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Right General Fractional Monotone Approximation

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis,
Memphis, TN 38152, U.S.A.,
ganastss@memphis.edu

ABSTRACT

Here is introduced a right general fractional derivative Caputo style with respect to a base absolutely continuous strictly increasing function g. We give various examples of such right fractional derivatives for different g. Let f be p-times continuously differentiable function on $[a, b]$, and let L be a linear right general fractional differential operator such that $L(f)$ is non-negative over a critical closed subinterval J of $[a, b]$. We can find a sequence of polynomials Q_n of degree less-equal n such that $L(Q_n)$ is non-negative over J, furthermore f is approximated uniformly by Q_n over $[a, b]$.

The degree of this constrained approximation is given by an inequality using the first modulus of continuity of $f^{(p)}$. We finish we applications of the main right fractional monotone approximation theorem for different g.

RESUMEN

Aquí introducimos una derivada fraccional derecha general al estilo de Caputo con respecto a una base de funciones absolutamente continuas estrictamente crecientes g. Damos varios ejemplos de dichas derivadas fraccionales derechas para diferentes g. Sea f una función p-veces continuamente diferenciable en $[a, b]$, y sea L un operador diferencial fraccional derecho general tal que $L(f)$ es no-negativo en un subintervalo cerrado crítico J de $[a, b]$. Podemos encontrar una sucesión de polinomios $L(Q_n)$ de grado menor o igual a n tal que $L(Q_n)$ es no-negativo en J, más aún f es aproximada uniformemente por Q_n en $[a, b]$. El grado de esta aproximación restringida es dada por una desigualdad usando el primer módulo de continuidad de $f^{(p)}$. Concluimos con aplicaciones del teorema principal de aproximación monótona fraccional derecha para diferentes g.

Keywords and Phrases: Right Fractional Monotone Approximation, general right fractional derivative, linear general right fractional differential operator, modulus of continuity.

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1 Introduction and Preparation

The topic of monotone approximation started in [11] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [4] the authors replaced the kth derivative with a linear ordinary differential operator of order k.

Furthermore in [1], the author generalized the result of [4] for linear right fractional differential operators.

To describe the motivating result here we need

Definition 1. *([5]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in C^m([-1,1])$. We define the right Caputo fractional derivative of f of order α as follows:*

$$\left(D_{1-}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^1 (t-x)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{1}$$

for any $x \in [-1,1]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$.

We set

$$D_{1-}^{0} f(x) = f(x), \tag{2}$$

$$D_{1-}^{m} f(x) = (-1)^m f^{(m)}(x), \quad \forall\, x \in [-1,1]. \tag{3}$$

In [1] we proved

Theorem 1.1. *Let h, k, p be integers, h is even, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)}, \delta\right)$, $\delta > 0$, there. Let $\alpha_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and assume for $x \in [-1,0]$ that $\alpha_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$. Let the real numbers $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < ... < \alpha_p < p$. Here $D_{1-}^{\alpha_j} f$ stands for the right Caputo fractional derivative of f of order α_j anchored at 1. Consider the linear right fractional differential operator*

$$L := \sum_{j=h}^{k} \alpha_j(x) \left[D_{1-}^{\alpha_j}\right] \tag{4}$$

and suppose, throughout $[-1,0]$,

$$L(f) \geq 0. \tag{5}$$

Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1,0], \tag{6}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1\left(f^{(p)}, \frac{1}{n}\right), \tag{7}$$

where C is independent of n or f.

Notice above that the monotonicity property is only true on $[-1, 0]$, see (5), (6). However the approximation property (7) it is true over the whole interval $[-1, 1]$.

In this article we extend Theorem 1.1 to much more general linear right fractional differential operators.

We use here the following right generalised fractional integral.

Definition 2. *(see also [8, p. 99]) The right generalised fractional integral of a function* f *with respect to given function* g *is defined as follows:*

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. *Here* $g \in AC([a, b])$ *(absolutely continuous functions) and is strictly increasing,* $f \in L_\infty([a, b])$. *We set*

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t) \, dt, \quad x \le b, \tag{8}$$

clearly $\left(I_{b-;g}^{\alpha} f\right)(b) = 0$.

When g *is the identity function* id, *we get that* $I_{b-;\mathrm{id}}^{\alpha} = I_{b-}^{\alpha}$, *the ordinary right Riemann-Liouville fractional integral, where*

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) \, dt, \quad x \le b, \tag{9}$$

$\left(I_{b-}^{\alpha} f\right)(b) = 0$.

