Semantic Reading in Mathematics and Mathematics Teaching

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Abstract

This author already published one paper about semantic reading in mathematics teaching (Naziev, 2014). The paper contained short instructions on what semantic reading is and several more or less simple examples of the application of semantic reading in school algebra, geometry, probability, and calculus. In this paper, we will give a more instructive definition of semantic reading and several more complicated and, we hope, more interesting examples.

Our work is connected with the results in artificial intelligence (Garrido, 2017). An important subject in the artificial intelligence is the automated (or mechanical) theorem proving, and, in particular, mechanical *geometry* theorem proving (Chou, 1988). The development of this last area has showed the evidence that in order to carry out proofs of geometry theorems mechanically, we have to *strictly* follow some rules and axioms (Chou, 1988). This is the main goal of our article, to show how important it is in mathematics, not only in artificial intelligence, to *strictly* follow axioms, definitions and theorems, that is, to read them semantically.

Keywords: Teaching Mathematics, Semantic Reading, Quantifiers, Quadratic Trinomial, Archimedes, Zeno.

1. Semantic reading: definition and first examples

To find the right approach to the definition of the notion of semantic reading, the following dictum of the great mathematician will help us:

> Mathematics is the art of calling different things by the same name."

(Henri Poincare)

Note that every speech act connects two sides: the sender (the speaker, the writer) and the recipient (the listener, the reader). The sender calls different things by the same name, but this does not mean that the recipient automatically sees these different things with the same name. He must be ready and capable of this. Thus, for *our purposes*, the above quotation of H. Poincare should be clarified:

> Mathematics is the art of naming different things with the same name, *combined with the art of seeing different things behind the same name*.

Note, further, that if the same name names different things, then each message, in the context of which this name is included, inevitably reports various things, that is, carries with it different meanings. Thus, the art of calling the same name different things should have the continuation in the art of seeing in the same message different meanings.

Now, in general, it is clear what needs to be called semantic reading.

> *Semantic reading* is the art of discovering in the readable all the variety of meanings contained in it and extract from it the meaning that most closely corresponds to the problem being solved.

REMARKS. 1) Semantic reading is an art. By this reason, we cannot teach semantic reading. You can teach the craft, but not the art. Art can only be mastered, studying patterns, imitating them and improving them. For this reason, both in our previous work, and in the now proposed one, there are almost no theoretical considerations. But there are many examples, imitating and improving, the reader can gradually, through tireless independent exercises, begin to master this difficult art.

2) Perhaps it would be more correct to talk here about the semantic *perception* or even about the semantic *comprehension*. But we will not go far from the usual terminology. Let it be a semantic *reading*. Only, let's not forget that one can read not only texts in the usual sense of the word.

Animals read by smelling. People sometimes read through the eyes. The musicians read the notes. Analyzes are read in chemical laboratories. Radio-operators read the audio signals transmitted by the radio. Seamen read messages sent using signal flags. And so on. So, the word "reading" is not a big limitation here.

We underline:

• semantic reading is an art;
• "all diversity" is, of course, ideally a more mundane formulation — "as many senses as possible";
• and also, to see; it is desirable to use as many senses as possible, but in each specific situation, as a rule, only one of them is needed.

Let's give the first examples of productive use of semantic reading.

EXAMPLE 1. In the sentence $a+b=0$, where a and b denote real numbers, one can see at least four senses. The first is straightforward: the sum of numbers a and b equals 0. But we can look at a and say what we see: we see that the sentence is telling to us that a is a number whose sum with b equals 0. What does this mean? This means that a is opposite to b, that is $a=-b$. Proceeding analogously with b we discover that the sentence is telling that $b=-a$. Then, combining, we will "hear" from the sentence that $a=-(-a)$ and $b=-(-b)$. Surely, it is quite important that pupils can see all these senses and use them in their work.

EXAMPLE 2. In the sentence $2^3=8$ we can see many senses. The first is straightforward: 2^3 is the same number as 8. But we can look at 2 and say what we see. This gives us the new sense of the above statement, namely that 2 is a number whose cube equals 8, that is, 2 is cubic root of 8, $2=\sqrt[3]{8}$. And this means that there exists cubic root of 8 — the third sense of the given sentence. Looking to 3 at the given sentence we will see that 3 is the exponent in which 2 gives 8, that means that 3 is the logarithm of 8 on the base 2, $3=\log_2 8$ — the forth sense. And this means that there exists the logarithm of 8 on the base 2 — the fifth sense! Five senses in one sentence!

EXAMPLE 3. And in the sentence $\sin' x=\cos x$ on $\mathbb{R}$ we can see at least four senses. The first is straightforward: in every point $x\in\mathbb{R}$ the derivative of sin equals cos. This gives to us a new sense: the function sin has derivative, that is, it is derivable on $\mathbb{R}$. Turning to integral calculus we learn a new sense of this statement according to which sin is antiderivative of cos on $\mathbb{R}$. And from here – the forth sense: the function cos has antiderivative, that is, it is integrable on $\mathbb{R}$.

In conclusion of this series of examples let me repeat one example from my previous work, my favorite example, my own proof of one well known identity.

