup_{t \ge 0}[e^{\gamma t}\varepsilon(t)] \in \mathbb{R}_+$$

it follows that

$$\int_{t_0}^{t} \frac{ke^{\gamma s}\varepsilon(s)}{r}ds \le \frac{kM}{r}t \quad \forall t \ge t_0.$$

Hence, there exist $s \in \mathbb{R}_+$ such that

$$\sup_{[s,+\infty)} \phi < \frac{\gamma}{k}$$

then we can apply the lemma 2, we deduce that

$$\int_{t_0}^{t} \phi \leq \int_{0}^{s} \phi + (\sup_{[s,+\infty)} \phi)(t - t_0) \quad \forall t \geq t_0$$

consequently, we obtain

$$\| x(t) \| \leq k e^{k \int_{0}^{s} \phi} \| x_0 \| e^{-[\gamma - k \sup_{[s,+\infty)} \phi](t - t_0)} + r e^{-[\gamma - k \sup_{[s,+\infty)} \phi - \frac{kM}{r}]t + k \int_{0}^{s} \phi}.$$

Taking

$$r > \frac{M}{\frac{\gamma}{k} - \sup_{[s,+\infty)} \phi}$$

$$L = k e^{k \int_{0}^{s} \phi}$$

$$N = r e^{k \int_{0}^{s} \phi)} = \frac{r}{k} L$$

$$\delta = \gamma - k \sup_{[s,+\infty)} \phi \in (0, \gamma]$$

$$\theta = \gamma - k \sup_{[s,+\infty)} \phi - \frac{kM}{r} \in (0, \delta).$$

Finally, we obtain

$$\| x(t) \| \leq L \| x_0 \| e^{-\delta(t - t_0)} + N e^{-\theta t} \quad \forall t \geq t_0.$$

Corollary 2. *Under the assumptions* $(1),(2),(3),(4)$ *and* (6) *of theorem 2, and by replacing the condition* (5) *by*

$$(5'): \quad \phi(t) \xrightarrow[t \to +\infty]{} 0$$

then, we obtain the same consequences of theorem 2.

Proof

Since $\lim_{t \to +\infty} \phi(t) = 0$, then there exist $s \geq 0$ such that

$$\forall t \geq s \quad \phi(t) \leq \frac{\gamma}{2k}$$

therefore

$$\sup_{[s,+\infty)} \phi < \frac{\gamma}{k}.$$

Thus, we can apply theorem 2 to prove the result.

Remark 2. *It is clear that if we choose* $\varepsilon = 0$ *in theorem* 1 *or* 2, *we obtain due to* $M = 0$:

$$\theta = \delta = \gamma - k \frac{p-1}{p} M_s$$

$$L = k e^{k(\frac{M_s}{p} + \int_0^s \phi)}$$

$$N = r e^{k(\frac{M_s}{p} + \int_0^s \phi)} \quad \forall r > 0$$

as $r \to 0^+$, *we get the classic result:*

$$\| x(t) \| \leq L \| x_0 \| e^{-\delta(t - t_0)} \quad \forall t \geq t_0.$$

We can see that the claim of the theorem 1 is true by examining a specific example, where a solution of the scalar equation can be found.

Example 1. *Consider the stability of following system :*

$$\begin{cases} \dot{x}_1 = -x_1 - tx_2 + \dfrac{1}{(1+t^2)^2} \dfrac{x_1^2}{1 + \sqrt{x_1^2 + x_2^2}} + \dfrac{e^{-2t}}{1 + x_1^2} \\[4mm] \dot{x}_2 = tx_1 - x_2 + \dfrac{t}{(1+t^2)^2} \dfrac{x_2^2}{1 + \sqrt{x_1^2 + x_2^2}} \end{cases} \tag{23}$$

which can be writing as

$$\dot{x} = A(t)x + h(t,x)$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$A(t) = \begin{pmatrix} -1 & -t \\ t & -1 \end{pmatrix}$$

and

$$h(t,x) = \begin{pmatrix} h_1(t,x) \\ h_2(t,x) \end{pmatrix}$$

with

$$\begin{cases} h_1(t,x) = \dfrac{1}{(1+t^2)^2} \dfrac{x_1^2}{1 + \sqrt{x_1^2 + x_2^2}} + \dfrac{e^{-2t}}{1 + x_1^2} \\[4mm] h_2(t,x) = \dfrac{t}{(1+t^2)^2} \dfrac{x_2^2}{1 + \sqrt{x_1^2 + x_2^2}} \end{cases}$$