When $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$, we get

Definition 3. *([8, p. 110]) Let* $0 < a < b < \infty$, $\alpha > 0$. *The right Hadamard fractional integral of order* α *is given by*

$$\left(J_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{y}{x}\right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x \le b, \tag{10}$$

where $f \in L_\infty([a, b])$.

We mention

Definition 4. *The right fractional exponential integral is defined as follows: Let* $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a, b])$. *We set*

$$\left(I_{b-;e^x}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(e^t - e^x\right)^{\alpha-1} e^t f(t) \, dt, \quad x \le b. \tag{11}$$

Definition 5. *Let* $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a, b])$, $A > 1$. *We introduce the right fractional integral*

$$\left(I_{b-;A^x}^{\alpha} f\right)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_x^b \left(A^t - A^x\right)^{\alpha-1} A^t f(t) \, dt, \quad x \le b. \tag{12}$$

We also give

Definition 6. *Let* $\alpha, \sigma > 0$, $0 \leq a < b < \infty$, $f \in L_\infty([a,b])$. *We set*

$$\left(K_{b-;x^\sigma}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t^\sigma - x^\sigma)^{\alpha-1} f(t)\, \sigma t^{\sigma-1} dt, \quad x \leq b. \tag{13}$$

We introduce the following general right fractional derivative.

Definition 7. *Let* $\alpha > 0$ *and* $\lceil \alpha \rceil = m$, *($\lceil \cdot \rceil$ ceiling of the number). Consider* $f \in AC^m([a,b])$ *(space of functions* f *with* $f^{(m-1)} \in AC([a,b])$*). We define the right general fractional derivative of* f *of order* α *as follows*

$$\left(D_{b-;g}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) f^{(m)}(t)\, dt, \tag{14}$$

for any $x \in [a,b]$, *where* Γ *is the gamma function.*

We set

$$D_{b-;g}^{m} f(x) = (-1)^m f^{(m)}(x), \tag{15}$$

$$D_{b-;g}^{0} f(x) = f(x), \quad \forall\, x \in [a,b]. \tag{16}$$

When $g = id$, *then* $D_{b-}^{\alpha} f = D_{b-;id}^{\alpha} f$ *is the right Caputo fractional derivative.*

So we have the specific general right fractional derivatives.

Definition 8.

$$D_{b-;\ln x}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \left(\ln \frac{y}{x}\right)^{m-\alpha-1} \frac{f^{(m)}(y)}{y} dy, \quad 0 < a \leq x \leq b, \tag{17}$$

$$D_{b-;e^x}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \left(e^t - e^x\right)^{m-\alpha-1} e^t f^{(m)}(t)\, dt, \quad a \leq x \leq b, \tag{18}$$

and

$$D_{b-;A^x}^{\alpha} f(x) = \frac{(-1)^m \ln A}{\Gamma(m-\alpha)} \int_x^b \left(A^t - A^x\right)^{m-\alpha-1} A^t f^{(m)}(t)\, dt, \quad a \leq x \leq b, \tag{19}$$

$$\left(D_{b-;x^\sigma}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (t^\sigma - x^\sigma)^{m-\alpha-1} \sigma t^{\sigma-1} f^{(m)}(t)\, dt, \quad 0 \leq a \leq x \leq b. \tag{20}$$

We mention

Theorem 1.2. *(Trigub, [12], [13]) Let* $g \in C^p([-1,1])$, $p \in \mathbb{N}$. *Then there exists real polynomial* $q_n(x)$ *of degree* $\leq n$, $x \in [-1,1]$, *such that*

$$\max_{-1 \leq x \leq 1} \left| g^{(j)}(x) - q_n^{(j)}(x) \right| \leq R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right), \tag{21}$$

$j = 0, 1, ..., p$, *where* R_p *is independent of* n *or* g.

In [2], based on Theorem 1.2 we proved the following useful here result

Theorem 1.3. *Let* $f \in C^p([a,b])$, $p \in \mathbb{N}$. *Then there exist real polynomials* $Q_n^*(x)$ *of degree* $\leq n \in \mathbb{N}$, $x \in [a,b]$, *such that*

$$\max_{a \leq x \leq b} \left| f^{(j)}(x) - Q_n^{*(j)}(x) \right| \leq R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \tag{22}$$

$j = 0, 1, ..., p$, *where* R_p *is independent of* n *or* g.