EXAMPLE 4. We know from the properties of the module of real number that $|a|\geqslant 0$ and that $|a|^2=a^2$. As far as we think about these properties separately they constitute only two separated properties, and nothing more. But let's combine them in conjunction

$$|a|\geqslant 0 \text{ and } |a|^2=a^2.$$

We will see that $|a|$ is a nonnegative number whose square equals a^2. What does this mean? This means that $|a|$ is the arithmetical square root of a^2, that is $\sqrt{a^2} = |a|$.

We obtained the new proof of well-known identity and obtained it by discovering a new sense in known statements, that is, by semantic reading. I do not know any mathematics textbooks that contains this remarkable proof (I hope, you will agree that this proof is remarkable). In my opinion, this is because traditional didactic of mathematics does not pay much attention to this side of mathematics teaching, to the semantic reading. The traditional didactics of mathematics is based on actions and "rules of processing" (Freudenthal, 1991, p. 3). Interested readers can find more detailed discussions of this subject in the first section of my above mentioned paper.

2. The role of mathematics in the formation of the ability to semantic reading

Much is written about using semantic reading in the teaching of native or foreign language, literature, history, chemistry, biology, and comparatively little about the semantic reading in the teaching of mathematics. The role of mathematics in the formation of the ability to semantic reading and focusing on it is underestimated. And in vain. In reality,

no academic subject awakens in a person the ability to understand the meaningful speech addressed to him as mathematics does.

To get a feel of it, let us look at an example.

Osip Mandelstam

SILENTIUM

She who has not yet been born
Is both word and music
And so the imperishable link
Between everything living.

The sea's chest breathes calmly,
But the mad day sparkles
And the foam's pale lilac
In its bowl of turbid blue.

May my lips attain
The primordial muteness,
Like a crystal-clear sound
Immaculate since birth!

Remain foam, Aphrodite,
And—word—return to music;
And, fused with life's core,
Heart be ashamed of heart!

1910

Besides the word "silentium" in the title, which means silence, we understand every word in this poem. But what is the sense of the poem in whole? The reader can find in the literature tens of works devoted to the discovering the sense of this poem. We suggest the reader to find his own solution to this problem. (We must only warn the reader that the translation does not accurately convey the meaning of the original. In Mandelstam, the poem begins the Russian word "она", which can be translated both as "she" and as "it". The control question to those who want to discover the sense of this poem would be: what is this she or it?)

Now, compare this poem with the following one:

If two sides of one triangle and the angle between them are equal to the sides and angle between them of another triangle then these triangles are congruent.

In this "poem" we can easily understand not only every word but the statement in the whole. And even if we could not, we know where we can find answers to all our questions. And in the process of learning mathematics teachers show us how one can search the answers to such questions.

3. Semantic reading in the problems of elementary geometry

One of the divisions of mathematics, where semantic reading is supposed to be vital but constantly ignored, is school geometry. The reason is quite understandable, although not always noticed. It is in a false sense of clarity, generating the illusion of evidence in geometry. Here are examples that support these statements.

The first of these examples we took from a remarkable book on the mechanical geometry theorem proving, (Chou, 1988) (although this example does not belong to Chou, it appears also in the book Adler C. F., 1958).

EXAMPLE. Every triangle is isosceles.

PROOF. Let ABC be a triangle as shown in Figure 1. We want to prove $CA = CB$. Let D be the intersection of the perpendicular bisector of AB and the internal bisector of angle ACB. Let $DE \perp AC$ and $DF \perp CB$. It is easy to see that $\triangle CDE \cong \triangle CDF$ and $\triangle BDF \cong \triangle ADE$. Hence $CE + EA = CF + FB$, i.e., $CA == CB$. QED!

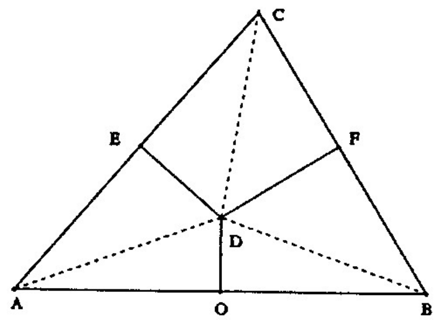

Figure 1

This example clearly reveals the shortcomings of teaching mathematics in isolation from the semantic reading. It follows this guideline: "Allow your hand to draw everything that it pleases, look at what happened, and base the solution of the problem on the fantasies of your hand!"

The example is really remarkable, but it is *invented* while the following examples are taken from *real life*, namely, from a textbook written by the famous Russian mathematician and mathematics didact A. V. Pogorelov, Pogorelov A. V. (2013).

§ 5, **PROBLEM 3**. Prove that if the diameter of a circle passes through the midpoint of the chord then the diameter is perpendicular to the chord.

And the author provides the following Figure.

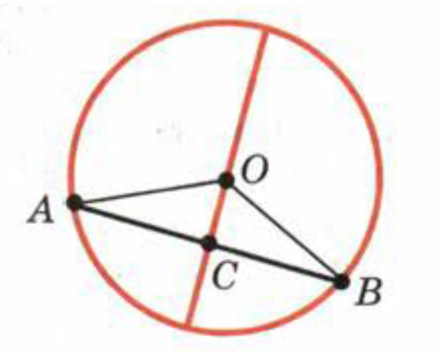

Figure 2. A circle, a chord, and a diameter

But the triangle *AOB* will not exist if the chord passes through the center of the circle, and the diameter in this case is not necessary perpendicular to the chord (draw the corresponding picture!). So, this famous mathematician and mathematics didact builds the solution not on axioms, definitions and theorems but also on extra considerations such as a Figure drawn by his hand! This mistake is especially astonishing because, first, it is evident, and, second, even in Euclid *Elements* one can see the correct statement: if a diameter of a circle passes through the midpoint of a chord **that is not a diameter**, then the diameter is perpendicular to the chord.

