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ON THE (COLORED) YANG-BAXTER EQUATION

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Abstract. The quantum Yang-Baxter equation first appeared in theoretical physics and statistical mechanics. Afterwards, it has proved to be important also in knot theory, quantum groups, etc. This paper deals with the (colored) Yang-Baxter equation and computational methods. A new result about the set-theoretical Yang-Baxter equation is presented.

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1. Introduction and Preliminaries

The quantum Yang-Baxter equation first appeared in theoretical physics and statistical mechanics. Afterwards, it has proved to be important also in knot theory, quantum groups, the quantization of integrable non-linear evolution systems, etc.

Throughout this paper k is a field. All tensor products appearing in this paper are defined over k. For V a k-space, we denote by $\tau : V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$, and by $I : V \to V$ the identity map of the space V.

We use the following notation concerning the Yang-Baxter equation.
If $R : V \otimes V \to V \otimes V$ is a k-linear map, then $R^{12} = R \otimes I$, $R^{23} = I \otimes R$, $R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition. An invertible k-linear map $R : V \otimes V \to V \otimes V$ is called a Yang-Baxter operator if it satisfies the equation

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{1}$$

Remark. The equation (1) is usually called the braid equation. It is a well-known fact that the operator R satisfies (1) if and only if $R \circ \tau$ satisfies the constant quantum Yang-Baxter equation (QYBE), if and only if $\tau \circ R$ satisfies the constant QYBE:

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} \tag{2}$$

Remark (i) $\tau : V \otimes V \to V \otimes V$ is an example of a Yang-Baxter operator.
(ii) An exhaustive list of invertible solutions for (2) in dimension 2 was obtained in [5].
(iii) Finding all Yang-Baxter operators in dimension greater than 2 is an unsolved problem.

Let A be an associative k-algebra, and $\alpha, \beta, \gamma \in k$. We define the k-linear map:

$$R^A_{\alpha,\beta,\gamma} : A \otimes A \to A \otimes A, \quad R^A_{\alpha,\beta,\gamma}(a \otimes b) = \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b$$

Theorem. (S. Dăscălescu and F. F. Nichita, [4]) Let A be an associative k-algebra with $\dim A \geq 2$, and $\alpha, \beta, \gamma \in k$. Then $R^A_{\alpha,\beta,\gamma}$ is a Yang-Baxter operator if and only if one of the following holds:
(i) $\alpha = \gamma \neq 0, \quad \beta \neq 0$;
(ii) $\beta = \gamma \neq 0, \quad \alpha \neq 0$;
(iii) $\alpha = \beta = 0, \quad \gamma \neq 0$.
If so, we have $(R^A_{\alpha,\beta,\gamma})^{-1} = R^A_{\frac{1}{\beta},\frac{1}{\alpha},\frac{1}{\gamma}}$ in cases (i) and (ii), and $(R^A_{0,0,\gamma})^{-1} = R^A_{0,0,\frac{1}{\gamma}}$ in case (iii).

Remark. In the next section, we will generalize the construction given in the previous theorem, in order to solve another type of Yang-Baxter equation.

2. The main result

There are many versions of the Yang-Baxter equation. We present a lesser known version of this equation (which is called the "colored Yang-Baxter equation" by some authors) in this paper.

We attempt to find solutions for it and to explain how computational methods could help us. In another paper, we will study its applications in theoretical physics.

Formally, a colored Yang-Baxter operator is defined as a function

$$R : k \times X \times X \to End_k V \otimes V,$$

where X is a set and V is a finite dimensional vector space over a field k.

Thus, for any $x \in k, \ u, v \in X, \ R(x,u,v) : V \otimes V \to V \otimes V$ is a linear operator.

As in the previous section, we consider three operators acting on a triple tensor product $V \otimes V \otimes V$, $R^{12}(x,u,v) = R(x,u,v) \otimes I$, $R^{23}(x,v,w) = I \otimes R(x,v,w)$, and similarly $R^{13}(x,u,w)$ as an operator that acts non-trivially on the first and third factor in $V \otimes V \otimes V$.

R is a colored Yang-Baxter operator if it satisfies the equation:

$$R^{12}(x,u,v)R^{13}(x+y,u,w)R^{23}(y,v,w) = R^{23}(y,v,w)R^{13}(x+y,u,w)R^{12}(x,u,v) \tag{3}$$

for all $x, y \in k,\ u, v, w \in X$.

We now apply our original method to find solutions for the equations (3).

We assume that X is equal to (a subset of) the ground field k. Our method of constructing solutions to equation (3) is based on the ideas applied before.

The key point of the construction is to suppose that $V = A$ is an associative k-algebra, and then to derive a solution to equation (3) from the associativity of the product in A.

Thus, we seek solutions to equation (3) of the following form

$$R(x, u, v)(a \otimes b) = \alpha_x(u, v)1 \otimes ab + \beta_x(u, v)ab \otimes 1 - \gamma_x(u, v)b \otimes a, \qquad (4)$$

where $\alpha_x, \beta_x, \gamma_x$ are k-valued functions on $X \times X$, for any $x \in k$.