Remark 1.4. *Here* $g \in AC([a,b])$ *(absolutely continuous functions),* g *is increasing over* $[a,b]$, $\alpha > 0$.

Let $g(a) = c$, $g(b) = d$. *We want to calculate*

$$I = \int_a^b (g(t) - g(a))^{\alpha-1} g'(t)\, dt. \tag{23}$$

Consider the function

$$f(y) = (y - g(a))^{\alpha-1} = (y-c)^{\alpha-1}, \ \ \forall\, y \in [c,d]. \tag{24}$$

We have that $f(y) \geq 0$, *it may be* $+\infty$ *when* $y = c$ *and* $0 < \alpha < 1$, *but* f *is measurable on* $[c,d]$. *By [9], Royden, p. 107, exercise 13 d, we get that*

$$(f \circ g)(t)\, g'(t) = (g(t) - g(a))^{\alpha-1} g'(t) \tag{25}$$

is measurable on $[a,b]$, *and*

$$I = \int_c^d (y-c)^{\alpha-1} dy = \frac{(d-c)^\alpha}{\alpha} \tag{26}$$

(notice that $(y-c)^{\alpha-1}$ *is Riemann integrable).*

That is

$$I = \frac{(g(b) - g(a))^\alpha}{\alpha}. \tag{27}$$

Similarly it holds

$$\int_x^b (g(t) - g(x))^{\alpha-1} g'(t)\, dt = \frac{(g(b) - g(x))^\alpha}{\alpha}, \quad \forall\, x \in [a,b]. \tag{28}$$

Finally we will use

Theorem 1.5. *Let* $\alpha > 0$, $\mathbb{N} \ni m = \lceil \alpha \rceil$, *and* $f \in C^m([a,b])$. *Then* $\left(D_{b-;g}^\alpha f\right)(x)$ *is continuous in* $x \in [a,b]$, $-\infty < a < b < \infty$.

Proof. By [3], Apostol, p. 78, we get that g^{-1} exists and it is strictly increasing on $[g(a), g(b)]$. Since g is continuous on $[a,b]$, it implies that g^{-1} is continuous on $[g(a), g(b)]$. Hence $f^{(m)} \circ g^{-1}$ is a continuous function on $[g(a), g(b)]$.

If $\alpha = m \in \mathbb{N}$, then the claim is trivial.

We treat the case of $0 < \alpha < m$.

It holds that

$$\left(D^{\alpha}_{b-;g} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) f^{(m)}(t) \, dt =$$

$$\frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) \left(f^{(m)} \circ g^{-1}\right)(g(t)) \, dt = \tag{29}$$

$$\frac{(-1)^m}{\Gamma(m-\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1}\right)(z) \, dz.$$

An explanation follows.

The function

$$G(z) = (z - g(x))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1}\right)(z)$$

is integrable on $[g(x), g(b)]$, and by assumption g is absolutely continuous : $[a, b] \to [g(a), g(b)]$.

Since g is monotone (strictly increasing here) the function

$$(g(t) - g(x))^{m-\alpha-1} g'(t) \left(f^{(m)} \circ g^{-1}\right)(g(t))$$

is integrable on $[x, b]$ (see [7]). Furthermore it holds (see also [7]),

$$\frac{(-1)^m}{\Gamma(m-\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1}\right)(z) \, dz =$$

$$\frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) \left(f^{(m)} \circ g^{-1}\right)(g(t)) \, dt \tag{30}$$

$$= \left(D^{\alpha}_{b-;g} f\right)(x), \quad \forall \, x \in [a, b].$$

And we can write

$$\left(D^{\alpha}_{b-;g} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1}\right)(z) \, dz,$$

$$\left(D^{\alpha}_{b-;g} f\right)(y) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{g(y)}^{g(b)} (z - g(y))^{m-\alpha-1} \left(f^{(m)} \circ g^{-1}\right)(z) \, dz. \tag{31}$$

Here $a \leq y \leq x \leq b$, and $g(a) \leq g(y) \leq g(x) \leq g(b)$, and $0 \leq g(b) - g(x) \leq g(b) - g(y)$.

Let $\lambda = z - g(x)$, then $z = g(x) + \lambda$. Thus

$$\left(D^{\alpha}_{b-;g} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_0^{g(b)-g(x)} \lambda^{m-\alpha-1} \left(f^{(m)} \circ g^{-1}\right)(g(x) + \lambda) \, d\lambda. \tag{32}$$

Clearly, see that $g(x) \leq z \leq g(b)$, and $0 \leq \lambda \leq g(b) - g(x)$.