4. Semantic reading and the explication of quantifiers

Quantifiers first appeared explicitly in 1879 in the little but great book *Begriffsschrift: eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* by Gottlob Frege (1848–1925). Thanks to this book, it became gradually obvious that mathematical language is literally filled with quantifiers. However, the methodology of teaching mathematics has not yet realized this. In the traditional formulation of statements, quantifiers are shamefully omitted, making it difficult to understand the real sense of what is asserted. And sometimes it takes a lot of effort to restore quantifiers in their rights and to reveal the true meaning of the statements. Let us illustrate these claims with examples. We begin with the traditional formulation of a well-known theorem about medians.

EXAMPLE. *Medians of a triangle intersect at one point.*

Regularly I show to my students and to teachers I am working with the following picture (see Figure 3)

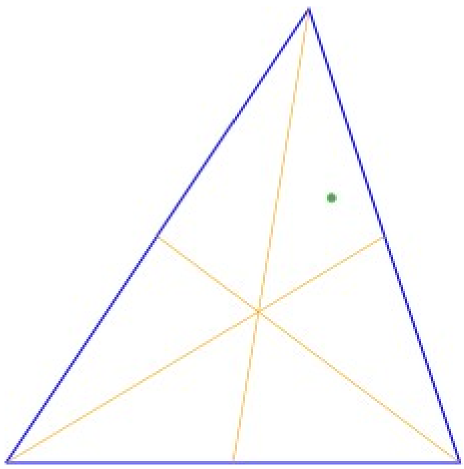

Figure 3. A triangle with medians and "one" point

and tell them: "This is a triangle, these are its medians and this is one point; but the medians do not want to pass through this one point!". It is funny, but this unpretentious example confuses many students and even many math teachers. As a rule, it is necessary to ask many questions before the "honest" formulation of the theorem appears. "Which triangle, this or that? What point, any or not?" After answering these questions, we, at last, came to the genuine sense of this theorem.

*In **every** triangle, **there exists** a point through which **all** its medians pass.*

Without this reformulation there is no and cannot be a correct understanding of the meaning of the theorem under consideration.

But this does not exhaust the role and significance of this reformulation. We see in it what this author calls "quantifiers zigzag", "*for every … there exists … such that for every …*", $\forall\,\exists\,\forall$. Precisely this logical structure has many theorems for analysis: $(\forall\varepsilon>0)(\exists\delta>0)(\forall x)(\ldots)$. It is well known that these theorems present great difficulties for beginners to study mathematical analysis at the university. These difficulties would not have been so great if the foundations for understanding such theorems were laid in the school course of mathematics. There exist a lot of theorems in school mathematics having such a logical structure: analogous statements about bisectors, altitudes, mediatrixes, statements about existing inscribed and circumscribed circles, spheres and so on.

Now, turn to the definition of rational numbers in its traditional form.

A number is said to be rational if it is a quotient of the division of a whole number by a natural number.

Or, more shortly:

A number α is said to be rational if $\alpha=\frac{m}{n}$ where m is whole, n is natural.

In my teaching practice, there was the following almost fantastic case. I proposed to the pupils of one mathematics (sic!) class the

PROBLEM. Prove that the number $\sin 1^\circ$ is irrational.

After short period, suddenly I heard from the pupils that the problem is trivial because it has the following

"**SOLUTION**". To say that $\sin 1^\circ$ is rational means to say that $\sin 1^\circ=\frac{m}{n}$ where m is whole, n is natural. So, to say that $\sin 1^\circ$ is irrational means to say that $\sin 1^\circ\neq\frac{m}{n}$ where m is whole, n is natural. This is true because $\sin 1^\circ\neq\frac{-3}{2}$ where -3 is whole and 2 is natural.

Why did this mistake appear? Because the teacher, when he defined the set of rational numbers, omitted the quantifiers and gave to his pupils the abovementioned "definition" instead of genuine one:

*A number α is said to be rational if **there exist** numbers m and n such that $\alpha=\frac{m}{n}$ and m is whole, n is natural.*

According to this definition, the rationality of a number is the fact of the *existence*, not of the *equality*!

5. About the meaning of the expression "roots of a square trinomial" and erroneous formulations of Vieta's theorem for quadratic trinomials

The following remarks have arisen from a comparison of two solutions of the problem

(Skanavi M. I. ,2013, PROBLEM 6.125) Find the coefficients A and B of the equation

$$x^2+Ax+B=0, \tag{1}$$

if it is known that the numbers A and B are also its roots.

FIRST SOLUTION. Let A and B be the roots of the equation (1). Then, by Vieta's theorem,

$$\begin{cases} A+B &= -A; \\ A\cdot B &= B. \end{cases} \tag{2}$$

By the first equation $B=-2A$, by the second one $B=0$ or $A=1$. Therefore 1) $A=0$ and $B=0$ or 2) $A=1$ and $B=-2$.