it is clear that the system

$$\dot{x} = A(t)x$$

is globally uniformly asymptotically stable. Indeed, the transition matrix $R(t, t_0)$ *satisfies :*

$$
\begin{aligned}
R(t, t_0) &= e^{(t-t_0)A} \\
&= e^{-(t-t_0)} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}
\end{aligned}
$$

thus, we obtain

$$
\| R(t, t_0) \| = k e^{-\gamma(t-t_0)}
$$

with $\gamma = k = 1$ *and* $\| \quad \|$ *represents the euclidean norm. On the other hand,*

$$
\begin{aligned}
\| h(t, x) \| &= h_1^2(t, x) + h_2^2(t, x) \\
&\leq \frac{1}{(1+t^2)^3}(x_1^2 + x_2^2) + 2e^{-2t}.
\end{aligned}
$$

By using the classic inequality

$$
\sqrt{a^2 + b^2} \leq a + b \quad \forall a, b \geq 0
$$

we get

$$
\| h(t, x) \| \leq \phi(t) \| x(t) \| + \varepsilon(t) \quad \forall t \geq 0
$$

where

$$
\phi(t) = \frac{1}{(1+t^2)^{\frac{3}{2}}}
$$

and

$$
\varepsilon(t) = \sqrt{2}e^{-t}.
$$

It is easy to verify that ϕ *and* ε *are continuous, positive and bounded on* $[0, +\infty)$, *in particular*

$$
\phi \in L^p(\mathbb{R}_+, \mathbb{R}_+) \quad \forall p \in [1, +\infty].
$$

To estimate $\| \phi \|_p$, *we use the inequality :*

$$
\phi^p(t) \leq \phi(t) \quad \forall t \geq 0
$$

since $\| \phi \|_\infty = 1$, *then*

$$
\int_0^{+\infty} \phi^p \leq \int_0^{+\infty} \phi
$$

however $\int_0^{+\infty} \phi = 1$, *then*

$$
\| \phi \|_p \leq 1 \quad \forall p \geq 1.
$$

Consequently $\| \phi \|_p < \frac{p}{p-1}$ $\forall p \geq 1$, *and we can apply theorem* 1 *to prove the following results :*
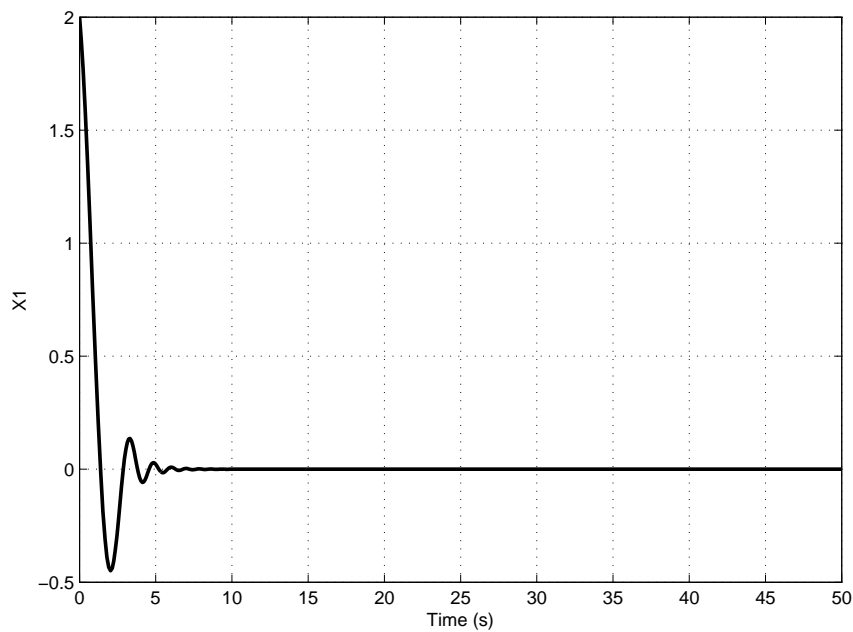• $\forall (t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^2$, *there exist a unique maximal solution* x *of* (8) *such that* $x(t_0) = x_0$.
Moreover, x *is defined on* $[t_0, +\infty)$.
• $\forall t \geq t_0, \forall p \geq 1$

$$
\| x(t) \| \leq e^{\frac{1}{p}} \| x_0 \| e^{-\frac{1}{p}(t-t_0)} + 2\sqrt{2}e^{-\frac{1}{2p}t+\frac{1}{p}}
$$

by choosing $r = 2\sqrt{2}p$.
In particular

$$\| x(t) \| \le e \| x_0 \| e^{-(t-t_0)} + 2\sqrt{2}e. \tag{24}$$

The estimation (24) *implies that the system* (23) *is globally uniformly practically asymptotically stable in the sense that the ball* $B_{2\sqrt{2}e}$ *is globally uniformly asymptotically stable.*

