Semantic Reading in the Context of the History of Mathematics

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Abstract
This article is adjacent to our two previous works on semantic reading (see the references), but may be read independently of them. It has three plots: from geometrical algebra to the solving the cubic equations; from ancient theory of ratios to modern theory of real numbers; from one problem in “Arithmetic” by Diophantus to modern problems of mathematics and mathematics teaching. Since all three plots originate in ancient Greece, before the main content of our article we give historical background on the part of the mathematics of ancient Greece that we need.

Keywords: Geometrical Algebra; Solving Of Cubic Equations, Eudoxus’ Ratios; Dedekind’ Sections; Fermat Last Theorem.

1. Introduction

This author already published two papers about semantic reading in mathematics and mathematics teaching (Naziev, 2014) and (Naziev, 2018). The first paper contained short instruction on what is semantic reading and several examples of its application in school algebra, geometry, probability, and calculus. In the second paper, we gave a more instructive definition of semantic reading and several examples that are more complicated. Recall the definition of semantic reading we give in the second of the mentioned papers.

Semantic reading is the art of discovering in the readable a variety of meanings and extract from it the meaning that most closely corresponds to the problem under solution.

Possible relations of so understandable semantic reading with artificial intelligence (Garrido, 2017) indicated in (Naziev, 2018).

History of mathematics is the area of knowledge where semantic reading is vital. For the convenience of the reader, let us illustrate this thought here by an example from our previous work.

From the past, we received the message

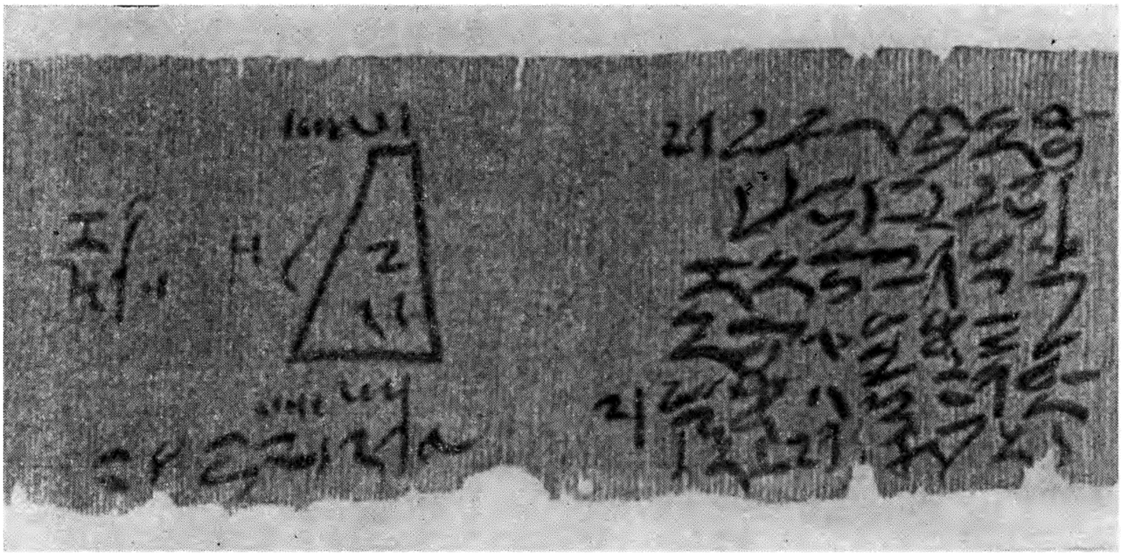


Figure 1

First, we must translate it into a modern language (this also requires semantic reading). Upon translation, we will get something like the following assembly of signs (with Russian words):