It is easy to verify that, conversely, in each of these two cases the numbers A and B are the roots of the given equation.

ANSWER. $A=0$ and $B=0$; or $A=1$ and $B=-2$.

It seems that the authors of the problem, solved it in exactly this way because they have the same answer.

Now imagine a pupil who has not yet studied the theorem of Vieta. He will solve this task in another way.

SECOND SOLUTION. Let A and B be the roots of the equation (1). Then, by virtue of the definition of the root of an equation,

$$\begin{cases} 2A^2+B &= 0; \\ B^2+AB+B &= 0. \end{cases} \tag{3}$$

By the second equation, $B=0$ or $B=-A-1$. In the first case, from the first equation, we find that $A=0$; in the second case the first equation takes the form

$$2A^2-A-1=0$$

and has two roots: $A=1$ and $A=-\frac{1}{2}$.

Thus, the system has three solutions: 1) $A=0,\ B=0$; 2) $A=0,\ B=-2$; 3) $A=B=-\frac{1}{2}$.

It is easy to verify that, conversely, in each of the three cases the numbers A and B are the roots of the given equation.

ANSWER. $A=0,\ B=0$; $A=0,\ B=-2$; $A=B=-\frac{1}{2}$.

We obtained the answer different from the above! Let's try to discover the cause of the discrepancy.

Let $A=B=-\frac{1}{2}$. Then the equation (1) takes the form:

$$x^2-\tfrac{1}{2}\cdot x-\tfrac{1}{2}=0 \tag{4}$$

By a direct calculation, we can convince ourselves that A and B are indeed the roots of the equation. However, their sum is *not* equal to the second coefficient and the product is *not* equal to the free term! Strange… In the school textbook we read (Freudenthal, 1991, p. 247):

> if x_1 and x_2 are the roots of the equation $x^2+px+q=0$, then $x_1+x_2=-p$ and $x_1x_2=q$.

Hence,

> if A and B are the roots of the equation $x^2+Ax+B=0$, then $A+B=-A$, $AB=B$.

In particular, taking $A=B=-\frac{1}{2}$, we obtain:

if $-\frac{1}{2}$ and $-\frac{1}{2}$ are the roots of the equation $x^2-\frac{1}{2}\cdot x-\frac{1}{2}=0$,
then $-\frac{1}{2}+-\frac{1}{2}=-\frac{1}{2}$ and $(-\frac{1}{2})(-\frac{1}{2})=-\frac{1}{2}$.

Since the numbers $-\frac{1}{2}$ and $-\frac{1}{2}$ really are the roots of the equation $x^2-\frac{1}{2}\cdot x-\frac{1}{2}=0$, must hold the equalities $-\frac{1}{2}+-\frac{1}{2}=-\frac{1}{2}$ and $(-\frac{1}{2})(-\frac{1}{2})=-\frac{1}{2}$. But they obviously do not hold!

And here is another theorem (Freudenthal, 1991, P. 20):

If x_1 and x_2 are the roots of the quadratic trinomial ax^2+bx+c,

then $ax^2+bx+c=a(x-x_1)(x-x_2)$.

Hence,

If A and B are the roots of the quadratic trinomial x^2+Ax+B, then $x^2+Ax+B=(x-A)(x-B)$.

In particular, for $A=B=-\frac{1}{2}$, we obtain:

if $-\frac{1}{2}$ and $-\frac{1}{2}$ are the roots of the equation $x^2-\frac{1}{2}\cdot x-\frac{1}{2}=0$ then $x^2-\frac{1}{2}\cdot x-\frac{1}{2}=(x+\frac{1}{2})^2$.

Again the premise of the theorem is correct, therefore, the conclusion must be true, but it is obviously wrong.

Let us estimate the situation. In the facts that $(-\frac{1}{2})+(-\frac{1}{2})\neq-\frac{1}{2}$, $(-\frac{1}{2})(-\frac{1}{2})\neq-\frac{1}{2}$ and $x^2-\frac{1}{2}x-\frac{1}{2}\neq(x+\frac{1}{2})^2$ there are no doubts. It turns out that it is not true that $-\frac{1}{2}$ and $-\frac{1}{2}$ are the roots of the equation $x^2-\frac{1}{2}\cdot x-\frac{1}{2}=0$. This is fantastic: $-\frac{1}{2}$ is the root of the equation but $-\frac{1}{2}$ and $-\frac{1}{2}$ are not the roots!

So the question naturally arises, what is this, the rootS of the equation?

In fact, what does it mean that x_1 and x_2 are the rootS of the equation? Weird question! Asserting that A and B are integers, it means to assert, exactly, that A is an integer and B is an integer. Asserting that A and B are non-empty sets, means to assert, precisely, that A is a nonempty set and B is a non-empty set. To assert, that A and B are Russians, means to assert, exactly, that A is a Russian and B is a Russian. So things are going everywhere, in mathematics and in real life. And only in one place, in the algebra textbook, in the study of quadratic equations, the language is used differently. To assert that that x_1 and x_2 are the roots of a square trinomial does not mean to assert that x_1 is a root and x_2 is a root, it means to assert something else! What exactly?

