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Mathematical Problems in the Theory of Bone Poroelasticity

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***Abstract*—This paper concerns with the quasi-static theory of bone poroelasticity for materials with double porosity. The system of equations of this theory based on the equilibrium equations, conservation of fluid mass, the effective stress concept and Darcy's law for material with double porosity. The internal and external basic boundary value problems (BVPs) are formulated and uniqueness of regular (classical) solutions are proved. The single-layer and double-layer potentials are constructed and their basic properties are established. Finally, the existence theorems for classical solutions of the BVPs are proved by means of the potential method (boundary integral method) and the theory of singular integral equations.**

***Keywords*-bone poroelasticity; double porosity; boundary value problems.**

I. Introduction

The concept of porous media is used in many areas of applied science (e.g., biology, biophysics, biomechanics) and engineering. Poroelasticity is a well-developed theory for the interaction of fluid and solid phases of a fluid saturated porous medium. It is an effective and useful model for deformation-driven bone fluid movement in bone tissue [1], [2], [3].

The theory of consolidation for elastic materials with double porosity was presented by Aifantis and his coworkers [4], [5]. The Aifantis' theory unifies the earlier proposed models of Barenblatt's for porous media with double porosity [6] and Biot's for porous media with single porosity [7].

However, Aifantis' quasi-static theory ignored the cross-coupling effects between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid by several authors [8], [9], [10].

The double porosity concept was extended for multiple porosity media by Bai et al. [11]. The theory of multiporous media, as originally developed for the mechanics of naturally fractured reservoirs, has found applications in blood perfusion. The double porosity model would consider the bone fluid pressure in the vascular porosity and the bone fluid pressure in the lacunar-canalicular porosity [1], [2], [3]. An extensive review of the results in the theory of bone poroelasticity can be found in the survey papers [1], [2]. For a history of developments and a review of main results in the theory of porous media, see de Boer [12].

The investigation of BVPs of mathematical physics by the classical potential method has a hundred year history. The application of this method to the 3D BVPs of the theory of elasticity reduces these problems to 2D singular integral equations [13]. Owing to the works of Mikhlin [14], Kupradze and his pupils (see [15], [16]), the theory of multidimensional singular integral equations has presently been worked out with sufficient completeness.

This theory makes it possible to investigate 3D problems not only of the classical theory of elasticity, but also problems of the theory of elasticity with conjugated fields. For an extensive review of works on the potential method, see Gegelia and Jentsch [17].

This paper concerns with the quasi-static theory of bone poroelasticity for materials with double porosity [8], [9], [10]. The system of equations of this theory based on the equilibrium equations, conservation of fluid mass, the effective stress concept and Darcy's law for a material with double porosity. The internal and external basic BVPs are formulated and uniqueness of classical solutions are proved. The single-layer and double-layer potentials are constructed and their basic properties are established. Finally, the existence theorems for classical solutions of the BVPs are proved by means of the boundary integral method and the theory of singular integral equations.

II. Basic Equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space R^3, let t denote the time variable, $t \geq 0$, $\mathbf{u}(\mathbf{x}, t)$ denote the displacement vector in solid, $\mathbf{u} = (u_1, u_2, u_3)$; $p_1(\mathbf{x}, t)$ and $p_2(\mathbf{x}, t)$ are the pore and fissure fluid pressures, respectively. We assume that subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range (1,2,3), and the dot denotes differentiation with respect to t.

In the absence of body force the governing system of field equations of the linear quasi-static theory of elasticity for solid with double porosity consists of the following equations [8], [9], [10].

1) *The equilibrium equations*

$$t_{lj,j} = 0, \quad l = 1, 2, 3, \tag{1}$$

where t_{jl} is the component of the total stress tensor.