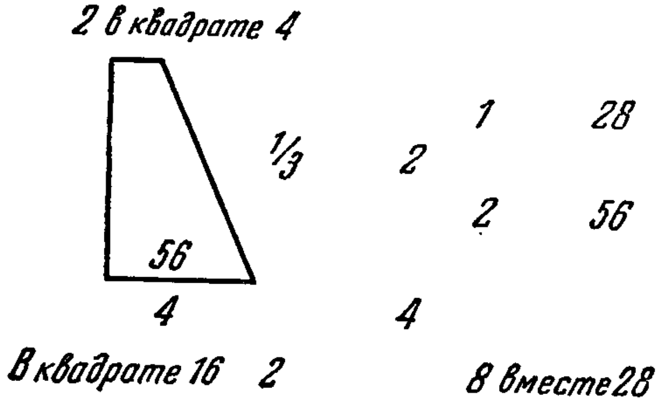


Figure 2

After that, we need to see sense in this assembly. It may be very hard. When Russian academician W. W. Struve made this, he saw the rule of evaluation of the volume of a regular truncated pyramid with a square base!

Working with texts from the past even in the simplest cases may require semantic reading. This is because of these texts written in the past while we write our works about these texts in today's language, so we must correctly interpret the content from these texts in contemporary terms.

For example, in geometrical texts until the end of XIX and even the beginning of XX century, a polygon was defined as a plane figure bounded by a finite many of straight *lines*. This definition is to be correctly understood. Nowadays, a straight line is infinite in both directions, while in the mentioned times it was *bounded* part of the line in the current sense, that is, a segment in today's language. This means that for the correct translation of the above definition to the contemporary language we have to use another word, namely, we must say that polygon is a part of a plane bounded by a finite number of *segments*.

This example clearly shows what we must do when working with the texts from the past. Namely, we must extract from these texts their *contemporary* sense. That is the main task in the remaining part of the article.

2. Geometric algebra and solving the cubic equations

2.1. Geometric algebra

Mathematics as a science created by the Greeks. They recognized geometry in the first half of the 6th century BC from the Egyptians, arithmetic — even earlier from the Phoenicians. The legend says that the Greek philosopher Thales from Egypt, the city of Miletus, visited Egypt and brought geometry information for the Greeks. He is considered the first of the thinkers who applied proof in mathematics.

Already the first steps of the Greeks in mathematics were marked by major achievements. The most remarkable of these were the awareness of unlimited continuation of natural number series and the transformation of geometry into a deductive science.

However, along with the successes, the Greeks faced with serious difficulties. The first of these was the detection in the school of Pythagoras of the incommensurability of the side and the diagonal of the square. Now we would regard this as a positive discovery — like the discovery of irrational numbers. However, for the Pythagoreans, this discovery had a different meaning.

The fact is that only quantities were called numbers at that time, i.e., what we now call natural numbers. The Pythagorean worldview was based on the idea that the universe consists of an infinite number of negligibly small indivisible particles — atoms. Connecting atoms form bodies, just as units, "connecting," form numbers. True, bodies have form. But the numbers of the Pythagoreans also had the form — were oblong, triangular, square-like, pyramidal, and so on. Therefore, the Pythagoreans believed that all knowledge could be expressed in the language of arithmetic. Hence, their slogan: "Everything is a number!"

Greek scientists themselves successfully resolved the crisis that struck them. The main outcome of the crisis were new remarkable discoveries of Greek mathematics.

It is precisely "due to the crisis" that geometric algebra and the theory of the ratios of magnitudes were created and the method of shards was developed (known as the exhaustion method since the 17th century).

The discovery of incommensurability showed that the means of arithmetic (in that understanding) are not enough to substantiate mathematics. This prompted the Greek scientists to seek support in geometry. The equation $x^2 = 2$ turned out to be unsolvable in the region of integers and their relations, but it was completely solvable in the domain of continuous quantities — segments. Its solution was a diagonal of a square, the side of which is equal to the unit of measurement. Replacing in a similar way numbers with segments, and actions with numbers as actions over the segments, the Pythagoreans (yes, namely they were the first among those who became so act) presented all the arithmetic in the form of geometric algebra.