Perhaps, in the above theorems from the school textbook it is implicitly assumed that x_1 is the smaller and x_2 is the larger root of a square trinomial? — If it was supposed so, this does not matter, because the expressions x_1+x_2 and $x_1\cdot x_2$ are symmetric relatively to x_1 and x_2.

Or maybe the matter is that the problem posed about the rootS of the equation (in the plural), that is implicitly assumed, that A and B are different? — No, they are talking about the roots also in those cases where the root is only one. For example, directly before the above theorem from P. 20, we read:

in the case where the discriminant of the quadratic trinomial equals zero,
they say that this trinomial has two equal roots.

To understand the matter here, let us compare the cases $A = B = 0$ and $A = B = -\frac{1}{2}$. In both cases, A and B are the roots of the equation $x^2 + Ax + B = 0$, but in the first case, these are **all** its roots while in the second one, **not all are its roots**. It turns out that the matter is just in this. The correct formulation of the above theorem is another.

THEOREM 1. Let x_1 and x_2 be the roots of a square trinomial $x^2 + px + q$ **and it has no other roots**. Then

$$\begin{cases} p &= -(x_1 + x_2); \\ q &= x_1 x_2. \end{cases}$$

PROOF. Suppose that the conditions of the theorem are satisfied. Then, since x_1 is a root, $x_1^2 + px_1 + q = 0$. Hence, for all x,

$$\begin{cases} x^2 + px + q &= x^2 + px + q - (x_1^2 + px_1 + q) \\ &= (x - x_1)(x + x_1 + p). \end{cases}$$

This shows that $-(x_1 + p)$ is the root of our trinomial. Let's show that it equals to x_2. We will argue as follows.

By the hypothesis, the trinomial has no roots other than x_1 and x_2. This means that $-(x_1 + p) = x_1$ or $-(x_1 + p) = x_2$. If $-(x_1 + p) = x_2$, then there is nothing to prove. If $-(x_1 + p) = x_1$ then, by virtue of the decompositions obtained, the trinomial has only one root. Hence, $x_1 = x_2$ and again, $-(x_1 + p) = x_2$.

Thus, $-(x_1 + p) = x_2$. Hence $p = -(x_1 + x_2)$ and $q = -(x_1^2 + px_1) = x_1 \cdot (-(x_1 + p)) = x_1 x_2$.

This is what we had to prove.

The converse is also true. (By the way, Vieta himself formulated this converse, which in the current school textbooks is called the converse theorem of Vieta.)

THEOREM 2. Let x_1 and x_2 be any real numbers and

$$\begin{cases} p &= -(x_1 + x_2); \\ q &= x_1 x_2. \end{cases}$$

Then x_1 and x_2 are the roots of the square trinomial $x_1^2 + px_1 + q$ and **this trinomial has no other roots**.

PROOF. Indeed, let the conditions of the theorem be satisfied. Then, for any x,

$$\begin{cases} x^2 + px + q &= x^2 - (x_1 + x_2)x + x_1 x_2 \\ &= x^2 - x_1 x - x_2 x + x_1 x_2 \\ &= x(x - x_1) - x_2(x - x_1) \\ &= (x - x_1)(x - x_2). \end{cases}$$

Hence, for any x,

$$x^2 + px + q = 0 \Leftrightarrow (x - x_1)(x - x_2) = 0.$$

From this, it is clear that x_1 and x_2 both are the roots of the square trinomial $x_1^2 + px_1 + q$ and **it has no other roots**.

REMARK. By the way, this theorem shows that traditional formulation of the so-called Vieta's theorem in the school textbooks in mathematics (provided at the beginning of this part of the paper) is not correct.

Now it is easy to explain the reason for the discrepancy in the answers obtained above. Strictly speaking, the first of the solutions we gave is erroneous, because, as shows the second of the theorems proved here, for the fulfillment of the Vieta's relations is not sufficient that the numbers A and B are the roots of the equation $x^2+Ax+B=0$, it is also necessary that there be no other roots.

Since nothing is said about this in the condition of the problem, the correct solution is the second one. This is exactly how any normal man would solve this problem, not spoiled by erroneous statements of Vieta's theorem, like cited above. (For no normal person after hearing that Ivan and Peter are Russians, will think that, thus, there are no other Russians!)

DIGRESSION (on formalization of Vieta's theorem). Let us formalize (for teachers, not for pupils!) our considerations. The assertion that x_1 and x_2 are the roots of the equation $x^2+px+q=0$ means that whenever $x=x_1$ or $x=x_2$, then the equation holds, that is $x^2+px+q=0$ or, in short,

$$(\forall x)(x=x_1 \vee x=x_2 \to x^2+px+q=0).$$

The assertion that the equation has no other roots than x_1 and x_2 means that the converse of this statement holds, that is, every root of the equation equals either x_1 or x_2, in short

$$(\forall x)(x^2+px+q=0 \to x=x_1 \vee x=x_2).$$

The assertion that $x=x_1$ and $x=x_2$ are the roots of the equation **and** there exist no other roots means that the conjunction of these two statements holds,

$$(\forall x)(x=x_1 \vee x=x_2 \to x^2+px+q=0) \wedge (\forall x)(x^2+px+q=0 \to x=x_1 \vee x=x_2).$$

