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Almost ω-continuous functions defined by ω-open sets due to Arhangel'skiĭ

E. Rosas [1], C. Carpintero[2], M. Salas[2], J. Sanabria[2] and L. Vásquez
[1] *Department of Mathematics, Universidad De Oriente,*
Núcleo De Sucre, Cumaná, Venezuela.
Departamento de Ciencias Naturales y Exactas, Universidad de la Costa
Barranquilla, Colombia.

[2] *Department of Mathematics, Universidad De Oriente,*
Núcleo De Sucre, Cumaná, Venezuela.

ennisrafael@gmail.com, carpintero.carlos@gmail.com, salasbrown@gmail.com,
jesanabri@gmail.com, eligiovm85@gmail.com

ABSTRACT

In this paper, we apply the notion of ω-open sets due to Arhangel'skiĭ [1] to present and study a new class of functions called almost ω-continuous functions. Relationships between this new class and other classes of functions are established.

RESUMEN

En este artículo, aplicamos la noción de ω-conjuntos abiertos dada por Arhangel'skiĭ [1] para presentar y estudiar una nueva clase de funciones llamadas funciones casi ω-continuas. Establecemos relaciones entre esta nueva clase y otras clases de funciones.

Keywords and Phrases: ω-open sets, almost ω-continuous functions.

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1 Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologist worldwide. Indeed, a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity and separation axioms, by utilizing generalized closed sets. Recently, as generalization of closed sets, the notion of β-closed sets were introduced and studied by Noiri et al. [12] and the notion of ω-closed sets were introduced and studied by Hdeib [8]. Let (X, τ) be a topological space and let A be a subset of X. We denote the closure and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A subset A is said to be ω-closed [8] if it contains all its condensation points. It is well known that a subset W of a space (X, τ) is ω-open[8] if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \setminus W$ is countable. Other notion of ω-closed sets were introduced and studied by Arhangel'skiĭ [1]. A subset A of X is called ω-closed [1], if $\mathrm{Cl}(B) \subset A$ whenever $B \subset A$ and B is a countable set. The complement of an ω-closed set is said to be an ω-open set [1]. In the sequel, we will use the ω-closed and ω-open sets in the sense of [1]. In the case that, we use the ω-closed and ω-open sets in the sense of [8], this will be explicitly stated. The family of all ω-open subsets of a topological space (X, τ) forms a topology on X which is finer than τ. The set of all ω-open sets of (X, τ) is denoted by $\omega O(X)$. The set of all ω-open sets of (X, τ) containing a point $x \in X$ is denoted by $\omega O(X, x)$. The intersection of all ω-closed sets containing A is called the ω-closure of A and is denoted by $\omega\mathrm{Cl}(A)$. The ω-interior of A is defined by the union of all ω-open sets contained in A and is denoted by $\omega\mathrm{Int}(A)$. A point $x \in X$ is called a θ-cluster point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-closed. The complement of a θ-closed set is said to be a θ-open set. The union of all θ-open sets contained in A is called the θ-interior of A and is denoted by $\mathrm{int}_\theta(A)$. It follows from [17] that the collection of all θ-open sets in a topological space (X, τ) forms a topology on X which is coarser than τ and is denoted by τ_θ. A subset A of X is said to be regular open [16] if $A = \mathrm{Int}(\mathrm{Cl}(A))$. A subset A of X is said to be δ-open [17] if it is the union of regular open sets of X. The complement of a regular open (resp. δ-open) set is called regular closed (resp. δ-closed). The intersection of all δ-closed sets of (X, τ) containing A is called the δ-closure [17] of A and is denoted by $\mathrm{Cl}_\delta(A)$. A subset A of a topological space (X, τ) is said to be β-open [2] (resp. semiopen [10], preopen [11]) if $A \subset \mathrm{Cl}(\mathrm{int}(\mathrm{Cl}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{int}(A))$, $A \subset \mathrm{int}(\mathrm{Cl}(A))$). The complement of a semiopen (resp. preopen, β-open) set is called a semiclosed (resp. preclosed, β-closed) set. The set of all regular open (resp. regular closed, δ-open, δ-closed, β-open, preopen, semiclosed, preclosed, β-closed) sets of (X, τ) is denoted by $RO(X)$ (resp. $RC(X)$, $\delta O(X)$, $\delta C(X)$, $\beta O(X)$, $PO(X)$, $SC(X)$, $PC(X)$, $\beta C(X)$). The intersection of all semiclosed sets of (X, τ) containing A is called the semiclosure [5] of A and is denoted by $\mathrm{sCl}(A)$. In this article, using the notions of ω-open sets given in [1], we introduce and study a new class of functions called almost ω-continuous functions. The connections between these functions and other existing well-known related functions are investigated.

The following two examples shows that the notions of ω-open set in sense of [1] and ω-open set in sense of [8] are independent. That means, the topologies τ_ω generated by the ω-open sets in the sense of [1] and [8] are different.

Example 1.1. Let $X = \mathbb{R}$ with the usual topology. Then $A = \mathbb{R} \setminus \mathbb{Q}$ is an ω-open set in the sense of [8], but A is not an ω-open set in the sense of [1].

Example 1.2. Consider the topology of the countable complement on $X = \mathbb{R}$. Then $A = \{1\}$ is an ω-open set in the sense of [1], but A is not an ω-open set in the sense of [8].

Definition 1.3. A topological space (X, τ) is said to be:

(1) ω-T_1 (resp. r-T_1 [7]) if for each pair of distinct points x and y of X, there exist ω-open (resp. regular open) sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

(2) ω-T_2 (resp. r-T_2 [7]) if for each pair of distinct points x and y of X, there exist ω-open (resp. regular open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Lemma 1.4. Let (X, τ) be a space and let A be a subset of X. The following statements are true:

(1) $A \in \mathrm{PO}(X)$ if and only if $\mathrm{sCl}(A) = \mathrm{int}(\mathrm{Cl}(A))$ [9].