Geometric algebra made it possible for the first time to prove in a very general way the basic properties of arithmetic operations and even some algebraic identities, for example, $(a+b)^2 = a^2 + 2ab + b^2$.

However, all this was done only for quantities within two dimensions. To multiply three segments, it was necessary to use spatial figures, which was not done by the ancient Greeks. And about the product of four segments, it was said that such a product does not exist.

Solving cubic equations

Cordano was the first from whom we have a text with a proof of identity having a deal with the product of three quantities. Surely, his predecessors Scipio del Ferro and Tartaglia also can do this, but from them, we do not have documents with such proof.

Strangely enough, Cardano writes about the cubes and parallelepipeds but draws the squares and rectangles. Due to this, we have a problem to restore a spatial drawing corresponding to his descriptions. Such a drawing represented in Figure 1. Explain what presented there.

Let $AD = u$ and $AE = v$, so that $ED = u - v$. Then $u^3 - v^3$ equals to the volume of the body obtained from the cube $ABCDA'B'C'D'$ by deleting the cube $AKHEA^*K^*H^*E^*$ (call it "the cube 1"). This body consists of the red cube $H^*F^*C^*L^*H'F'C'L'$ and three parallelepipeds, the yellow one, "lying on the side" behind the cube 1, the green one, standing in front to the yellow one and to the right of the cube 1, and the blue one, lying over the yellow parallelepiped and cube 1to the left of the green parallelepiped. The volume of the red cube equals $(u-v)^3$, the volume of each of the mentioned parallelepipeds equals $uv(u-v)$. These considerations yield the identity

$$(u-v)^3 + 3uv(u-v) = u^3 - v^3.$$

After that, looking at this identity, Cardano see

$$\underbrace{(u-v)^3}_{x^3} + \underbrace{3uv}_{p}\,\underbrace{(u-v)}_{x} = \underbrace{u^3 - v^3}_{q},$$

that is, the equality of the form

$$x^3 + px = q,$$

(in words, "cube with things equal number") with $x = u - v$, $p = 3uv$, $q = u^3 - v^3$.