Applying to this conjunction the law of distributivity of the general quantifier and the conjunction we get

$$(\forall x)(x^2+px+q=0 \leftrightarrow x=x_1 \vee x=x_2).$$

So, Vieta's theorem consists in that whenever this hold, then $x_1+x_2=-p$ and $x_1 \cdot x_2=q$, that is

$$(\forall x)(x^2+px+q=0 \leftrightarrow x=x_1 \vee x=x_2) \to x_1+x_2=-p \wedge x_1 \cdot x_2=q.$$

The converse of the Vieta's theorem means that the converse of this implication holds, what in conjunction with this implication gets the equivalence

$$(\forall x)(x^2+px+q=0 \leftrightarrow x=x_1 \vee x=x_2) \leftrightarrow x_1+x_2=-p \wedge x_1 \cdot x_2=q.$$

Taking into account that this is true for all x_1 and x_2, we get, so to say, the bilateral Vieta's theorem,

$$(\forall x_1)(\forall x_2)((\forall x)(x^2+px+q=0 \leftrightarrow x=x_1 \vee x=x_2) \leftrightarrow x_1+x_2=-p \wedge x_1 \cdot x_2=q).$$

Of course, there can be no question of treating these considerations with schoolchildren. But the teacher needs to look at the full wording at least once, so that he understands that it is not so simple with Vieta's theorem. The simple formulations and easy proofs take a place only because of additional, difficultly formalized considerations are added to them unnoticed.

6. Semantic reading in the history of mathematics

History of mathematics is the area of knowledge where semantic reading is vital. In this section, we adduce examples of the results obtained in the past with the help of what is now named semantic reading, and examples of the contemporary application of semantic reading to the solution of the problems left to us from the past.

From the past we have the message (see Figure 4):

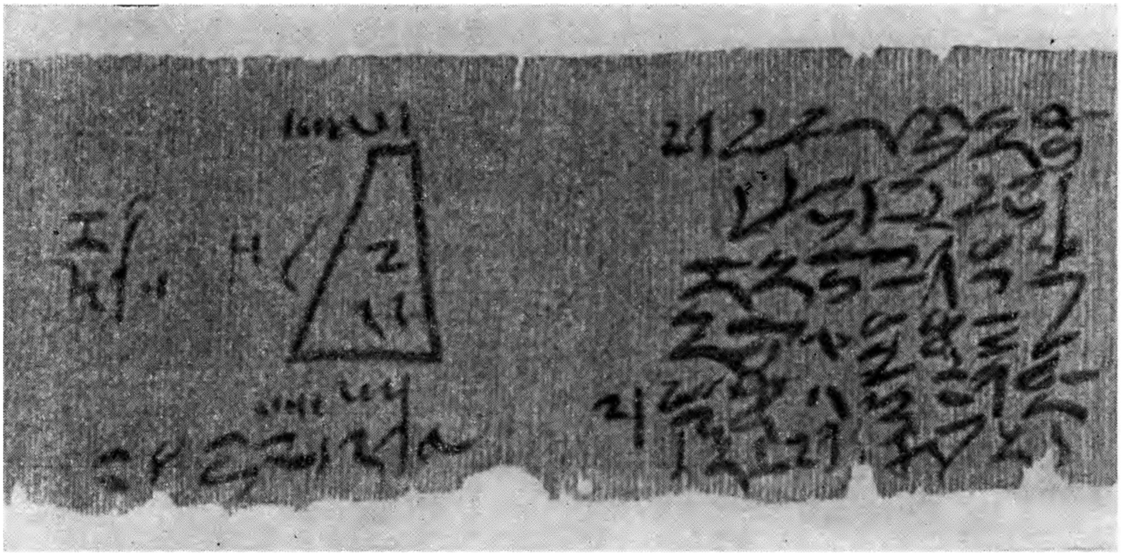

Figure 4

First of all, we must translate it to a modern language (and this also requires semantic reading). Upon translation we get something like this (the text is in Russian, see Figure 5):

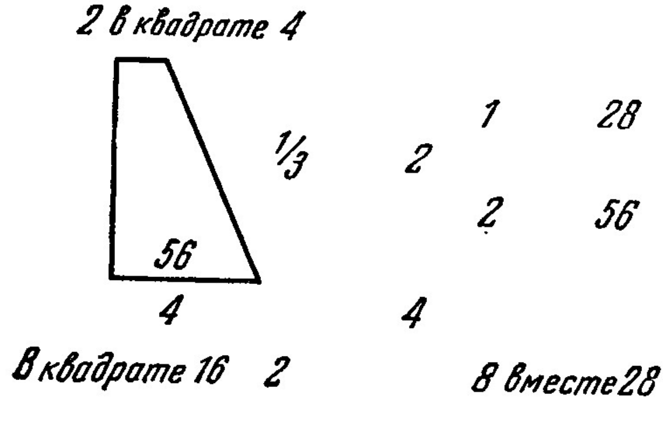

Figure 5

After that we need to include a sense into this set of signs. When Russian academician W. W. Struve made this, he saw the rule of evaluation of the volume of a regular truncated pyramid with a square base! That is what semantic reading is!

6.1. Semantic reading in “The Method” by Archimedes

The greatest master of semantic reading was Archimedes (287–212 BC). Let us look on his discovering of, how they say, the formula of the volume of the sphere, or, more precisely, the interrelation between volumes of a cylinder and a sphere inscribed in it, displayed in his work “The method” (Archimedes, 1962).