(2) $A \in \beta\mathrm{O}(X)$ if and only if $\mathrm{Cl}(A)$ is regular closed [3].

Definition 1.5. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(1) ω-continuous [6] if $f^{-1}(V)$ is ω-open in X for every open set V of Y.

(2) almost continuous [15] if $f^{-1}(V)$ is open in X for every regular open set V of Y.

(3) R-map [4] if $f^{-1}(V)$ is regular open in X for every regular open set V of Y.

(4) weakly ω-continuous [6] if for each point $x \in X$ and each open subset V in Y containing $f(x)$, there exists $U \in \omega\mathrm{O}(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$.

The proof of the following Lemma is a direct consequence of Definition 1.5(1).

Lemma 1.6. A function $f : (X, \tau) \to (Y, \sigma)$ is ω-continuous if and only if $f^{-1}(V) \in \omega\mathrm{C}(X)$ for every closed set V of Y.

2 Almost ω-continuous functions

Definition 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost ω-continuous if for each point $x \in X$ and each open subset V of Y containing $f(x)$, there exists $U \in \omega\mathrm{O}(X, x)$ such that $f(U) \subset \mathrm{int}(\mathrm{Cl}(V))$.

Theorem 2.2. For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(1) f is almost ω-continuous,

(2) $f^{-1}(V) \in \omega O(X)$ for every $V \in RO(Y)$,

(3) $f^{-1}(F) \in \omega C(X)$ for every $F \in RC(Y)$,

(4) $f(\omega \operatorname{Cl}(A)) \subset \operatorname{Cl}_\delta(f(A))$ for every subset A of X,

(5) $\omega \operatorname{Cl}(f^{-1}(B)) \subset f^{-1}(\operatorname{Cl}_\delta(B))$ for every subset B of Y,

(6) $f^{-1}(F) \in \omega C(X)$ for every $F \in \delta C(Y)$,

(7) $f^{-1}(V) \in \omega O(X)$ for every $V \in \delta O(Y)$.

Proof.
(1) $\Rightarrow$ (2) Suppose that $V \in RO(Y)$ and let $x \in f^{-1}(V)$, then $f(x) \in V$. Since V is an open set and f is an almost ω-continuous function, there exists $U \in \omega O(X, x)$ such that $f(U) \subset \operatorname{int}(\operatorname{Cl}(V)) = V$. Thus $x \in U \subset f^{-1}(f(U)) \subset f^{-1}(V)$ and hence, we obtain that $f^{-1}(V) \in \omega O(X)$.
(2) $\Rightarrow$ (3) Let $F \in RC(Y)$, then $Y \setminus F \in RO(Y)$. By hypothesis, $f^{-1}(Y \setminus F) \in \omega O(X)$ and since $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$, we have $X \setminus f^{-1}(F) \in \omega O(X)$. Therefore $f^{-1}(F) \in \omega C(X)$.
(3) $\Rightarrow$ (4) Suppose that K is a δ-closed set in Y containing $f(A)$. Observe that $K = \operatorname{Cl}_\delta(K) = \bigcap\{F : K \subset F \text{ and } F \in RC(Y)\}$ and so $f^{-1}(K) = \bigcap\{f^{-1}(F) : K \subset F \text{ and } F \in RC(Y)\}$. Now, by part (3), we have that $f^{-1}(K) \in \omega C(X)$ and $A \subset f^{-1}(K)$. Hence $\omega \operatorname{Cl}(A) \subset f^{-1}(K)$, and it follows that $f(\omega \operatorname{Cl}(A)) \subset K$. Since this is true for any δ-closed set K containing $f(A)$, we have $f(\omega \operatorname{Cl}(A)) \subset \operatorname{Cl}_\delta(f(A))$.
(4) $\Rightarrow$ (5) Let B be a subset of Y, then $f^{-1}(B)$ is a subset of X. By part (4), $f(\omega \operatorname{Cl}(f^{-1}(B))) \subset \operatorname{Cl}_\delta(f(f^{-1}(B))) \subset \operatorname{Cl}_\delta(B)$ and so, $\omega \operatorname{Cl}(f^{-1}(B)) \subset f^{-1}(f(\omega \operatorname{Cl}(f^{-1}(B)))) \subset f^{-1}(\operatorname{Cl}_\delta(B))$.
(5) $\Rightarrow$ (6) Suppose that $F \in \delta C(Y)$, then

$$\omega \operatorname{Cl}(f^{-1}(F)) \subset f^{-1}(\operatorname{Cl}_\delta(F)) = f^{-1}(F).$$

In consequence, $\omega \operatorname{Cl}(f^{-1}(F)) = f^{-1}(F)$ and hence $f^{-1}(F) \in \omega C(X)$.
(6) $\Rightarrow$ (7) Let $V \in \delta O(Y)$, then $Y \setminus V \in \delta C(Y)$. By hypothesis, $f^{-1}(Y \setminus V) \in \omega C(X)$ and since $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$, we have $X \setminus f^{-1}(V) \in \omega C(X)$. Therefore $f^{-1}(V) \in \omega O(X)$.
(7) $\Rightarrow$ (1) Let $x \in X$ and let V any open set in Y such that $f(x) \in V$. Put $W = \operatorname{int}(\operatorname{Cl}(V))$ and $U = f^{-1}(W)$. Since $\operatorname{Cl}(V)$ is a closed set in Y, we have $W = \operatorname{int}(\operatorname{Cl}(V)) \in \delta O(Y)$ and by part (7), $U = f^{-1}(W) \in \omega O(X)$. Now, $f(x) \in V = \operatorname{int}(V) \subset \operatorname{int}(\operatorname{Cl}(V)) = W$, it follows that $x \in f^{-1}(W) = U$ and $f(U) = f(f^{-1}(W)) \subset W = \operatorname{int}(\operatorname{Cl}(V))$. □

We note that Nour [13], has also defined a type of function which he calls almost ω-continuous. But this definition is given by using the ω-open sets in sense of [8]. The following example shows that the notions of almost ω-continuous function in the sense of this paper and almost ω-continuous function in the sense of [13], are independent.