Similarly

$$\left(D_{b-;g}^{\alpha}f\right)(y)=\frac{(-1)^{m}}{\Gamma(m-\alpha)}\int_{0}^{g(b)-g(y)}\lambda^{m-\alpha-1}\left(f^{(m)}\circ g^{-1}\right)(g(y)+\lambda)\,d\lambda. \tag{33}$$

Hence it holds

$$\left(D_{b-;g}^{\alpha}f\right)(y)-\left(D_{b-;g}^{\alpha}f\right)(x)=\frac{(-1)^{m}}{\Gamma(m-\alpha)}\cdot$$

$$\left[\int_{0}^{g(b)-g(x)}\lambda^{m-\alpha-1}\left(\left(f^{(m)}\circ g^{-1}\right)(g(y)+\lambda)-\left(f^{(m)}\circ g^{-1}\right)(g(x)+\lambda)\right)d\lambda+\right.$$

$$\left.\int_{g(b)-g(x)}^{g(b)-g(y)}\lambda^{m-\alpha-1}\left(f^{(m)}\circ g^{-1}\right)(g(y)+\lambda)\,d\lambda\right]. \tag{34}$$

Thus we obtain

$$\left|\left(D_{b-;g}^{\alpha}f\right)(y)-\left(D_{b-;g}^{\alpha}f\right)(x)\right|\leq\frac{1}{\Gamma(m-\alpha)}\cdot$$

$$\left[\frac{(g(b)-g(x))^{m-\alpha}}{m-\alpha}\omega_{1}\left(f^{(m)}\circ g^{-1},|g(y)-g(x)|\right)+\right. \tag{35}$$

$$\left.\frac{\left\|f^{(m)}\circ g^{-1}\right\|_{\infty,[g(a),g(b)]}}{m-\alpha}\left((g(b)-g(y))^{m-\alpha}-(g(b)-g(x))^{m-\alpha}\right)\right]=:(\xi).$$

As $y\to x$, then $g(y)\to g(x)$ (since $g\in AC([a,b])$). So that $(\xi)\to 0$. As a result

$$\left(D_{b-;g}^{\alpha}f\right)(y)\to\left(D_{b-;g}^{\alpha}f\right)(x), \tag{36}$$

proving that $\left(D_{b-;g}^{\alpha}f\right)(x)$ is continuous in $x\in[a,b]$. □

2 Main Result

We present

Theorem 2.1. *Here we assume that* $g(b)-g(a)>1$. *Let* h,k,p *be integers,* h *is even,* $0\leq h\leq k\leq p$ *and let* $f\in C^{p}([a,b])$, $a<b$, *with modulus of continuity* $\omega_{1}\left(f^{(p)},\delta\right)$, $0<\delta\leq b-a$. *Let* $\alpha_{j}(x)$, $j=h,h+1,...,k$ *be real functions, defined and bounded on* $[a,b]$ *and assume for* $x\in\left[a,g^{-1}(g(b)-1)\right]$ *that* $\alpha_{h}(x)$ *is either* $\geq$ *some number* $\alpha^{*}>0$, *or* $\leq$ *some number* $\beta^{*}<0$. *Let the real numbers* $\alpha_{0}=0<\alpha_{1}\leq 1<\alpha_{2}\leq 2<...<\alpha_{p}\leq p$. *Consider the linear right general fractional differential operator*

$$L=\sum_{j=h}^{k}\alpha_{j}(x)\left[D_{b-;g}^{\alpha_{j}}\right], \tag{37}$$

and suppose, throughout $\left[a,g^{-1}(g(b)-1)\right]$,

$$L(f)\geq 0. \tag{38}$$

Then, for any $n \in \mathbb{N}$*, there exists a real polynomial* $Q_n(x)$ *of degree* $\leq n$ *such that*

$$L(Q_n) \geq 0 \text{ throughout } \left[a, g^{-1}(g(b)-1)\right], \tag{39}$$

and

$$\max_{x\in[a,b]} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right), \tag{40}$$

where C *is independent of* n *or* f*.*

Proof. of Theorem 2.1.

Here $h, k, p \in \mathbb{Z}_+$, $0 \leq h \leq k \leq p$. Let $\alpha_j > 0$, $j = 1, ..., p$, such that $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3... < ... < \alpha_p \leq p$. That is $\lceil \alpha_j \rceil = j$, $j = 1, ..., p$.

Let $Q_n^*(x)$ be as in Theorem 1.3.