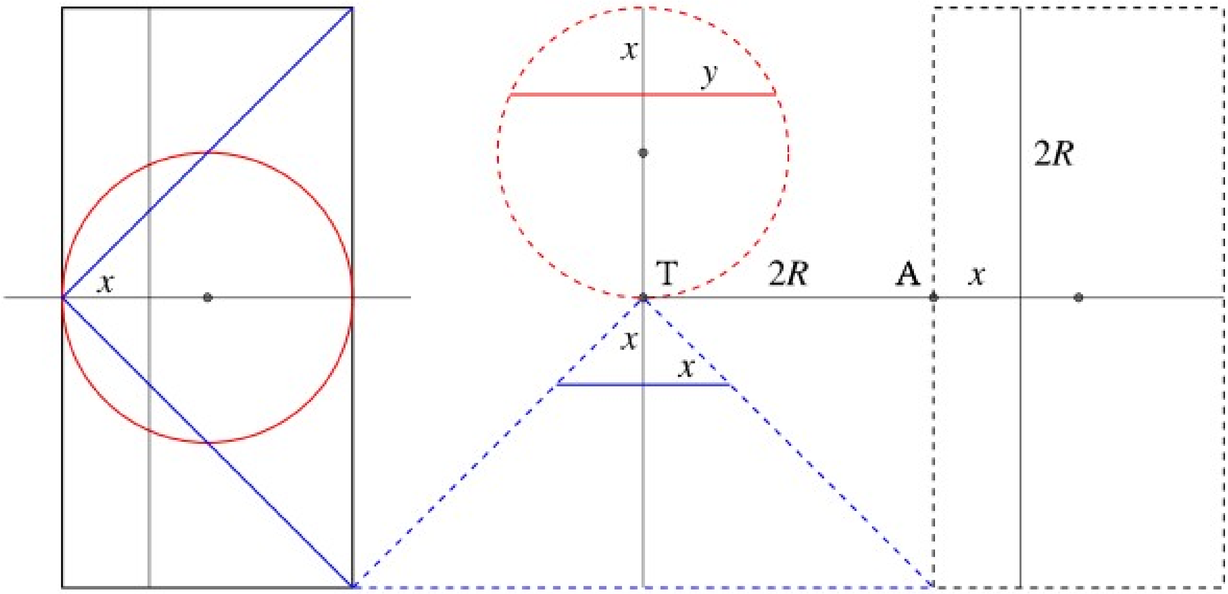

Figure 6. Archimedes: determining the volume of the ball inscribed to the cylinder

Archimedes begins with a sphere of radius R and a cylinder and a cone of radiuses $2R$ and of heights also $2R$, placed as it is shown in section on the left picture above (axes of the cylinder and the cone are horizontal). He intersects this combination of bodies by a plane perpendicular to the axes and detached to the distance x from the vertex of the cone (black segment on the left picture below). Then the cylinder gives in the section a circle of radius $2R$, the cone gives a circle of radius x, and the sphere gives a circle of the radius, say, y. Well known statement about products of the segments of two intersecting chords in a circle gives in this particular case the equality $y^2 = x \cdot (2R - x)$, or, equivalently, $x^2 + y^2 = 2Rx$. Multiplying both sides of this equality by π and $2R$ he obtained that $\pi(x^2 + y^2) \cdot 2R = 4\pi R^2 \cdot x$. All this does not require semantic reading. But what did Archimedes do further? In the last equality he sees the statement about equilibrium of the lever. Namely, he takes the lever with the support, say, A and arms AT of longitude $2R$ and AX of longitude x, and places the circle of radius x and the circle of radius y so that the center of gravity of each turns out to be below T, and places the circle of radius $2R$ so that it is perpendicular to the lever and the center of its gravity turns out to be in X. Then turns out that the equality $\pi(x^2 + y^2) \cdot 2R = 4\pi R^2 \cdot x$ expresses, exactly, the condition of equilibrium of the described lever (look at Figure 6 where the first two circles are represented by red and blue horizontal segments while the third one by the black vertical segment; in order to reduce the size of picture we placed one circle above T, not below).

After that Archimedes writes:

“If now, taking such circles, fill them as the cylinder as the ball with the cone, then the cylinder, remaining in the same position, will be in equilibrium with the ball and the cone together with respect to the point A if we transfer them to the lever in T and placed so that the center of gravity of each turns out to be below T.”

This gives to him the equality

$$(v_{\text{ball}} + V_{\text{cone}}) \cdot 2R = V_{\text{cyl}} \cdot R$$

where all denotations are naturally interpreted. Now denote by v_{cyl} the volume of the cylinder circumscribed around the sphere. We obtain that

$$\begin{aligned} v_{\text{ball}} &= \tfrac{1}{2}V_{\text{cyl}} - V_{\text{cone}} \\ &= \tfrac{1}{2}V_{\text{cyl}} - \tfrac{1}{3}V_{\text{cyl}} \\ &= \tfrac{1}{6}V_{\text{cyl}} \\ &= \tfrac{2}{3}v_{\text{cyl}}, \end{aligned}$$

and, finally,

$$\boxed{v_{\text{ball}} = \tfrac{2}{3}v_{\text{cyl}}}.$$

In words, a ball takes two thirds of the volume of the circumscribed cylinder.