Example 2.3. Consider $X = \mathbb{R}$ with the countable complement topology τ_c and $Y = \mathbb{R}$ with the discrete topology τ_d. Then, the function $f : (X, \tau_c) \to (Y, \tau_d)$ defined as $f(x) = x$, is almost ω-continuous in the sense of this paper, but f is not almost ω-continuous in the sense [13].

Example 2.4. Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $Y = \{a, b, c\}$ with the topology $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Consider the function $f : (X, \tau) \to (Y, \sigma)$ defined as follows:

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q} \\ b, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then, f is an almost ω-continuous function in the sense [13], but f is not almost ω-continuous in the sense of this paper.

Proposition 2.5. Every almost ω-continuous function is weakly ω-continuous.

Proof.
Let $x \in X$ and let V an open subset of Y such that $f(x) \in V$. Since f is an almost ω-continuous function, there exists $U \in \omega O(X)$ such that $x \in U$ and $f(U) \subset \mathrm{int}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(V)$. Therefore, f is a weakly ω-continuous function. □

The following examples show that the converse of Proposition 2.5 is not true in general.

Example 2.6. Consider the function f in Example 2.4. It is easy to see that f is weakly ω-continuous but is not almost ω-continuous.

Example 2.7. *Let* $X = \mathbb{R}$ *with the topology* $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$ *and* $Y = \{a, b, c\}$ *with the topology* $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. *Take* $A \subset \mathbb{Q}$ *and define the function* $F : (X, \tau) \to (Y, \sigma)$ *as follows:*

$$f(x) = \begin{cases} a, & \textit{if } x \in \mathbb{Q} \setminus A. \\ c, & \textit{if } x \in \mathbb{R} \setminus \mathbb{Q}. \\ b, & \textit{if } x \in A. \end{cases}$$

Then, F *is weakly* ω*-continuous, but* F *is not almost* ω*-continuous in in the sense [13].*

Theorem 2.8. For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(1) f is almost ω-continuous,

(2) for each $x \in X$ and each open set V of Y containing $f(x)$ there exists $U \in \omega O(X, x)$ such that $f(U) \subset \mathrm{sCl}(V)$,

(3) for each $x \in X$ and each regular open set V of Y containing $f(x)$ there exists $U \in \omega O(X, x)$ such that $f(U) \subset V$,

(4) for each $x \in X$ and each δ-open set V of Y containing $f(x)$ there exists $U \in \omega O(X, x)$ such that $f(U) \subset V$.

Proof. (1)⇒(2): Let $x \in X$ and V be an open set of Y containing $f(x)$. By part (1), there exists $U \in \omega O(X, x)$ such that $f(U) \subset \operatorname{int}(\mathrm{Cl}(V))$. Since V is a preopen set, then by Lemma 1.4, $f(U) \subset \mathrm{sCl}(V)$.
(2)⇒(3): Let $x \in X$ and V be a regular open set of Y containing $f(x)$. Then V is an open set of Y containing $f(x)$. By part (2), there exists $U \in \omega O(X, x)$ such that $f(U) \subset \mathrm{sCl}(V)$. Since V is a preopen set, then by Lemma 1.4, $f(U) \subset \operatorname{int}(\mathrm{Cl}(V)) = V$.
(3)⇒(4). Let $x \in X$ and V be a δ-open set of Y containing $f(x)$. Then, there exists an open set G containing $f(x)$ such that $G \subset \operatorname{int}(\mathrm{Cl}(G)) \subset V$. Since $\operatorname{int}(\mathrm{Cl}(G))$ is a regular open set of Y containing $f(x)$, by part (3), there exists $U \in \omega O(X, x)$ such that $f(U) \subset \operatorname{int}(\mathrm{Cl}(G)) \subset V$.
(4)⇒(1). Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $\operatorname{int}(\mathrm{Cl}(V))$ is a δ-open set of Y containing $f(x)$. By part (4), there exists $U \in \omega O(X, x)$ such that $f(U) \subset \operatorname{int}(\mathrm{Cl}(V))$. Therefore, f is almost ω-continuous. □

Theorem 2.9. For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(1) f is almost ω-continuous,

(2) $f^{-1}(\operatorname{int}(\mathrm{Cl}(V))) \in \omega O(X)$ for every open set V of Y,

(3) $f^{-1}(\mathrm{Cl}(\operatorname{int}(F))) \in \omega C(X)$ for every closed set F of Y.

Proof. (1)⇒(2): Let V be an open set in Y. We have to show that $f^{-1}(\operatorname{int}(\mathrm{Cl}(V)))$ is an ω-open set in X. Let $x \in f^{-1}(\operatorname{int}(\mathrm{Cl}(V)))$. Then $f(x) \in \operatorname{int}(\mathrm{Cl}(V))$ and $\operatorname{int}(\mathrm{Cl}(V))$ is a regular open set in Y. Since f is almost ω-continuous, there exists $U \in \omega O(X, x)$ such that $f(U) \subset \operatorname{int}(\mathrm{Cl}(V))$. These implies that $x \in U \subset f^{-1}(\operatorname{int}(\mathrm{Cl}(V)))$, in consequence, $f^{-1}(\operatorname{int}(\mathrm{Cl}(V)))$ is ω-open set in X.
(2)⇒(3): Let F be a closed set of Y. Then $Y \setminus F$ is an open set of Y. By part (2), we have $f^{-1}(\operatorname{int}(\mathrm{Cl}(Y \setminus F))))$ is ω-open set in X and as $f^{-1}(\operatorname{int}(\mathrm{Cl}(Y \setminus F))) = f^{-1}(\operatorname{int}(Y \setminus \operatorname{int}(F))) = f^{-1}(Y \setminus \mathrm{Cl}(\operatorname{int}(F))) = X \setminus f^{-1}(\operatorname{int}(\mathrm{Cl}(F)))$ then $f^{-1}(\operatorname{int}(\mathrm{Cl}(F)))$ is an ω-closed set in X.
(3)⇒(1): Let F be a regular closed set of Y. Then F is a closed set of Y. By part (3), $f^{-1}(\mathrm{Cl}(\operatorname{int}(F)))$ is an ω-closed set in X. Since F is a regular closed set, then $f^{-1}(\mathrm{Cl}(\operatorname{int}(F))) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is an ω-closed set in X. By Theorem 2.2, f is an almost ω-continuous function. □