Inserting this ansatz into equation (3), we obtained the following system of equations (whose solutions produce colored Yang-Baxter operators):

$$\begin{aligned}(\beta_y(v, w) - \gamma_y(v, w))(\alpha_x(u, v)\beta_{x+y}(u, w) - \alpha_{x+y}(u, w)\beta_x(u, v))\\ +(\alpha_x(u, v) - \gamma_x(u, v))(\alpha_y(v, w)\beta_{x+y}(u, w) - \alpha_{x+y}(u, w)\beta_y(v, w)) = 0\end{aligned} \qquad (5)$$

$$\begin{aligned}\beta_y(v, w)(\beta_x(u, v) - \gamma_x(u, v))(\alpha_{x+y}(u, w) - \gamma_{x+y}(u, w))\\ +(\alpha_y(v, w) - \gamma_y(v, w))(\beta_{x+y}(u, w)\gamma_x(u, v) - \beta_x(u, v)\gamma_{x+y}(u, w)) = 0\end{aligned} \qquad (6)$$

$$\begin{aligned}\alpha_x(u, v)\beta_y(v, w)(\alpha_{x+y}(u, w) - \gamma_{x+y}(u, w)) + \alpha_y(v, w)\gamma_{x+y}(u, w)(\gamma_x(u, v) - \alpha_x(u, v))\\ +\gamma_y(v, w)(\alpha_x(u, v)\gamma_{x+y}(u, w) - \alpha_{x+y}(u, w)\gamma_x(u, v)) = 0\end{aligned} \qquad (7)$$

$$\begin{aligned}\alpha_x(u, v)\beta_y(v, w)(\beta_{x+y}(u, w) - \gamma_{x+y}(u, w)) + \beta_y(v, w)\gamma_{x+y}(u, w)(\gamma_x(u, v) - \beta_x(u, v))\\ +\gamma_y(v, w)(\beta_x(u, v)\gamma_{x+y}(u, w) - \beta_{x+y}(u, w)\gamma_x(u, v)) = 0\end{aligned} \qquad (8)$$

$$\begin{aligned}\alpha_x(u, v)(\alpha_y(v, w) - \gamma_y(v, w))(\beta_{x+y}(u, w) - \gamma_{x+y}(u, w))\\ +(\beta_x(u, v) - \gamma_x(u, v))(\alpha_{x+y}(u, w)\gamma_y(v, w) - \alpha_y(v, w)\gamma_{x+y}(u, w)) = 0\end{aligned} \qquad (9)$$

Remark.

(i) The system of equations (5–9) is rather non-trivial. It is an open problem to classify its solutions.

(ii) That system has some remarkable symmetry properties which can be used to find some solutions. For example, the equations (6) and (9) are in some sense dual to each other. Likewise, (7) and (8) are in some sense dual to each other.

(iii) If we look at the system of equations (5–9), one simplification that produces many solutions is to let α_x, β_x and γ_x be functions that do not depend on x. In this case we may call them α, β, and γ. Letting γ be an arbitrary function and a an arbitrary element of the base field k, one verifies that setting $\alpha = a\gamma$ and $\beta = \gamma$ gives a solution. Similarly, letting γ and $b \in k$ be arbitrary, setting $\alpha = \gamma$ and $\beta = b\gamma$ gives a solution.

3. The set-theoretical Yang-Baxter equation

If X is a set, let $S : X \times X \to X \times X$ be a function, $S^{12} = S \times I$ and $S^{23} = I \times S$.

Definition. Using the above notation, the set-theoretical Yang-Baxter equation reads:

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23} \tag{10}$$

We obtained the following theorem, which will be proved in another paper.

Theorem. Let X be a set and $R \subset X \times X$ a reflexive relation on X.
We define the function $S = S_R : X \times X \to X \times X$, by
$S(u, v) = (u, v)$ if $(u, v) \in R$,
$S(u, v) = (v, u)$ if $(u, v) \notin R$.
Then, S satisfies (10) if and only if $R \cup R^{op}$ is an equivalence relation, and $\bar{R}$ is a strict partial order relation on each class of $R \cup R^{op}$.
(We let R^{op} denote the opposite relation of R and let $\bar{R}$ denote the complementary relation of R.)

4. Conclusions

Computational methods are very important in solving the Yang-Baxter equations. For example, Hietarinta found all R-matrices in the case n=2 using Grobner basis and computer calculations (see [5], [6]). These papers are very much cited by many authors.

However, the computational methods are not powerful enough to fully classify the solutions for other small dimensions: a complete computer calculation for n=3 is still out of reach at this time.

On the other hand, the computational methods can be very useful in approaching problems in Quantum Group Theory. For example, one can explicitly solve the set-theoretical Yang-Baxter equation (10) for small sets be exhaustive search. Computational methods are helpful for solving other equations and problems in Quantum Group Theory.

In Section 3, we give a new theorem about the set-theoretical Yang-Baxter equation which will be proved and studied in future papers. We anticipate interesting connections of it with the constructions given in Section 2.

Finally, we propose the problem of finding algorithms for solving the system of equations (5–9).

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