Figure 1:

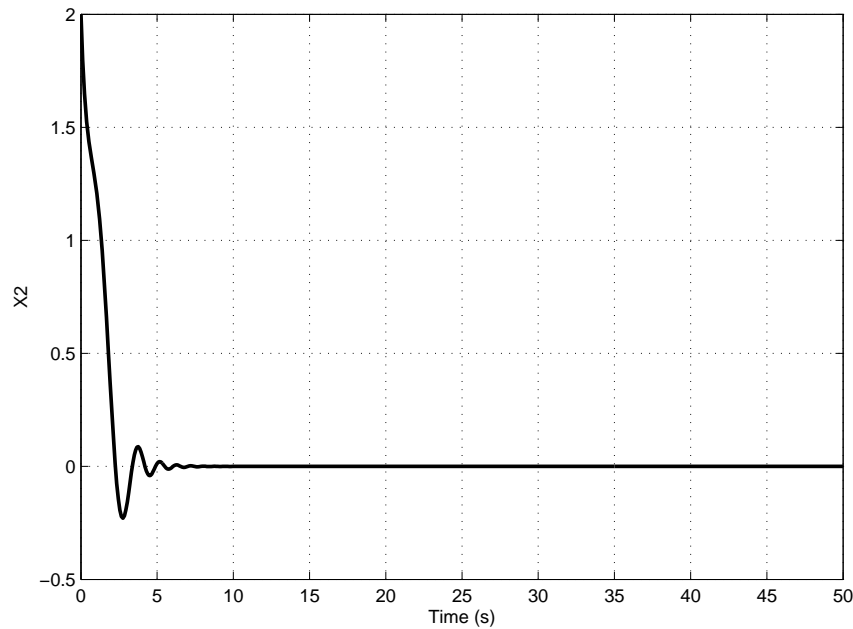Time evolution of the state $x_1(t)$ of system (23)

Figure 2:

Time evolution of the state $x_2(t)$ of system (23)

References

[1] R. Bellman, Stability Theory of Differential Equations, McGraw Hill, New York, (1953).

[2] A. Benabdallah, M. Dlala and M.A. Hammami. A new lyapunov function for stability of time-varying nonlinear perturbed systems, *Systems and Control Letters* 56, (2007) 179-187 .

[3] A.Ben Abdallah, I.Ellouze and M.A.Hammami, Practical stability of nonlinear time-varying cascade systems, *Journal of Dynamical and Control Systems* 15, No. 1, (2009) 45-62.

[4] A.Chaillet and A.Loria, Necessary and sufficient conditions for uniform semiglobal practical asymptotic stability: Application to cascaded systems, *Automatica* 42 (2006), 1899-1906.

[5] M. Corless. Guaranteed rates of exponential convergence for uncertain systems, *J. Optim. Theory Appl.* Vol 64, No 3, (1900) 481-494.

[6] T.H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. Math., 20(2) (1919), 293-296.

[7] M.A. Hammami. On the stability of nonlinear control systems with uncertainly , *J. Dynamical Control systems* 7 (2) (2001) 171-179.

[8] H.K.Khalil, Nonlinear Systems, Prentice-Hall, New York (2002).

[9] Y.Lin, D.Sontag, Y.Wang, A smouth converse Lyapunov theorem for robust stability, *SIAM Journal on control and optimisation* 34 No. 1, (1996), 124-160.

[10] M. Malisoff. Further results on Lyapunov functions and domains of attraction for perturbed asymptotically stable systems, *Dynamics of Continuous, Discrete and Impulsive Systems Ser. A* 12 (2), (2005) 193-225.

[11] X. Mu, D. Cheng, On Stability and Stabilization of Time-varying Nonlinear Control Systems, *Asian J. Control*, Vol. 7, No. 3, (2005) 244-255.

[12] E.Panteley and A.Loria, On global uniform asymptotic stability of nonlinear time-varying systems in cascade, *Systems Control Letters* 33 (1998), 131-138.

[13] E.Panteley and A.Loria, Growth rate conditions for uniform asymptotic stability of cascade time-varying systems, *Automatica* 37 No. 3, (2001), 453-460.

[14] M.Vidyasagar, Nonlinear Systems Analysis, Practice Hall 2nd edition, (1993).

[15] V. I. Zubov, Methods of A.M. Lyapunov and their Application, P. Noordhoff, Groningen, The Netherlands, (1964).