Theorem 2.10. A function $f : (X, \omega O(X)) \to (Y, \sigma)$ is almost ω-continuous if and only if it is almost continuous.

Proof. This is an immediate consequence of Theorem 2.2. □

Theorem 2.11. For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(1) f is almost ω-continuous,

(2) $\omega\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every $V \in \beta O(Y)$,

(3) $f^{-1}(\operatorname{int}(F)) \subset \omega \operatorname{int}(f^{-1}(F))$ for every $F \in \beta C(Y)$,

(4) $f^{-1}(\text{int}(F)) \subset \omega\,\text{int}(f^{-1}(F))$ for every $F \in SC(Y)$,

(5) $\omega\,\text{Cl}(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V))$ for every $V \in SO(Y)$,

(6) $f^{-1}(V) \subset \omega\,\text{int}(f^{-1}(\text{int}(\text{Cl}(V))))$ for every $V \in PO(Y)$.

Proof. (1)⇒ (2): Let V be any β-open set of Y. Since $\text{Cl}(V) \in RC(Y)$, by Theorem 2.2, $f^{-1}(\text{Cl}(V))$ is ω-closed in X and $f^{-1}(V) \subset f^{-1}(\text{Cl}(V))$. Therefore, $\omega\,\text{Cl}(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V))$.
(2)⇒(3): Let F be any β-closed set of Y. Then $Y\backslash F$ is β-open set of Y. By part (2), $\omega\,\text{Cl}(f^{-1}(Y\backslash F)) \subset f^{-1}(\text{Cl}(Y\backslash F))$ and $\omega\,\text{Cl}(X\backslash f^{-1}(F)) \subset f^{-1}(Y\backslash \text{int}(F))$ and hence, $X\backslash \omega\text{Int}(f^{-1}(F)) \subset X\backslash f^{-1}(\text{int}(F))$. Therefore, $f^{-1}(\text{int}(F)) \subset \omega\text{Int}(f^{-1}(F))$.
(3)⇒(4): This is obvious since $SC(Y) \subset \beta C(Y)$.
(4)⇒(5): Let V be any semiopen set of Y. Then $Y \setminus V$ is a semiclosed set in Y. By part (4), $f^{-1}(\text{int}(Y \setminus V)) \subset \omega\,\text{int}(f^{-1}(Y \setminus V))$ and $f^{-1}(Y \setminus \text{Cl}(V)) \subset \omega\,\text{int}(X \setminus f^{-1}(V))$ and hence, $X \setminus f^{-1}(\text{Cl}(V)) \subset X \setminus \omega\,\text{Cl}(f^{-1}(V))$. Therefore, $\omega\,\text{Cl}(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V))$.
(5)⇒(1): Let $K \in RC(Y)$. Then $K \in SO(Y)$ and by part (5), $\omega\,\text{Cl}(f^{-1}(K)) \subset f^{-1}(Cl(K)) = f^{-1}(K)$. Therefore, $f^{-1}(K)$ is ω-closed in X and hence f is almost ω-continuous by Theorem 2.2.
(1)⇒(6): Let V be any preopen set of Y. Since $\text{int}(\text{Cl}(V)) \in RO(Y)$, by Theorem 2.2, we have $f^{-1}(\text{int}(\text{Cl}(V))) \in \omega O(X)$ and hence

$$f^{-1}(V) \subset f^{-1}(\text{int}(\text{Cl}(V))) = \omega\,\text{int}(f^{-1}(\text{int}(\text{Cl}(V)))).$$

(6)⇒(1): Let V be any regular open set of Y. Since $V \in PO(Y)$, $f^{-1}(V) \subset \omega\,\text{int}(f^{-1}(\text{int}(\text{Cl}(V)))) = \omega\,\text{int}(f^{-1}(V))$ and hence $f^{-1}(V) \in \omega O(X)$. It follows from Theorem 2.2, that f is almost ω-continuous. □

As a direct consequence of Theorem 2.11, we obtain the following two corollaries

Corollary 2.12. For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(1) f is almost ω-continuous,

(2) $f^{-1}(V) \subset \omega\,\text{int}(f^{-1}(\text{sCl}(V)))$ for each preopen set V of Y,

(3) $\omega\,\text{Cl}(f^{-1}(\text{Cl}(\text{int}(F)))) \subset f^{-1}(F)$ for each preclosed set F of Y,

(4) $\omega\,\text{Cl}(f^{-1}(\text{sInt}(F))) \subset f^{-1}(F)$ for each preclosed set F of Y.

Corollary 2.13. For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(1) f is almost ω-continuous,

(2) for each neighborhood V of $f(x)$, $x \in \omega\,\text{int}(f^{-1}(\text{sCl}(V)))$,

(3) for each neighborhood V of $f(x)$, $x \in \omega\,\text{int}(f^{-1}(\text{int}(\text{Cl}(V))))$.

Theorem 2.14. Let $f : (X, \tau) \to (Y, \sigma)$ be an almost ω-continuous function and let V be any open subset of Y. If $x \in \omega\,\mathrm{Cl}(f^{-1}(V)) \setminus f^{-1}(V)$, then $f(x) \in \omega\,\mathrm{Cl}(V)$.