2) *The equations of fluid mass conservation*

$$\begin{aligned} &\operatorname{div} \mathbf{v}^{(1)} + \dot{\zeta}_1 + \beta_1 \dot{e}_{rr} + \gamma(p_1 - p_2) = 0, \\ &\operatorname{div} \mathbf{v}^{(2)} + \dot{\zeta}_2 + \beta_2 \dot{e}_{rr} - \gamma(p_1 - p_2) = 0, \end{aligned} \tag{2}$$

where $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are fluid flux vectors for the pores and fissures, respectively; e_{lj} is the component of the strain tensor,

$$e_{lj} = \frac{1}{2}\left(u_{l,j} + u_{j,l}\right), \quad l, j = 1, 2, 3, \tag{3}$$

β_1 and β_2 are the effective stress parameters, γ is the internal transport coefficient and corresponds to a fluid transfer rate respecting the intensity of flow between the pores and fissures, $\gamma > 0$; ζ_1 and ζ_2 are the increment of fluid in the pores and fissures, respectively, and defined by

$$\zeta_1 = \alpha_1 p_1 + \alpha_3 p_2, \quad \zeta_2 = \alpha_3 p_1 + \alpha_2 p_2, \tag{4}$$

α_1 and α_2 measure the compressibilities of the pore and fissure systems, respectively, α_3 is the cross-coupling compressibility for fluid flow at the interface between the two pore systems at a microscopic level [8], [9], [10]. However, the coupling effect (α_3) is often neglected in the literature [4], [5], [6].

3) *The equations of the effective stress concept*

$$t_{lj} = t'_{lj} - (\beta_1 p_1 + \beta_2 p_2)\,\delta_{lj}, \quad l, j = 1, 2, 3, \tag{5}$$

where $t'_{lj} = 2\mu e_{lj} + \lambda e_{rr}\delta_{lj}$ is the component of effective stress tensor, λ and μ are the Lamé constants, δ_{lj} is the Kronecker's delta.

4) *The Darcy's law for material with double porosity*

$$\mathbf{v}^{(1)} = -\frac{\kappa_1}{\mu'}\operatorname{grad} p_1, \quad \mathbf{v}^{(2)} = -\frac{\kappa_2}{\mu'}\operatorname{grad} p_2, \tag{6}$$

where μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with the matrix and fissure porosity, respectively. We note that in the real porous media the fissure permeability κ_2 is much greater than the matrix permeability κ_1, while the fracture porosity is much smaller than the matrix porosity.

Substituting equations (3)-(6) into (1) and (2), we obtain the following system of homogeneous equations in the linear quasi-static theory of elasticity for solids with double porosity expressed in terms of the displacement vector $\mathbf{u}$, pressures p_1 and p_2.

$$\begin{aligned} &\mu \Delta \mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} - \beta_1 \nabla p_1 - \beta_2 \nabla p_2 = \mathbf{0}, \\ &k_1 \Delta p_1 - \alpha_1 \dot{p}_1 - \alpha_3 \dot{p}_2 - \gamma(p_1 - p_2) - \beta_1 \operatorname{div}\dot{\mathbf{u}} = 0, \\ &k_2 \Delta p_2 - \alpha_3 \dot{p}_1 - \alpha_2 \dot{p}_2 + \gamma(p_1 - p_2) - \beta_2 \operatorname{div}\dot{\mathbf{u}} = 0, \end{aligned} \tag{7}$$

where Δ and ∇ are the Laplacian and gradient operators, respectively, and $k_j = \dfrac{\kappa_j}{\mu'}$ $(j = 1, 2)$.

In the follows we assume that the inertial energy density of solid with double porosity is a positive definite quadratic form. Thus, the constitutive coefficients satisfy the conditions:

$$\begin{aligned} &\mu > 0, \quad 3\lambda + 2\mu > 0, \quad k_1 > 0, \quad k_2 > 0, \\ &\alpha_1 > 0, \quad \alpha_1 \alpha_2 - \alpha_3^2 > 0. \end{aligned}$$