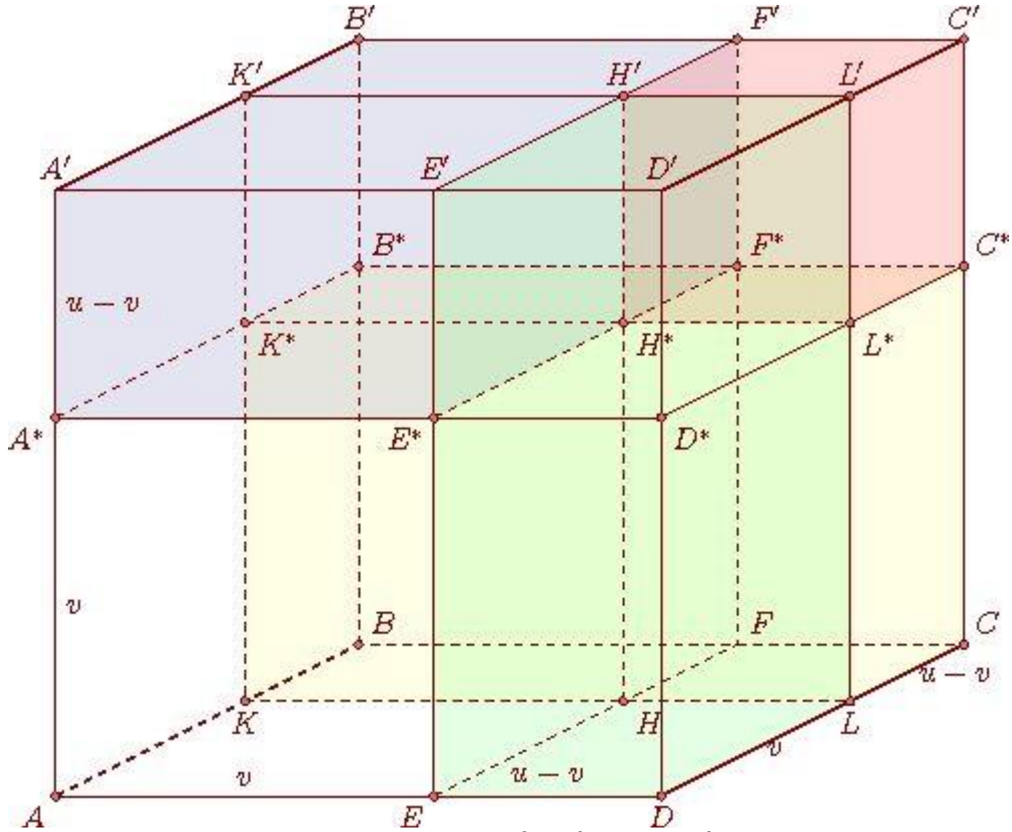


Figure 3. Proof of the identity $u^3 - v^3 = (u-v)^3 + 3uv(u-v)$

Again, in words, he describes the rule of solution for such equations and applies the rule to the two very similar to one another concrete equations, one of which is

$$x^3 + 6x = 20.$$

For the sake of completeness, bring here the solution of this equation.

Let $u^3 - v^3 = 20$ and $uv = 2$. Then $u^3 - v^3 = 20$ and $u^3v^3 = 8$. Solving these two equations simultaneously, that is, as the system

$$\begin{cases} \alpha - \beta = 20, \\ \alpha\beta = 8, \end{cases}$$

with $\alpha = u^3$, $\beta = v^3$, shows that

$$\alpha = \sqrt{108} + 10,\ \beta = \sqrt{108} - 10,$$

that is,

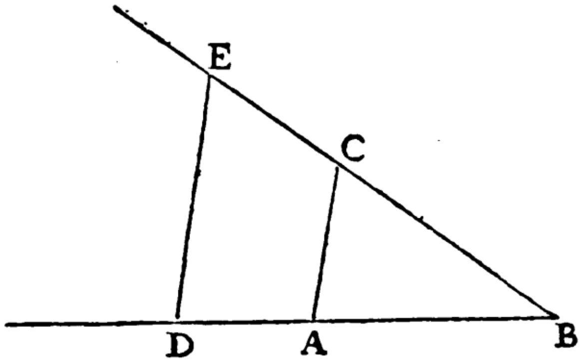
$$u = \sqrt[3]{\sqrt{108} + 10},\ v = \sqrt[3]{\sqrt{108} - 10}$$

and thus

$$x = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}.$$

Descartes' "Geometry"

In the previous subsection, we saw how the looking at the multiplication as the operation which leads from segments to rectangles or rectangular parallelepipeds forced mathematicians from the Ancient Greeks to Cardano perform our usual actions with the help of complicated geometric constructions. Descartes' genius was needed in order to mathematicians could see that it was possible to act much easier and more naturally. Namely, Descartes showed that multiplication could be defined so that the product of segments is again a segment. And what is very surprising, the means for this were for almost two thousand years before everyone's eyes! Reproduce here the corresponding piece from Descartes book "Geometry."



For example, let AB be taken as unity, and let it be required to multiply BD by BC. I have only to join the points A and C, and draw DE parallel to CA; then BE is the product of BD and BC.

If it be required to divide BE by BD, I join E and D, and draw AC parallel to DE; then BC is the result of the division.

Figure 4

All this enabled Descartes to suggest a universal method for solving geometrical problems, the method of equations; see about this [Naziev A. (2017)].

3. Eudoxus' ratios and modern theory of real numbers

Eudoxus' theory of ratios has come down to us in the presentation by Euclid, contained in the book V of his *Elements*.

Before outlining the main content of the book V, let us try to understand the fundamental difference in the positions of the Greek and modern mathematicians in the question of relationships. For modern mathematics, *every ratio* of two quantities is expressed by a number. Insofar as for numbers the basic arithmetic operations are defined, there is no need to create a special theory of operations on relationships.