Proof. Let $x \in X$ such that $x \in \omega\,\mathrm{Cl}(f^{-1}(V)) \setminus f^{-1}(V)$ and suppose $f(x) \notin \omega\,\mathrm{Cl}(V)$. Then there exists an ω-open set H containing $f(x)$ such that $H \cap V = \emptyset$. Then $\mathrm{Cl}(H) \cap V = \emptyset$ implies $\mathrm{int}(\mathrm{Cl}(H)) \cap V = \emptyset$ and $\mathrm{int}(\mathrm{Cl}(H))$ is a regular open set. Since f is almost ω-continuous, there exists an ω-open set U in X containing x such that $f(U) \subset \mathrm{int}(\mathrm{Cl}(H))$. Therefore, $f(U) \cap V = \emptyset$. However, since $x \in \omega\,\mathrm{Cl}(f^{-1}(V)$, $U \cap f^{-1}(V) \neq \emptyset$ for every ω-open set U in X containing x, so that $f(U) \cap V \neq \emptyset$. We have a contradiction. It follows that $f(x) \in \omega\,\mathrm{Cl}(V)$. □

Recall that the family of all ω-open subsets of a topological space (X, τ) forms a topology on X finer than τ. From this fact we obtain immediately the following result.

Lemma 2.15. Let A and B be subsets of a topological space (X, τ). If $A \in \omega O(X)$ and $B \in \tau$, then $A \cap B \in \omega O(B)$.

Proof. Since τ is a topology, then the induced topology on B, denoted by τ_B is $\{K \cap B : K \in \tau\}$. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $A \in \omega O(X)$, there exists $U \in \tau$ with $x \in U$ and $U \setminus A$ is countable. Since $U \cap B$ is an open subset in τ_B with $x \in U \cap B$, follows that $(U \cap B) \setminus (A \cap B) = (U \setminus A) \cap B$. Since $U \setminus A$ is countable, $(U \cap B) \setminus (A \cap B)$ is countable, in consequence, $A \cap B \in \omega O(B)$. □

Theorem 2.16. Let $f : (X, \tau) \to (Y, \sigma)$ be an almost ω-continuous function and $A \subset X$. If $A \in \tau$, then $f|_A : (A, \tau_A) \to (Y, \sigma)$ is almost ω-continuous.

Proof. It follows from Lemma 2.15. □

Theorem 2.17. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of X. If $f|_{U_i}$ is almost ω-continuous for each $i \in I$, then f is almost ω-continuous.

Proof. Suppose that V is a regular open set of Y. Since $f|_{U_i}$ is almost ω-continuous for each $i \in I$, it follows that $(f|_{U_i})^{-1}(V)$ is ω-open in U_i. Since $f^{-1}(V) = X \cap f^{-1}(V) = (\bigcup_{i \in I} U_i) \cap f^{-1}(V) = \bigcup_{i \in I}(U_i \cap f^{-1}(V)) = \bigcup_{i \in I}(f|_{U_i})^{-1}(V)$, then $f^{-1}(V) \in \omega O(X)$, which means that f is almost ω-continuous. □

Definition 2.18. Let (X, τ) be a topological space. A filter base Λ is said to be:

(1) ω-convergent to a point x in X if for every $U \in \omega O(X, x)$, there exists $B \in \Lambda$ such that $B \subset U$.

(2) r-convergent to a point x in X if for every regular open set U of X containing x, there exists $B \in \Lambda$ such that $B \subset U$.

Theorem 2.19. If $f : (X, \tau) \to (Y, \sigma)$ is an almost ω-continuous function, then for each point $x \in X$ and each filter base Λ in X ω-converging to x, the filter base $f(\Lambda)$ is r-convergent to $f(x)$.

Proof. Let $x \in X$ and Λ be any filter base in X, ω-converging to x. By Theorem 2.8, for any regular open set V of (Y, σ) containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(U) \subset V$. Since Λ is ω-converging to x, there exists $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and hence the filter base $f(\Lambda)$ is r-convergent to $f(x)$. □

Definition 2.20. A net (x_λ) is said to be ω-convergent to a point x if for every ω-open set V containing x, there exists an index λ_0 such that for $\lambda \geq \lambda_0$, $x_\lambda \in V$.

Theorem 2.21. If $f : (X, \tau) \to (Y, \sigma)$ is an almost ω-continuous function, then for each point $x \in X$ and each net (x_λ) which is ω-convergent to x, the net $f((x_\lambda))$ is r-convergent to $f(x)$.

Proof. The proof is similar to that of Theorem 2.19. □

Theorem 2.22. If $f : (X, \tau) \to (Y, \sigma)$ is an almost ω-continuous injective function and (Y, σ) is r-T_1, then (X, τ) is ω-T_1.

Proof. Suppose that (Y, σ) is r-T_1. For any distinct points x and y in X, using the injectivity of f, $f(x) \neq f(y)$ and then, there exist regular open sets V and W such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is almost ω-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are ω-open subsets of (X, τ) such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that (X, τ) is ω-T_1. □

Theorem 2.23. If $f : (X, \tau) \to (Y, \sigma)$ is an almost ω-continuous injective function and (Y, σ) is r-T_2, then (X, τ) is ω-T_2.

Proof. For any pair of distinct points x and y in X, using the injectivity of f, $f(x) \neq f(y)$ and then, there exist disjoint regular open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is almost ω-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are ω-open sets in X containing x and y, respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ because $U \cap V = \emptyset$. This shows that (X, τ) is ω-T_2. □

Theorem 2.24. If $f : (X, \tau) \to (Y, \sigma)$ is an almost continuous function and $g : (X, \tau) \to (Y, \sigma)$ is an almost ω-continuous function and Y is a r-T_2-space, then the set $E = \{x \in X : f(x) = g(x)\}$ is an ω-closed set in (X, τ).