If the displacement vector $\mathbf{u}$, the pressures p_1 and p_2 are postulated to have a harmonic time variation, that is,

$$\{\mathbf{u}, p_1, p_2\}(\mathbf{x}, t) = \mathrm{Re}\left[\{\mathbf{u}', p_1', p_2'\}(\mathbf{x})e^{-i\omega t}\right],$$

then from system (7) we obtain the following system of homogeneous equations of steady vibrations in the linear quasi-static theory of elasticity for solids with double porosity

$$\begin{aligned} &\mu\Delta\mathbf{u}' + (\lambda+\mu)\nabla\mathrm{div}\mathbf{u}' - \beta_1\nabla p_1' - \beta_2\nabla p_2' = \mathbf{0},\\ &(k_1\Delta + a_1)p_1' + a_3 p_2' + i\omega\,\beta_1\,\mathrm{div}\,\mathbf{u}' = 0,\\ &a_3 p_1' + (k_2\,\Delta + a_2)p_2' + i\omega\,\beta_2\,\mathrm{div}\,\mathbf{u}' = 0, \end{aligned} \tag{8}$$

where $a_j = i\omega\,\alpha_j - \gamma$, $a_3 = i\omega\,\alpha_3 + \gamma$ $(l, j = 1, 2)$; ω is the oscillation frequency, $\omega > 0$.

III. Boundary value problems

Let S be the closed surface surrounding the finite domain Ω^+ in R^3, $S \in C^{2,\lambda_0}$, $0 < \lambda_0 \le 1$, $\bar{\Omega} = \Omega \cup S$, $\Omega^- = R^3 \backslash \bar{\Omega}^+$.

Definition 1. A vector function $\mathbf{U} = (\mathbf{u}', p_1', p_2') = (U_1, U_2, \cdots, U_5)$ is called regular in Ω^- (or Ω^+) if

1) $U_l \in C^2(\Omega^-) \cap C^1(\bar{\Omega}^-)$ (or $U_l \in C^2(\Omega^+) \cap C^1(\bar{\Omega}^+)$),

2)
$$U_l(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad U_{l,j}(\mathbf{x}) = o(|\mathbf{x}|^{-1}),$$

where $|\mathbf{x}| \gg 1, l = 1, 2, \cdots, 5,\ j = 1, 2, 3$.

The basic BVPs of steady vibrations in the linear quasi-static theory of elasticity for solid with double porosity are formulated as follows.

Find a regular (classical) solution $\mathbf{U} = (\mathbf{u}', p_1', p_2')$ to system (8) satisfying the boundary condition

$$\lim_{\Omega^+ \ni \mathbf{x} \to\ \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the *Problem* $(I)^+_{\mathbf{f}}$, and

$$\lim_{\Omega^- \ni \mathbf{x} \to\ \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the *Problem* $(I)^-_{\mathbf{f}}$, where $\mathbf{f}$ is the known five-component vector function.

IV. Uniqueness theorems

We have the following results.

Theorem 1. *The internal homogeneous BVP* $(I)^+_{\mathbf{f}}$ *admits at most one regular solution.*

Theorem 2. *The external BVP* $(I)^-_{\mathbf{f}}$ *admits at most one regular solution.*

Theorems 1 and 2 can be proved similarly to the corresponding theorems in the classical theory of thermoelasticity (for details see [13, Chapter III]).

V. Basic properties of elastopotentials

The system (8) may be written as $\mathbf{B}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{B}(\mathbf{D_x})$ is the matrix differential operator corresponding left-hand side of (8) and $\mathbf{D_x} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$.

We introduce the following notations:

1) $\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) = \int\limits_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the single-layer potential,

2) $\mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) = \int\limits_S [\tilde{\mathbf{P}}(\mathbf{D_y}, \mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{y})]^\top \mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$

is the double-layer potential, where $\mathbf{\Gamma} = (\Gamma_{lj})_{5\times 5}$ is the fundamental matrix of the operator $\mathbf{B}(\mathbf{D_x})$, $\tilde{\mathbf{P}} = \left(\tilde{P}_{lj}\right)_{5\times 5}$ is the matrix differential operator of the first order, $\mathbf{g}$ is five-component vector, the superscript $\top$ denotes transposition.

Remark 1. In the Aifantis' quasi-static theory ($\alpha_3 = 0$), the fundamental matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed by Svanadze [18].

We have the following basic properties of elastopotentials.