We must underline that Archimedes did not consider the above reasoning a proof of this interrelation. For him this was only a way to guess what should be the result, after what the result must be proved through mathematical, not mechanical, reasoning (in the times of Archimedes these were the so called "apagogical reasoning").

6.2. Semantic reading for the solution of Zeno's aporia "Achilles"

From Archimedes turn to Zeno with his aporia "Achilles". We know about this aporia from the mention in Aristotle's *Physics* and from the commentary Simplicius(b) *On Aristotle's Physics*, 1014.10 (see Stanford Encyclopedia of Pholosophy in our references). Here we try to resolve this aporia with the help of semantic reading.

First, recall the formulation of the aporia. Achilles and the turtle are moving in a straight line in one direction. At the initial moment, Achilles is on some distance behind the turtle, but it is clear that he must catch up with her (because he is fleet-footed!). No, argued Zeno. Before Achilles catches up with the turtle, he must reach the place where the turtle was at the moment of their launch. In the time that he needs for this, the turtle will have time to move on, so that when Achilles gets to the point where the tortoise was at the moment of launch, the turtle will again be ahead of Achilles, and everything will start from the beginning. So, concluded Zeno, Achilles will never overtake the tortoise.

What is paradoxical in this reasoning? We are sure that Achilles must overtake tortoise but Zeno show us that this will not be the case. We are sure, let us repeat it, that Achilles will overtake tortoise. How will this happen? Do we think that Achilles suddenly equals a turtle? No, familiar notions tell us that if Achilles catches up with the turtle, then the distance between them will *gradually* decrease to zero. Even this simple remark removes a valued part of the paradox in Zeno's reasoning. Zeno simply lists some points in the process of chase, and the moments he lists them are *preceding* the moment when the turtle will be overtaken. So far, nothing is paradoxical.

And after that, the conclusion is suddenly made: "So, Achilles will **never** overtake the tortoise!"

There exists vast literature devoted to this aporia. They constructed theories of space and time, theories of motion but nobody thought to ask the

QUESTION: What does the word "never" mean here?

To make it clear why we ask this question, we must turn for the minute to another question. What does the word "always" mean in the statement: "The number $n(n+1)$ is always divisible by 2"? Does it mean "at all times" or "for all n"? Although, of course, every mathematician is sure that he establishes truths for all time, nevertheless it is not a question of times, but of natural numbers, and "always" in this statement it means "for all n". So, what does "never" mean in Zeno's reasoning? If it means "in none of the above moments", then the conclusion to which Zeno inclines us is correct, but nothing in it is paradoxical. All the reasoning of Zeno is then a particular case of the following:

"Let T_1, T_2, …, T_n, … be the moments, in which the distance between Achilles and the tortoise is still different from zero. Then in none of these moments does Achilles overtake the tortoise."

Very good, and no paradox! If, however, "never" refers to the time that is flowing regardless of enumerations chosen by Zeno, then Zeno's conclusion is unfounded. To justify it, it would be necessary to establish that for any moment of time T there exists a moment T_n such that $T < T_n$. Now we know that to prove this is impossible (because the sequence (T_n) is a sequence of partial sums of an infinitely decreasing geometric progression, and therefore it is limited). The Greeks in Zeno's time did not know this, but this is unimportant. In order to compel the "bi-tongue" (as ancient Greeks called Zeno) to say no more, they did not need to know this. It was enough to ask Zeno to justify his conclusion. He would not have the success, and the problem would have been lifted.

That is all! I think that this is the final solution of the "Achilles" aporia.

References

Archimedes (1962). A message to Eratosthenes. On mechanical theorems. In Veselovskii I. N. (ed.) *Archimedes. Collected works.* Moscow: State Publishing of Physical and Mathematical Literature (in Russian).

Adler C. F. (1958) *Modern Geometry*, McGraw-Hill Book Company Inc., 1958.

Chou, Shang-Ching (1988) *Mechanical Geometry Theorem Proving*. Dordrecht, Holland: D. Reidel Publishing Company.

Frege G. (1879) *Begriffsschrift: eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle: Verlag von Louis Nebert.

Freudenthal H. (1991) *Revisiting mathematical education*. Dordrecht, Boston, London: Kluwer Academic Publishers.

Garrido, A. (2017). Brain and Artificial Intelligence. BRAIN – Broad Research in Artificial Intelligence and Neuroscience, Volume 8, Issue 4, pp. 85-90.

Merzlyak A. G., Polonskii V.B., Yakir M. S. (2013) *Algebra. 8 grade*. Moscow: Ventana-Graph (in Russian).

Naziev A. (2014) Semantic reading in mathematics teaching. In *Методички Аспекти Наставе Математике III: Зборник радова са трећег међународног научног скупа одржаног 14–15 јуна 2014 године на Факултету педагошких наука Универзитета у Крагујевцу*. Jagodina: Универзитет у Крагујевцу, P. 107–119.

Pogorelov A. V. (2013) *Geometry. 7–9 grades*. Moscow: Prosvescschenie.

Skanavi M. I. (ed.) (2013) *A Collection of Problems in Mathematics for Applicants to Technical Colleges*. Moscow: Publishig House "Mir i Obrazovanie" (in Russian).

Stanford Encyclopedia of Philosophy: https://plato.stanford.edu/entries/paradox-zeno/

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