A completely different state of affairs was in Greek mathematics. For Greek mathematician number, $\alpha\rho\iota\theta\mu o\varsigma$, is, first of all, what is now called a natural number, collection of units, in addition, specially arranged (so-called figured numbers: triangular, square-like, oval, pyramidal…). But very soon the Greek mathematicians understood that numbers in this sense and even their ratios can express not all elements of regular geometric shapes, by that what we now call fractions. How in such cases to understand the ratio of two quantities?

So long as ratios were expressed be integers, for determining the ratio of two quantities, it was necessary to repeat the smaller quantity so many times as necessary to that the result of the repetition equals the larger quantity. If this number was equal to m, then:

the smaller quantity, taken m times, was equal to the larger value;

the smaller quantity taken $m-1$ times turned out to be less than the larger value,

and taken $m+1$ times it turned out to be more than the larger value.

From here, it is clear how to change the definition of a ratio in the case of fractional numbers. Let the ratio of two quantities a and b be expressed by a fractional number $\frac{m}{n}$. Then:

if we take m times b and n times a, then the results will be equal;

if we take $m-1$ times b, then we will get the result less than na;

if we take $m+1$ times$b$, we get the result larger than na.

Thus, to determine fractional ratios, we use these relations:

$(m-1)b < na,$
$mb = na,$
$(m+1)b > na.$

Suppose now that the ratio of a and b is not expressed by a fractional number, so that the quantities and b are incommensurable. Then the middle relation is impossible, but both extremes remain fair. Namely them uses Eudox for defining the ratio of two quantities in the most common case.

As already mentioned, the Eudox theory of relations reached us in Euclidean presentation ("Elements", book V). It was based on a common concept of magnitude, covering and numbers (in the sense described above), and continuous values: length, area, volume. This concept is introduced using axioms describing the basic properties of equality and inequality. In Euclid, these axioms are formulated as follows.

1. *Things equal to the same thing are also equal to one another.*
2. *And if equal things are added to equal things then the wholes are equal.*
3. *And if equal things are subtracted from equal things then the remainders are equal.*
4. *And things coinciding with one another are equal to one another.*
5. *And the whole [is] greater than the part.*

However, the range of magnitudes obeying these axioms is still too wide so that you can define ratios over them. Before that, it was necessary to specify which magnitudes can have a ratio. The following definition does this.

BOOK V, DEFINITION 4.*Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another*.

The purpose of this definition is twofold. First, it breaks everything magnitudes on classes of homogeneous quantities so that the magnitudes of one class relate to each other, and values of different classes not relate. For example, lengths are related to each other, and areas have a relationship with each other, but the length and area not have.

Indeed, taking the appropriate number of times the same area (here we must bear in mind that the word "area" means to the Greeks a piece of the plane), you can cover the segment, but no matter how many times you repeat the segment, the area you will not cover.

And secondly, acceptance of this definition guarantees that in each class, the quantities are Archimedean, i. e. satisfy to the so-called "Archimedes' axiom": whatever quantities $a \neq 0$ and b may be selected, there exists an integer m such that $b < ma$.

BOOK V, DEFINITION 5. *Quantities are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.*

In order to extract understandable sense from this definition, represent it in more appropriate notation. To the left, we wrote original formulation split into several rows, to the right corresponding representation in modern expressions.

Quantities	For each quantities a_1, a_2, a_3, a_4,
are said	it is said that
to be in the same ratio, the first to the second and the third to the fourth,	$a_1 : a_2 = a_3 : a_4,$
when,	when,
if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth,	for every m and n,
the former equimultiples alike exceed,	$ma_1 > na_2$ and $ma_3 > na_4$
	or
are alike equal to,	$ma_1 = na_2$ and $ma_3 = na_4$
or	or
alike fall short of, the latter equimultiples respectively taken in	$ma_1 < na_2$ and $ma_3 < na_4$.

corresponding order.