We have that

$$\left(D_{b-;g}^{\alpha_j} f\right)(x) = \frac{(-1)^j}{\Gamma(j-\alpha_j)} \int_x^b (g(t) - g(x))^{j-\alpha_j-1} g'(t) f^{(j)}(t)\, dt, \tag{41}$$

and

$$\left(D_{b-;g}^{\alpha_j} Q_n^*\right)(x) = \frac{(-1)^j}{\Gamma(j-\alpha_j)} \int_x^b (g(t) - g(x))^{j-\alpha_j-1} g'(t) Q_n^{*(j)}(t)\, dt, \tag{42}$$

$j = 1, ..., p$.

Also it holds

$$\left(D_{b-;g}^{j} f\right)(x) = (-1)^j f^{(j)}(x), \quad \left(D_{b-;g}^{j} Q_n^*\right)(x) = (-1)^j Q_n^{*(j)}(x), \quad j = 1, ..., p. \tag{43}$$

By [10], we get that there exists g' a.e., and g' is measurable and non-negative.

We notice that

$$\left|\left(D_{b-;g}^{\alpha_j} f\right)(x) - D_{b-;g}^{\alpha_j} Q_n^*(x)\right| =$$

$$\frac{1}{\Gamma(j-\alpha_j)} \left|\int_x^b (g(x) - g(t))^{j-\alpha_j-1} g'(t) \left(f^{(j)}(t) - Q_n^{*(j)}(t)\right) dt\right| \leq$$

$$\frac{1}{\Gamma(j-\alpha_j)} \int_x^b (g(x) - g(t))^{j-\alpha_j-1} g'(t) \left|f^{(j)}(t) - Q_n^{*(j)}(t)\right| dt \overset{(22)}{\leq}$$

$$\frac{1}{\Gamma(j-\alpha_j)} \left(\int_x^b (g(x) - g(t))^{j-\alpha_j-1} g'(t)\, dt\right) R_p \left(\frac{b-a}{2n}\right)^{p-j} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right)$$

$$\overset{(28)}{=} \frac{(g(b) - g(x))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \left(\frac{b-a}{2n}\right)^{p-j} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right) \leq$$

$$\frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \left(\frac{b-a}{2n}\right)^{p-j} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right). \tag{44}$$

Hence $\forall$ $x \in [a, b]$, it holds

$$\left|\left(D^{\alpha_j}_{b-;g}f\right)(x) - D^{\alpha_j}_{b-;g}Q^*_n(x)\right| \leq$$

$$\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \left(\frac{b-a}{2n}\right)^{p-j} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right), \tag{45}$$

and

$$\max_{x\in[a,b]}\left|D^{\alpha_j}_{b-;g}f(x) - D^{\alpha_j}_{b-;g}Q^*_n(x)\right| \leq$$

$$\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \left(\frac{b-a}{2n}\right)^{p-j} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right), \tag{46}$$

$j = 0, 1, ..., p$.

Above we set $D^0_{b-;g}f(x) = f(x)$, $D^0_{b-;g}Q^*_n(x) = Q^*_n(x)$, $\forall$ $x \in [a, b]$, and $\alpha_0 = 0$, i.e. $\lceil\alpha_0\rceil = 0$.

Put

$$s_j = \sup_{a\leq x\leq b}\left|\alpha_h^{-1}(x)\,\alpha_j(x)\right|, \quad j = h, ..., k, \tag{47}$$

and

$$\eta_n = R_p\omega_1\left(f^{(p)}, \frac{b-a}{2n}\right)\left(\sum_{j=h}^{k} s_j \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2n}\right)^{p-j}\right). \tag{48}$$

I. Suppose, throughout $\left[a, g^{-1}(g(b)-1)\right]$, $\alpha_h(x) \geq \alpha^* > 0$. Let $Q_n(x)$, $x \in [a, b]$, be a real polynomial of degree $\leq n$, according to Theorem 1.3 and (46), so that

$$\max_{x\in[a,b]}\left|D^{\alpha_j}_{b-;g}\left(f(x) + \eta_n (h!)^{-1} x^h\right) - \left(D^{\alpha_j}_{b-;g}Q_n\right)(x)\right| \leq \tag{49}$$

$$\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p \left(\frac{b-a}{2n}\right)^{p-j} \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right),$$

$j = 0, 1, ..., p$.