Theorem 3. *If* $S \in C^{2,\lambda_0}$, $\mathbf{g} \in C^{1,\lambda'}(S)$, $0 < \lambda' < \lambda_0 \le 1$, *then:*

(a)
$$\mathbf{Z}^{(1)}(\cdot, \mathbf{g}) \in C^{0,\lambda'}(R^3) \cap C^{2,\lambda'}(\bar{\Omega}^\pm) \cap C^\infty(\Omega^\pm),$$

(b)
$$\mathbf{B}(\mathbf{D_x})\,\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}, \quad \mathbf{x} \in \Omega^\pm,$$

(c) $\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g})$ *is a singular integral,*

(d)
$$\begin{aligned} \{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g})\}^\pm &= \mp\,\frac{1}{2}\,\mathbf{g}(\mathbf{z})\\ &+\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}), \quad \mathbf{z} \in S, \end{aligned}$$

where $\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))$ *is the stress operator in the linear theory of elasticity for solids with double porosity.*

Theorem 4. *If* $S \in C^{2,\lambda_0}$, $\mathbf{g} \in C^{1,\lambda'}(S)$, $0 < \lambda' < \lambda_0 \le 1$, *then:*

(a)
$$\mathbf{Z}^{(2)}(\cdot, \mathbf{g}) \in C^{1,\lambda'}(\bar{\Omega}^\pm) \cap C^\infty(\Omega^\pm),$$

(b)
$$\mathbf{B}(\mathbf{D_x})\,\mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}, \quad \mathbf{x} \in \Omega^\pm,$$

(c) $\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})$ *is a singular integral,*

(d)
$$\{\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^\pm = \pm\,\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}), \quad \mathbf{z} \in S.$$

Theorems 3 and 4 can be proved similarly to the corresponding theorems in the classical theory of thermoelasticity (for details see [13, Ch. X]).

VI. Existence theorem

We introduce the notation

$$\mathcal{K}_1 \, \mathbf{g}(\mathbf{z}) \equiv \frac{1}{2} \, \mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}),$$

$$\mathcal{K}_2 \, \mathbf{g}(\mathbf{z}) \equiv -\frac{1}{2} \, \mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})$$

for $\mathbf{z} \in S$. Obviously, on the basis of theorem 4, $\mathcal{K}_1$ and $\mathcal{K}_2$ are the singular integral operators.

Lemma 1. *The singular integral operators $\mathcal{K}_1$ and $\mathcal{K}_2$ are of the normal type with an index equal to zero.*

Lemma 1 leads to the following existence theorems.

Theorem 5. *If $S \in C^{2,\lambda_0}$, $\mathbf{f} \in C^{1,\lambda'}(S)$, $0 < \lambda' < \lambda_0 \leq 1$, then a regular (classical) solution of the internal BVP $(I)^+_{\mathbf{f}}$ exists, is unique and is represented by double-layer potential*

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) \quad \text{for} \quad \mathbf{x} \in \Omega^+,$$

where $\mathbf{g}$ is a solution of the singular integral equation

$$\mathcal{K}_1 \, \mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

Theorem 6. *If $S \in C^{2,\lambda_0}$, $\mathbf{f} \in C^{1,\lambda'}(S)$, $0 < \lambda' < \lambda_0 \leq 1$, then a regular (classical) solution of the external BVP $(I)^-_{\mathbf{f}}$ exists, is unique and is represented by sum*

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) - i \, \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) \quad \text{for} \quad \mathbf{x} \in \Omega^-,$$

where $\mathbf{g}$ is a solution of the singular integral equation

$$\mathcal{K} \, \mathbf{g}(\mathbf{z}) - i \, \mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

Theorem 5 and 6 are proved by means of the potential method and the theory of singular integral equations (for details see [13]).

VII. Conclusion

1. By the above mentioned method it is possible to prove the existence and uniqueness theorems in the modern linear theories of elasticity and thermoelasticity for materials with microstructure.

2. On the basis of theorems 1 to 6 it is possible to obtain numerical solutions of the BVPs of the quasi-static theory of elasticity for solids with double porosity by using of the boundary element method.

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