Note that the statement

$$\begin{gathered}\text{for every } m \text{ and } n,\\ ma_1 > na_2 \text{ and } ma_3 > na_4\\ \text{or}\\ ma_1 = na_2 \text{ and } ma_3 = na_4\\ \text{or}\\ ma_1 < na_2 \text{ and } ma_3 < na_4\end{gathered}$$

is equivalent to the statement

$$\begin{gathered}\text{for every } m \text{ and } n,\\ ma_1 > na_2 \leftrightarrow ma_3 > na_4.\end{gathered}$$

Thanks to this, the whole statement, after translation it to the contemporary language, means
For each quantities a_1, a_2, a_3, a_4,

$$a_1 : a_2 = a_3 : a_4,$$

when,

$$(\forall m, n \in \mathbb{N})(ma_1 > na_2 \leftrightarrow ma_3 > na_4).$$

Now, compare this statement with the modern theory of real numbers. It is well known that rational numbers constitute a dense subset in the set of all real numbers. This means that between every two (distinct) real numbers there must be at least one rational number. In other words, whatever real numbers α and β will be selected, if $\alpha < \beta$, then there exists at least one rational number r such that $\alpha < r < \beta$. From this, for all real numbers α and β,

$$\alpha < \beta \leftrightarrow (\forall r \in \mathbb{Q})(r < \alpha \leftrightarrow r < \beta).$$

Let now $\alpha = \frac{a_1}{a_2}$, $\beta = \frac{a_3}{a_4}$. Then given the above equivalence we have

$$\alpha < \beta \leftrightarrow (\forall m,\ n \in \mathbb{N}) \left(\frac{a_1}{a_2} < \frac{n}{m} \leftrightarrow \frac{a_3}{a_4} < \frac{n}{m} \right)$$
$$\leftrightarrow (\forall m,\ n \in \mathbb{N})(ma_1 < na_2 \leftrightarrow ma_3 < na_4).$$

This is, how we have seen, equivalent to what is written in Euclid. Highly remarkably! Four centuries before the new era, Eudoxus was able to formulate a statement equivalent to the modern proposition about the density of the set of rational numbers in the set of real numbers!

4. Diophantus, school geometry, Fermat and modern mathematics

Let us begin with a citation from “Arithmetic” by Diophantus.

BOOK II, PROBLEM 8. The given square to decompose into two squares.

To this problem, Diophantus gives the following solution.

Let it be necessary to decompose 16 into two squares. Suppose that the 1st is x^2; then the 2nd will be $16 - x^2$; therefore, $16 - x^2$ is also equal to a square. I am making a square of some amount of x minus as many units as there are in the side of 16; let it be $2x - 4$. Then this square itself is $4x^2 + 16 - 16x$; it should be .$16 - x^2$. Add to both sides the missing and subtract the like from similar ones. Then $5x^2$ is equal to $16x$ and x will be equal to 16 fifths. One square is 256/25 and the other is 144/25; both stacked give 400/25, or 16, and each will be a square.

So it is stated at the Diophantus. In order to better understand what is being done here, let us turn to more familiar notation.

Pay attention to the phrase: “I am making a square of *some* quantities x minus *as many units* as there are in the side of 16. *Let* it be $2x - 4$”.

The words we have highlighted show that numbers 2 and 4 are used by Diophantus as parameters, and only a disadvantage symbolism does not allow him to lead a decision in the whole community. Thus, we will better understand Diophantus, if instead of 16 we take a^2, and instead of $2x-4$ taking $kx-a$ with an arbitrary integer k. Then the equation takes the form

$$x^2+(kx-a)^2=a^2.$$

Solving it according to the scheme proposed by Diophant, for every $k \neq 0$ gives:

$$x=\frac{2k}{k^2+1}a, \quad y=\frac{k^2-1}{k^2+1}a.$$

From this is seen that given problem has infinitely many solutions in rational numbers. Diophantus himself tells here nothing about the number of solutions, but in the solution of the problem 19 of book III he says like about well known, “we know that decomposition of a given square into two squares can be made in infinite ways.”