Proof. If $x \in X \setminus E$, then it follows that $f(x) \neq g(x)$. Since Y is r-T_2, there exist disjoint regular open sets V and W of Y such that $f(x) \in V$ and $g(x) \in W$. Since f is almost continuous and g is almost ω-continuous, then $f^{-1}(V)$ is open and $g^{-1}(W)$ is ω-open in X with $x \in f^{-1}(V)$ and $x \in g^{-1}(W)$. Put $A = f^{-1}(V) \cap g^{-1}(W)$. Since $\omega O(X)$ is a topology on X finer than τ, we have A is ω-open in X. Therefore, $f(A) \cap g(A) = \emptyset$ and it follows that $x \notin \omega \operatorname{Cl}(E)$. This shows that E is ω-closed in X. □

Definition 2.25. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(1) ω-irresolute if $f^{-1}(V)$ is ω-open in X for every ω-open set V of Y.

(2) faintly ω-continuous if for each point $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(U) \subset V$.

Theorem 2.26. A function $f : (X, \tau) \to (Y, \sigma)$ is faintly ω-continuous if and only if $f^{-1}(V) \in \omega O(X)$ for every θ-open set V of Y.

Proof. Suppose that f is faintly ω-continuous. Let V be a θ-open set of Y and $x \in f^{-1}(V)$. Since $f(x) \in V$ and f is faintly ω-continuous, there exists $U \in \omega O(X, x)$ such that $f(U) \subset V$. It follows that $x \in U \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is ω-open in X.
Conversely, let $x \in X$ and V be a θ-open set of Y containing $f(x)$, by hypothesis $f^{-1}(V)$ is an ω-open set containing x. Take $U = f^{-1}(V)$, then $f(U) \subset V$. This shows that f is faintly ω-continuous. □

As a direct consequence of the Theorem 2.26, we obtain the following corollary.

Corollary 2.27. A function $f : (X, \tau) \to (Y, \sigma)$ is faintly ω-continuous if and only if $f^{-1}(V) \in \omega C(X)$ for every θ-closed set V of Y.

Theorem 2.28. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ hold for the following properties of a function $f : (X, \tau) \to (Y, \sigma)$:

(1) f is ω-continuous.

(2) $f^{-1}(\mathrm{Cl}_\delta(B))$ is ω-closed in X for every subset B of Y.

(3) f is almost ω-continuous.

(4) f is weakly ω-continuous.

(5) f is faintly ω-continuous.

If, in addition, Y is regular, then the five properties are equivalent of one another.

Proof. $(1) \Rightarrow (2)$: Since $\mathrm{Cl}_\delta(B)$ is closed in Y for every subset B of Y, by Theorem 2.2, $f^{-1}(\mathrm{Cl}_\delta(B))$ is ω-closed in X.
$(2) \Rightarrow (3)$: For any subset B of Y, $f^{-1}(\mathrm{Cl}_\delta(B))$ is ω-closed in X and hence we have $\omega\,\mathrm{Cl}(f^{-1}(B)) \subset \omega\,\mathrm{Cl}(f^{-1}(\mathrm{Cl}_\delta(B)) = f^{-1}(\mathrm{Cl}_\delta(B))$. It follows from Theorem 2.2 that f is almost ω-continuous.
$(3) \Rightarrow (4)$: This is obvious.
$(4) \Rightarrow (5)$: Let F be any θ-closed set of Y. Since F is closed, it follows from Lemma 1.6 that, $f^{-1}(F)$ is ω-closed in X and hence, by Theorem 2.26, f is faintly ω-continuous.
Suppose that Y is regular. We prove that $(5) \Rightarrow (1)$. Let V be any open set of Y. Since Y is regular, V is θ-open in Y. By the faintly ω-continuity of f, $f^{-1}(V)$ is ω-open in X. Therefore, f is ω-continuous. □

Definition 2.29. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω-preopen if $f(U) \in PO(Y)$ for every ω-open set U of X.

Theorem 2.30. If a function $f : (X, \tau) \to (Y, \sigma)$ is ω-preopen and weakly ω-continuous, then f is almost ω-continuous.

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$. Since f is weakly ω-continuous, there exists $U \in \omega O(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Since f is ω-preopen, $f(U) \subset \mathrm{int}(\mathrm{Cl}(f(U))) \subset \mathrm{int}(\mathrm{Cl}(V))$ and hence f is almost ω-continuous. □

Theorem 2.31. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be functions. Then the composition $g \circ f : (X, \tau) \to (Z, \eta)$ is almost ω-continuous if f and g satisfy one of the following conditions:

(1) f is almost ω-continuous and g is R-map.

(2) f is ω-irresolute and g is almost ω-continuous.

(3) f is ω-continuous and g is almost continuous.

Proof. (1) Follows from Theorem 2.2 and Definition 1.5.

(2) Follows from Theorem 2.2 and Definition 2.25.

(3) Follows from Theorem 2.2 and Definition 1.5.

□

Theorem 2.32. If $f : X \to \prod_{i \in I} Y_i$ is almost ω-continuous function then $p_i \circ f : X \to Y_i$ is almost ω-continuous for each $i \in I$, where p_i is the projection of $\prod_{i \in I} Y_i$ onto Y_i.

Proof. Let V be a regular open set of Y_i. Since p_i is continuous open, it is an R-map and hence $p_i^{-1}(V)$ is regular open in $\prod_{i \in I} Y_i$, it follows that $f^{-1}(p_i^{-1}(V)) = (p_i \circ f)^{-1}(V) \in \omega O(X)$. This shows that $p_i \circ f$ is almost ω-continuous for each $i \in I$. □

Definition 2.33. A topological space (X, τ) is said to be almost regular [14] if for any regular closed set F of X and any point $x \in X \setminus F$ there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.

Theorem 2.34. If $f : (X, \tau) \to (Y, \sigma)$ is a weakly ω-continuous function and Y is almost regular, then f is almost ω-continuous.

