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Locally compact space and continuity

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Abstract

Topological spaces for being T_0, T_1, T_2 and regular space have been discussed. The conditions for a topological space to be locally compact have also been studied. We have found that a continuous function preserves locally compactness.

Keywords: Topological spaces; Compactness; Regular space

1. Introduction

It has been seen that any product of compact spaces is compact. It has been also seen that most of the spaces turn out to be closed subspaces of products of compact spaces and such spaces are necessarily compact. The n-dimensional Euclidean space R^n is the most important type of topological space which has a great importance in Modern analysis.

A topological space is locally compact if each of it's point has a neighborhood with compact closure. As a result, R^n is locally compact for any open sphere centred on any point, is the neighborhood of the point whose closure, being a closed and bounded subspace of R^n is compact. It's application is in the field of geometry and Analysis.

2. Definitions

2.1 T_0- space

A topological space (X, J) is called T_0 space or Kolmogorff space iff given any pair of points $x, y \in x$ (distinct) there exists an open set containing one of them but not the other.

2.2 T_1- space

A topological space (X, J) is called T_1- space or Frechet space

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iff given any pair of points $x, y \in X$ there exist two open set, one containing x not y & other containing y not x.

2.3 T_2- space

A topological space (X, J) is said to be T_2-space or Hausdroff space iff given any pair of points $x, y \in X$ there exits two disjoint open sets one containing x and other y.

Regular space : A topological space X is called regular if for each closed subset F of X and $x \in X$ such that $x \notin F$, there exist disjoint open sets G and H such $F \subseteq G$ and $x \in H$.

2.4 T_3 - space

A regular T_1-space is called T_3-space.

2.5 Locally compact space

A top. space (X, J) is said to locally compact if given $x \in X$ and any nbd. U of x, there is a compact set A such that $x \in \overset{0}{A} \subset A \subset U$.

$\overset{0}{A}$: It is union of all open sets contained in A called interior of A, obviously $\overset{0}{A} \subset A$.

3. Formalism

Proposition: Let (X, J) be a T_2-space then X is locally compact iff given $x \in X$, there is a compact set A such that $x \in \overset{0}{A}$.

Proposition: Any compact T_2-space is locally compact.

Proposition: Any locally compact T_2-space X is T_3.

Proposition: If a space X is T_2 and locally compact then every open and closed subspace is also T_2 space and locally compact.

Proposition: A substance Y of a locally compact T_2-space X is locally compact iff it is the intersection of an open set and a closed set.

Proposition: Let f be a continuous and open function from one topological space (X, J) to another topological space (Y, J') then X is locally compact $\Rightarrow$ Y is locally compact.

Proof: Let (X, J) and (Y, J') be two topological spaces and f be a continuous and open mapping from X onto Y. Let X is locally compact. To show that Y is also locally compact, let $y \in Y$ and N be the neighborhood of y then $y = f(x)$ for some $x \in X$. Since f is continuous so there exists a neighborhood v of x such that f(v) $\subset$ N. Since X is locally compact so there exists a compact set B such that $x \in \overset{0}{B} \subset B \subset N$.

Then $f(x) = y \subset f(\overset{0}{B}) \subset f(B) \subset N.$

But $f(\overset{0}{B})$ is open, being f is open mapping and also compactness is invariant under continuous mapping so f(B) is compact.

Thus $x \in f(\overset{0}{B}) \subset f(B) \subset N$.

Which shows that Y is locally compact.

Proposition: Let $\{x_i, J_i\}_{\in I}$ be a countable families of non-empty spaces and $\prod X_i$ be the product spaces. Then $\prod X_i$ is locally compact iff each component spaces is locally compact and all of the component spaces except atmost finitely many are compact.

Proof: Let p_i be the projection mapping $p_i : \prod_{i\in I} X_i \to X_i$ which is continuous onto and open so each X_i is locally compact. Let A be any compact subset of $\prod X_i$ such that some point y of $\prod_{i\in I} X_i$ is in $\overset{0}{A}$. Then there is a basis neighborhood $\prod_{i\in I} V_i$ of y such that $V_i = X_i$ for all but at most finitely many i and $\prod_{i\in I} V_i \subset \overset{0}{A}$. Thus $p_i(A) = X_i$ for all but at most finitely many i, since p_i is continuous and A is compact so X_i is compact for all but at most finitely many i.

Now we assume that each X_i is locally compact and all but finitely many of X_i are compact. Let $X, Y \in \prod_{i\in I} X_i$ and let Y_i be the i th co-ordinate of Y. If U is any neighborhood of y then U contains a basis neighborhood of Y of the form $\prod_{i\in I} V_i$, where V_i is open in X_i for each i and $V_i = X_i$ for all $i \in I$, except for at most finitely many say $i_1, i_2, ...i_n$.

Since each X_i is locally compact so for each $i \in I$ there is a compact subset A_i of X_i. Such that $Y_i \in \overset{0}{A}_i \subset A_i \subset V_i$. There are at most finitely many more $i \in I$, other than $i_1, i_2, ...i_n$ say $i_{n+1}, i_{n+2}, ...i_m$ such that $Xi_{n+1}, Xi_{n+2}, ..., Xi_m$ are not compact. For any i not in $i_1, i_2, ...i_n$, $i_{n+1}, i_{n+2}, ...i_m$ we may let $A_i = X_i$ then $y \in \prod_{i\in I} \overset{0}{A}_i \subset (\prod_{i\in I} A_i)^0 \subset \prod_{i\in I} A_i \subset \prod_{i\in I} V_i$. But $\prod_{i\in I} A_i$ is product of compact sets and is therefore compact. Hence $\prod_{i\in I} X_i$ is locally compact.

4. Conclusion

Locally compactness is not invariant under continuous mapping but in under certain assumptions which are openness, a continuous function preserves local compactness.

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