Now, look at this problem and its solution from a geometrical point of view. The equation “$x^2+y^2=a^2$” determines a circle of the radius a with center at the origin, while the equation “$y=kx-a$” determines the sheaf of lines with tangents k throw the point $(0, \ -a)$ on this circle (look at the following drawing).

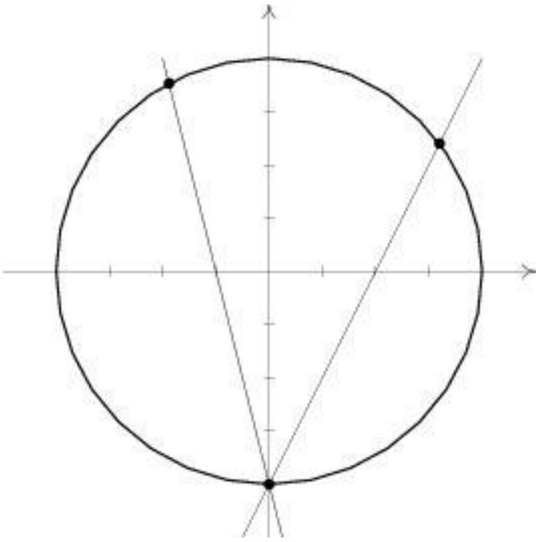


Figure 5. Diophantine circle

Moreover, since all the straight lines of the sheaf pass through one point, different lines intersect the circle at different points, so that the intersection points are “as many” as the straight lines. It is clear from the form of the found solution that any rational (and not just an integer) number can be taken as k so that the straight lines with arbitrary rational coefficients included in the indicated bundle. This shows that the points of intersection of the lines of the sheaf with the circle are densely located on the circle.

Thus, the solution of the above problem given by Diophantus shows that if the radius of a circle centered at the origin is a rational number, then on this circle there are infinitely many points with rational coordinates, and these points are everywhere dense on the circle. And from the given geometric interpretation it is clear that the same will be the case if the square of the radius is the sum of the squares of two rational numbers, the case considered by Diophantus in the next problem of the same book.

We chose this particular task because it is closely connected, firstly, with the school course of mathematics and, secondly, with the events in modern mathematics.

First, say about the school course. In the current textbooks in geometry for secondary schools, among other things, they prove that if the distance from the center of the circle to the line is less than the radius of the circle, then the circle and the line have two points in common. Do we need this proof? Is it not obvious?

Considering the set $\mathbb{Q}^2$ of all ordered pairs of *rational* numbers will help us to answer this question. We will use for this set the usual interpretation by points of the coordinate plane; and we will define straight lines and circles by ordinary equations (with rational coefficients, of course). Let us see how these figures will look like on the drawings.

Let us start with straight lines. Will the image of a straight line over the field of rational numbers differ from the image of a straight line over the field of real numbers? Nothing will be different. No matter how sharp the pencil is, its point will still have a non-zero thickness. And rational numbers form a dense set everywhere: between any two rational numbers, no matter how close they are to each other, there is an infinite number of other rational numbers. It turns out that rational straight line in the drawing looks the same as the real one.

With circles, the situation is more complicated. What some of them look like, Diophantus helped us to see. Namely, the above solution and its commentary showed that if the right side of the equation defining the circle is the square of a rational number or the sum of the squares of two rational numbers, then there are infinitely many rational points on this circle, and they are located everywhere densely. Therefore, such circles over the field of rational numbers in the drawing look the same as in case of the field of real numbers.

We now consider the image on the coordinate plane of the rational bisector of the first and third coordinate angles and the Diophantus' circle. "With our eyes in the forehead," we see two points of intersection (drawing on the left below), and with the help of "mental eyes," we see that there are no such points (drawing on the right)!