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$. By the almost regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset \mathrm{Cl}(G) \subset \mathrm{int}(\mathrm{Cl}(V))$ [14, Theorem 2.2]. Since f is weakly ω-continuous, there exists $U \in \omega O(X, x)$ such that $f(U) \subset \mathrm{Cl}(G) \subset \mathrm{int}(\mathrm{Cl}(V))$. Therefore, f is almost ω-continuous. □

Definition 2.35. An ω-frontier of a subset A of (X, τ), denoted by $\omega\mathrm{Fr}(A)$, is defined by $\omega\mathrm{Fr}(A) = \omega\,\mathrm{Cl}(A) \cap \omega\,\mathrm{Cl}(X \setminus A)$.

Theorem 2.36. The set of all points $x \in X$ in which a function $f : (X, \tau) \to (Y, \sigma)$ is not almost ω-continuous is identical with the union of ω-frontier of the inverse images of regular open sets containing $f(x)$.

Proof. Suppose that f is not almost ω-continuous at $x \in X$. Then there exists a regular open set V of Y containing $f(x)$ such that $U \cap (X \setminus f^{-1}(V)) \neq \emptyset$ for every $U \in \omega O(X, x)$. Therefore, $x \in \omega \operatorname{Cl}(X \setminus f^{-1}(V)) = X \setminus \omega \operatorname{int}(f^{-1}(V))$ and $x \in f^{-1}(V)$. Thus, $x \in \omega \operatorname{Fr}(f^{-1}(U))$. Conversely, suppose that f is almost ω-continuous at $x \in X$ and let V be a regular open set of Y containing $f(x)$. Then there exists $U \in \omega O(X, x)$ such that $U \subset f^{-1}(V)$. That is $x \in \omega \operatorname{int}(f^{-1}(V))$. Therefore, $x \in X \setminus \omega \operatorname{Fr}(f^{-1}(V))$. $\square$

Definition 2.37. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be complementary almost ω-continuous if for each regular open set V of Y, $f^{-1}(\operatorname{Fr}(V))$ is ω-closed in X, where $\operatorname{Fr}(V)$ denotes the frontier of V.

Theorem 2.38. If $f : (X, \tau) \to (Y, \sigma)$ is weakly ω-continuous and complementary almost ω-continuous, then f is almost ω-continuous.

Proof. Let $x \in X$ and V be a regular open set of Y containing $f(x)$. Then $f(x) \in Y \setminus \operatorname{Fr}(V)$ and hence $x \in X \setminus f^{-1}(\operatorname{Fr}(V))$. Since f is weakly ω-continuous there exists $G \in \omega O(X, x)$ such that $f(G) \subset \operatorname{Cl}(V)$. Put $U = G \cap (X \setminus f^{-1}(\operatorname{Fr}(V)))$. Then $U \in \omega O(X, x)$ and $f(U) \subset f(G) \cap (Y \setminus \operatorname{Fr}(V)) \subset \operatorname{Cl}(V) \cap (Y \setminus \operatorname{Fr}(V)) = V$. This shows that f is almost ω-continuous. $\square$

Theorem 2.39. If $f : (X, \tau) \to (Y, \sigma)$ is almost ω-continuous, $g : (X, \tau) \to (Y, \sigma)$ is weakly ω-continuous and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is ω-closed in (X, τ).

Proof. Let $A = \{x \in X : f(x) = g(x)\}$ and $x \in X \setminus A$. Then $f(x) \neq g(x)$. Since (Y, σ) is Hausdorff, there exist open sets V and W of Y such that $f(x) \in V$, $g(x) \in W$ and $V \cap W = \emptyset$, hence $\operatorname{int}(\operatorname{Cl}(V)) \cap \operatorname{Cl}(W) = \emptyset$. Since f is almost ω-continuous, there exists $G \in \omega O(X, x)$ such that $f(G) \subset \operatorname{int}(\operatorname{Cl}(V))$. Since g is weakly ω-continuous, there exists $H \in \omega O(X)$ such that $g(H) \subset \operatorname{Cl}(W)$. Now put $U = G \cap H$, then $U \in \omega O(X, x)$ and $f(U) \cap g(U) \subset \operatorname{int}(\operatorname{Cl}(V)) \cap \operatorname{Cl}(W) = \emptyset$. Therefore, we obtain $U \cap A = \emptyset$ and hence A is ω-closed in X. $\square$

Theorem 2.40. Suppose that the product of two ω-open sets is ω-open. If $f_1 : (X_1, \tau) \to (Y, \sigma)$ is weakly ω-continuous, $f_2 : (X_2, \tau) \to (Y, \sigma)$ is almost ω-continuous and (Y, σ) is Hausdorff, then the set $\{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}$ is ω-closed in $X_1 \times X_2$.

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = f(x_2)\}$. If $(x_1, x_2) \in (X_1 \times X_2) \setminus A$, then we have $f(x_1) \neq f(x_2)$. Since (Y, σ) is Hausdorff, there exist disjoint open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$ and $\operatorname{Cl}(V_1) \cap \operatorname{int}(\operatorname{Cl}(V_2)) = \emptyset$. Since f_1 (resp. f_2) is weakly ω-continuous (resp. almost ω-continuous), there exists $U_1 \in \omega O(X_1, x_1)$ such that $f(U_1) \subset \operatorname{Cl}(V_1)$ (resp. $U_2 \in \omega O(X_2, x_2)$ such that $f(\omega \operatorname{Cl}(U_1)) \subset \operatorname{int}(\operatorname{Cl}(V_2))$). Thus, we obtain $(x_1, x_2) \in U_1 \times U_2 \subset X_1 \times X_2 \setminus A$. Therefore, $(X_1 \times X_2) \setminus A$ is ω-open and so A is ω-closed in $X_1 \times X_2$. $\square$

Theorem 2.41. If $g : (X, \tau) \to (Y, \sigma)$ is almost ω-continuous and S is a δ-closed subset of $X \times Y$, then $p_X(S \cap G(g))$ is ω-closed in X, where p_X represents the projection of $X \times Y$ onto X and $G(g)$ denotes the graph of g.