In particular ($j = 0$) holds

$$\max_{x\in[a,b]}\left|\left(f(x) + \eta_n (h!)^{-1} x^h\right) - Q_n(x)\right| \leq R_p\left(\frac{b-a}{2n}\right)^p \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right), \tag{50}$$

and

$$\max_{x\in[a,b]}|f(x) - Q_n(x)| \leq \eta_n (h!)^{-1}(\max(|a|,|b|))^h + R_p\left(\frac{b-a}{2n}\right)^p \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right)$$

$$= \eta_n (h!)^{-1}\max\left(|a|^h, |b|^h\right) + R_p\left(\frac{b-a}{2n}\right)^p \omega_1\left(f^{(p)}, \frac{b-a}{2n}\right) = \tag{51}$$

$$R_p\omega_1\left(f^{(p)}, \frac{b-a}{2n}\right)\left(\sum_{j=h}^{k} s_j \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2n}\right)^{p-j}\right)(h!)^{-1}\max\left(|a|^h, |b|^h\right)$$

$$+R_p\left(\frac{b-a}{2n}\right)^p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right)\le$$

$$R_p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right)n^{k-p}\cdot$$

$$\left[\left(\sum_{j=h}^{k}s_j\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2}\right)^{p-j}\right)(h!)^{-1}\max\left(|a|^h,|b|^h\right)+\left(\frac{b-a}{2}\right)^p\right]. \tag{52}$$

We have found that

$$\max_{x\in[a,b]}|f(x)-Q_n(x)|\le R_p\left[\left(\frac{b-a}{2}\right)^p+(h!)^{-1}\max\left(|a|^h,|b|^h\right)\cdot\right.$$

$$\left.\left(\sum_{j=h}^{k}s_j\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2}\right)^{p-j}\right)\right]n^{k-p}\omega_1\left(f^{(p)},\frac{b-a}{2n}\right), \tag{53}$$

proving (40).

Notice for $j=h+1,...,k$, that

$$\left(D_{b-;g}^{\alpha_j}x^h\right)=\frac{(-1)^j}{\Gamma(j-\alpha_j)}\int_x^b(g(t)-g(x))^{j-\alpha_j-1}g'(t)\left(t^h\right)^{(j)}dt=0. \tag{54}$$

Here

$$L=\sum_{j=h}^{k}\alpha_j(x)\left[D_{b-;g}^{\alpha_j}\right],$$

and suppose, throughout $\left[a,g^{-1}(g(b)-1)\right]$, $Lf\ge0$. So over $a\le x\le g^{-1}(g(b)-1)$, we get

$$\alpha_h^{-1}(x)L(Q_n(x))\overset{(54)}{=}\alpha_h^{-1}(x)L(f(x))+\frac{\eta_n}{h!}\left(D_{b-;g}^{\alpha_h}\left(x^h\right)\right)+$$

$$\sum_{j=h}^{k}\alpha_h^{-1}(x)\alpha_j(x)\left[D_{b-;g}^{\alpha_j}Q_n(x)-D_{b-;g}^{\alpha_j}f(x)-\frac{\eta_n}{h!}D_{b-;g}^{\alpha_j}x^h\right]\overset{(49)}{\ge} \tag{55}$$

$$\frac{\eta_n}{h!}\left(D_{b-;g}^{\alpha_h}\left(x^h\right)\right)-\left(\sum_{j=h}^{k}s_j\frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\left(\frac{b-a}{2n}\right)^{p-j}\right)R_p\omega_1\left(f^{(p)},\frac{b-a}{2n}\right) \tag{56}$$

$$\overset{(48)}{=}\frac{\eta_n}{h!}\left(D_{b-;g}^{\alpha_h}\left(x^h\right)\right)-\eta_n=\eta_n\left(\frac{D_{b-;g}^{\alpha_h}\left(x^h\right)}{h!}-1\right)= \tag{57}$$

$$\eta_n\left(\frac{1}{\Gamma(h-\alpha_h)h!}\int_x^b(g(t)-g(x))^{h-\alpha_h-1}g'(t)\left(t^h\right)^{(h)}dt-1\right)=$$

$$\eta_n\left(\frac{h!}{h!\Gamma(h-\alpha_h)}\int_x^b(g(t)-g(x))^{h-\alpha_h-1}g'(t)\,dt-1\right)\overset{(28)}{=}$$

$$\eta_n \left(\frac{(g(b) - g(x))^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} - 1 \right) = \tag{58}$$

$$\eta_n \left(\frac{(g(b) - g(x))^{h-\alpha_h} - \Gamma(h - \alpha_h + 1)}{\Gamma(h - \alpha_h + 1)} \right) \geq$$

$$\eta_n \left(\frac{1 - \Gamma(h - \alpha_h + 1)}{\Gamma(h - \alpha_h + 1)} \right) \geq 0. \tag{59}$$

Clearly here $g(b) - g(x) \geq 1$.