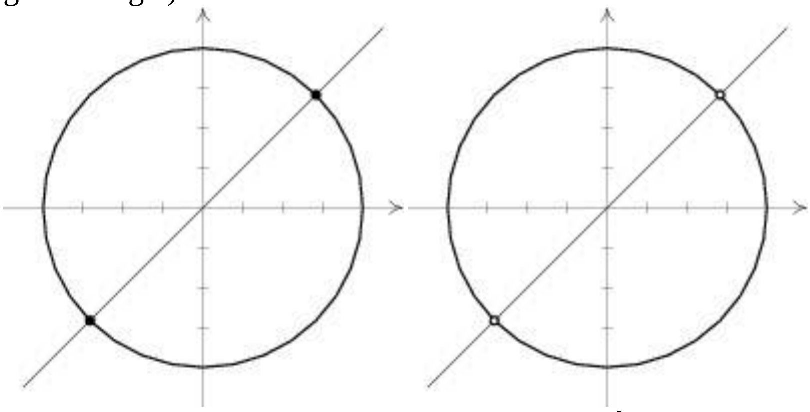


Figure 6. A circle and the lines in $\mathbb{Q}^2$

Thus Diophantus helps to understand that the "fact" of the intersection of a straight line with a circle only seems obvious and, therefore, must be proved. This explains why a corresponding proof is included in the geometry textbook. In addition — and this may be much more important — Diophantus helps to understand what a big role play numbers in geometry and how much the laws of geometry change depending on what field it is considered. But not only that.

Consider on the real plane the circle determined by the equation

$$x^2 + y^2 = 3. \tag{1}$$

How many rational points are there on this circle? It turns that no one!
In fact, let with some integers k, l, m and n it turns out that

$$\left(\frac{k}{l}\right)^2 + \left(\frac{m}{n}\right)^2 = 3.$$

Then

$$(kn)^2 + (ml)^2 = 3(mn)^2. \tag{2}$$

Without loss of generality, we can assume that the numbers kn, ml and mn are coprime. In this case, the numbers kn and ml cannot be both multiples of 3, because then the left side of the equality (2) would be a multiple of 9, and mn should be a multiple of 3, contrary to the assumption of mutual simplicity of the numbers kn, ml and mn. This means that either one of the numbers kn, ml has the form $3s$, and the other one is of the form $3t \pm 1$, or both of them are of the form $3p \pm 1$. In any of these cases, the left side of the equality (2) is not divisible by 3, while the right one is.

Amazing! Points with rational coordinates are located on the plane everywhere densely, and, nevertheless, the circle defined by the equation (1}), "manages" slip past them! But for straight lines this is impossible: on every straight line there are both rational and irrational points.

This is about the connection of the considered problem from Diophantus with the school course of mathematics. And now — about the connection with modern mathematics, namely — with the Great Fermat Theorem.

As is known, Fermat did not publish his results on number theory but wrote down them in the fields of his copy of the 1671 edition of Diophantus' Arithmetic. To the considered problem of the Book II, Fermat attributed:

"On the contrary, it is impossible to decompose neither a cube of two cubes, nor a biquadrat of two biquadrats, and generally no a degree greater than a square by two degrees with the same indicator. I discovered to this truly miraculous proof, but these fields are too small to contain it".

For 325 years after that, many outstanding mathematicians unsuccessfully tried to prove this statement made by Fermat. Finally, in 1995, this was able to do English mathematician Andrew Wiles. The interesting reader can find in the Internet dramatical history of Wiles' searchers for the proof.

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As a professor, he has developed and implemented in the practice of teaching at the university level more than ten author’s courses in mathematics and mathematics teaching; in particular, “A Course in Foundations of Mathematics” for students of Mathematical faculties of Pedagogical Universities (4 semesters); “A Course in Financial Calculations” for economical faculties; “Proofs in Mathematics and Mathematics Teaching” for Masters program in mathematical faculty, etc.. He has prepared ten Ph.D. students successfully defended their thesis on the methodology of mathematics teaching.