Proof. Let S be a δ-closed set of $X \times Y$ and $x \in \omega \operatorname{Cl}(p_X(S \cap G(g)))$. Let U be an open set of X containing x and V an open set of Y containing $g(x)$. Since g is almost ω-continuous, we have $x \in g^{-1}(V) \subset \omega \operatorname{int}(g^{-1}(\operatorname{int}(\operatorname{Cl}(V))))$ and $U \cap \omega \operatorname{int}(g^{-1}(\operatorname{int}(\operatorname{Cl}(V)))) \in \omega O(X, x)$. Since $x \in \omega \operatorname{Cl}(p_X(S \cap G(g)))$, $(U \cap \omega \operatorname{int}(g^{-1}(\operatorname{int}(\operatorname{Cl}(V))))) \cap p_X(S \cap G(g))$ contains some point u of X. This implies that $(u, g(u)) \in S$ and $g(u) \in \operatorname{int}(\operatorname{Cl}(V))$. Thus, we have $\emptyset \neq (U \times \operatorname{int}(\operatorname{Cl}(V))) \cap S \subset \operatorname{int}(\operatorname{Cl}(U \times V)) \cap S$ and hence $(x, g(x)) \in \operatorname{Cl}_\delta(S)$. Since S is δ-closed, $(x, g(x)) \in p_X(S \cap G(g))$ and $x \in p_X(S \cap G(g))$. Then $p_X(S \cap G(g))$ is ω-closed. $\square$

Corollary 2.42. If $f : (X, \tau) \to (Y, \sigma)$ has a δ-closed graph and $g : (X, \tau) \to (Y, \sigma)$ is almost ω-continuous, then the set $\{x \in X : f(x) = g(x)\}$ is ω-closed in X.

Proof. Since $G(f)$ is δ-closed and $p_X(G(f) \cap G(g)) = \{x \in X : f(x) = g(x)\}$ it follows from Theorem 2.41, that $\{x \in X : f(x) = g(x)\}$ is ω-closed in X. $\square$

Theorem 2.43. If for each pair of points x_1, x_2 distinct in a topological space (X, τ) there exists a function f on (X, τ) into a Hausdorff space (Y, σ) such that $f(x_1) \neq f(x_2)$, f is weakly ω-continuous at x_1 and f is almost ω-continuous at x_2, then X is ω-T_2.

Proof. Since (Y, σ) is Hausdorff, for each pair of point x_1, x_2 distinct, there exist disjoint open sets V_1 and V_2 of Y containing $f(x_1)$ and $f(x_2)$, respectively; hence $\operatorname{Cl}(V_1) \cap \operatorname{int}(\operatorname{Cl}(V_2)) = \emptyset$. Since f is weakly ω-continuous at x_1, there exists $U_1 \in \omega O(X, x_1)$ such that $f(U_1) \subset \operatorname{Cl}(V_1)$. Since f is almost ω-continuous at x_2, there exists $U_2 \in \omega O(X, x_2)$ such that $f(U_2) \subset \operatorname{int}(\operatorname{Cl}(V_2))$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$. This shows that (X, τ) is ω-T_2. $\square$

Definition 2.44. A function $f : (X, \tau) \to (Y, \sigma)$ is said to have an ω-strongly closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists an ω-open subset U of X and an open subset V of Y such that $(U \times \operatorname{Cl}(V)) \cap G(f) = \emptyset$.

As a consequence of Definition 2.44, we obtain easily the following Lemma.

Lemma 2.45. A function $f : (X, \tau) \to (Y, \sigma)$ has ω-strongly closed graph $G(f)$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$ there exists an ω-open set U and an open set V containing x and y, respectively such that $f(U) \cap \operatorname{Cl}(V) = \emptyset$.

Theorem 2.46. If $f : (X, \tau) \to (Y, \sigma)$ is an almost ω-continuous function and (Y, σ) is Hausdorff, then f has an ω-strongly closed graph.

Proof. Let $(x, y) \in X \times Y$ such that $y \neq f(x)$. Since (Y, σ) is Hausdorff, there exist open sets V and W of Y containing $f(x)$ and y, respectively, such that $V \cap W = \emptyset$. Then $f(x) \in Y \setminus \operatorname{Cl}(W)$ and

$Y \setminus \mathrm{Cl}(W)$ is regular open in Y. Since f is almost ω-continuous function, there exists $U \in \omega O(X, x)$ such that $f(U) \subset Y \setminus \mathrm{Cl}(W)$ and hence $f(U) \cap \mathrm{Cl}(W) = \emptyset$. Therefore, by Lemma 2.45 f has an ω-strongly closed graph. □

The following corollary is an immediate consequence of Lemma 2.45, as we can see.

Corollary 2.47. If $f : (X, \tau) \to (Y, \sigma)$ is an ω-continuous function and (Y, σ) is Hausdorff, then f has an ω-strongly closed graph.

Proof. Let $(x, y) \in X \times Y$ such that $y \neq f(x)$. Since (Y, σ) is Hausdorff, there exist open sets V and W of Y containing $f(x)$ and y, respectively such that $V \cap W = \emptyset$. Then $f(x) \in Y \setminus \mathrm{Cl}(W)$ and $Y \setminus \mathrm{Cl}(W)$ is an open set in Y. Since f is weakly ω-continuous function, there exists $U \in \omega O(X, x)$ such that $U \subset f^{-1}(Y \setminus \mathrm{Cl}(W))$ and then $f(U) \subset Y \setminus \mathrm{Cl}(W)$, hence $f(U) \cap \mathrm{Cl}(W) = \emptyset$. Therefore, by Lemma 2.45 f has an ω-strongly closed graph. □

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