Hence

$$L(Q_n(x)) \geq 0, \text{ for } x \in \left[a, g^{-1}(g(b) - 1)\right]. \tag{60}$$

A further explanation follows: We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h - \alpha_h < 1$ and $1 \leq h - \alpha_h + 1 < 2$. Thus

$$\Gamma(h - \alpha_h + 1) \leq 1 \text{ and } 1 - \Gamma(h - \alpha_h + 1) \geq 0. \tag{61}$$

II. Suppose, throughout $\left[a, g^{-1}(g(b) - 1)\right]$, $\alpha_h(x) \leq \beta^* < 0$.

Let $Q_n(x)$, $x \in [a, b]$ be a real polynomial of degree $\leq n$, according to Theorem 1.3 and (46), so that

$$\max_{x \in [a,b]} \left| D_{b-;g}^{\alpha_j} \left(f(x) - \eta_n (h!)^{-1} x^h \right) - \left(D_{b-;g}^{\alpha_j} Q_n \right)(x) \right| \leq \tag{62}$$

$$\frac{(g(b) - g(a))^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p \left(\frac{b-a}{2n} \right)^{p-j} \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right),$$

$j = 0, 1, ..., p$.

In particular ($j = 0$) holds

$$\max_{x \in [a,b]} \left| \left(f(x) - \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \tag{63}$$

and

$$\max_{x \in [a,b]} |f(x) - Q_n(x)| \leq \eta_n (h!)^{-1} (\max(|a|, |b|))^h + R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right)$$

$$= \eta_n (h!)^{-1} \max\left(|a|^h, |b|^h \right) + R_p \left(\frac{b-a}{2n} \right)^p \omega_1 \left(f^{(p)}, \frac{b-a}{2n} \right), \tag{64}$$

etc.

We find again that

$$\max_{x \in [a,b]} |f(x) - Q_n(x)| \leq R_p \left[\left(\frac{b-a}{2} \right)^p + (h!)^{-1} \max\left(|a|^h, |b|^h \right) \cdot \right.$$

$$\left(\sum_{j=h}^{k} s_j \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \left(\frac{b-a}{2}\right)^{p-j}\right)\Bigg] n^{k-p} \omega_1 \left(f^{(p)}, \frac{b-a}{2n}\right), \tag{65}$$

reproving (40).

Here again

$$L = \sum_{j=h}^{k} \alpha_j(x) \left[D_{b-;g}^{\alpha_j}\right],$$

and suppose, throughout $\left[a, g^{-1}(g(b)-1)\right]$, $Lf \geq 0$. So over $a \leq x \leq g^{-1}(g(b)-1)$, we get

$$\alpha_h^{-1}(x) L(Q_n(x)) \overset{(54)}{=} \alpha_h^{-1}(x) L(f(x)) - \frac{\eta_n}{h!}\left(D_{b-;g}^{\alpha_h}\left(x^h\right)\right) +$$

$$\sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left[D_{b-;g}^{\alpha_j} Q_n(x) - D_{b-;g}^{\alpha_j} f(x) + \frac{\eta_n}{h!} D_{b-;g}^{\alpha_j} x^h\right] \overset{(62)}{\leq} \tag{66}$$

$$-\frac{\eta_n}{h!}\left(D_{b-;g}^{\alpha_h}\left(x^h\right)\right) + \left(\sum_{j=h}^{k} s_j \frac{(g(b)-g(a))^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \left(\frac{b-a}{2n}\right)^{p-j}\right) R_p \omega_1 \left(f^{(p)}, \frac{b-a}{2n}\right) \tag{67}$$

$$\overset{(48)}{=} -\frac{\eta_n}{h!}\left(D_{b-;g}^{\alpha_h}\left(x^h\right)\right) + \eta_n = \eta_n \left(1 - \frac{D_{b-;g}^{\alpha_h}\left(x^h\right)}{h!}\right) = \tag{68}$$

$$\eta_n \left(1 - \frac{1}{\Gamma(h-\alpha_h) h!} \int_x^b (g(t)-g(x))^{h-\alpha_h-1} g'(t) \left(t^h\right)^{(h)} dt\right) =$$

$$\eta_n \left(1 - \frac{h!}{h!\Gamma(h-\alpha_h)} \int_x^b (g(t)-g(x))^{h-\alpha_h-1} g'(t)\, dt\right) \overset{(28)}{=}$$

$$\eta_n \left(1 - \frac{(g(b)-g(x))^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right) = \tag{69}$$

$$\eta_n \left(\frac{\Gamma(h-\alpha_h+1) - (g(b)-g(x))^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right) \overset{(61)}{\leq}$$

$$\eta_n \left(\frac{1 - (g(b)-g(x))^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right) \leq 0. \tag{70}$$

Hence again

$$L(Q_n(x)) \geq 0, \ \forall x \in \left[a, g^{-1}(g(b)-1)\right].$$

The case of $\alpha_h = h$ is trivially concluded from the above. The proof of the theorem is now over. □

We make

Remark 2.2. *By Theorem 1.5 we have that* $D^{\alpha_j}_{b-;g} f$ *are continuous functions,* $j = 0, 1, ..., p$. *Suppose that* $\alpha_h(x), ..., \alpha_k(x)$ *are continuous functions on* $[a, b]$, *and* $L(f) \geq 0$ *on* $\left[a, g^{-1}(g(b) - 1)\right]$ *is replaced by* $L(f) > 0$ *on* $\left[a, g^{-1}(g(b) - 1)\right]$. *Disregard the assumption made in the main theorem on* $\alpha_h(x)$. *For* $n \in \mathbb{N}$, *let* $Q_n(x)$ *be the* $Q^*_n(x)$ *of Theorem 1.3, and* f *as in Theorem 1.3 (same as in Theorem 2.1). Then* $Q_n(x)$ *converges to* $f(x)$ *at the Jackson rate* $\frac{1}{n^{p+1}}$ *([6], p. 18, Theorem VIII) and at the same time, since* $L(Q_n)$ *converges uniformly to* $L(f)$ *on* $[a, b]$, $L(Q_n) > 0$ *on* $\left[a, g^{-1}(g(b) - 1)\right]$ *for all* n *sufficiently large.*

3 Applications (to Theorem 2.1)

1) When $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$.

Here we would assume that $b > ae$, $\alpha_h(x)$ restriction true on $\left[a, \frac{b}{e}\right]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x) \left[D^{\alpha_j}_{b-;\ln x} f\right] \geq 0, \tag{72}$$

throughout $\left[a, \frac{b}{e}\right]$.

Then $L(Q_n) \geq 0$ on $\left[a, \frac{b}{e}\right]$.

2) When $g(x) = e^x$ on $[a, b]$, $a < b < \infty$.

Here we assume that $b > \ln(1 + e^a)$, $\alpha_h(x)$ restriction true on $\left[a, \ln\left(e^b - 1\right)\right]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x) \left[D^{\alpha_j}_{b-;e^x} f\right] \geq 0, \tag{73}$$

throughout $\left[a, \ln\left(e^b - 1\right)\right]$.

Then $L(Q_n) \geq 0$ on $\left[a, \ln\left(e^b - 1\right)\right]$.

3) When, $A > 1$, $g(x) = A^x$ on $[a, b]$, $a < b < \infty$.

Here we assume that $b > \log_A(1 + A^a)$, $\alpha_h(x)$ restriction true on $\left[a, \log_A\left(A^b - 1\right)\right]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x) \left[D^{\alpha_j}_{b-;A^x} f\right] \geq 0, \tag{74}$$

throughout $\left[a, \log_A\left(A^b - 1\right)\right]$.

Then $L(Q_n) \geq 0$ on $\left[a, \log_A\left(A^b - 1\right)\right]$.

4) When $\sigma > 0$, $g(x) = x^\sigma$, $0 \leq a < b < \infty$.

Here we assume that $b > (1 + a^\sigma)^{\frac{1}{\sigma}}$, $\alpha_h(x)$ restriction true on $\left[a, (b^\sigma - 1)^{\frac{1}{\sigma}}\right]$, and

$$Lf = \sum_{j=h}^{k} \alpha_j(x) \left[D^{\alpha_j}_{b-;x^\sigma} f\right] \geq 0 \tag{75}$$

throughout $\left[a, (b^{\sigma} - 1)^{\frac{1}{\sigma}}\right]$.

Then $L(Q_n) \geq 0$ on $\left[a, (b^{\sigma} - 1)^{\frac{1}{\sigma}}\right]$.

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