Matrices Associated to 3-Interval Exchange Transformation and their Spectra

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A three by three integer matrix M is said to have the MEME Property if $\mathbf{MEM}^T = \pm\mathbf{E}$, *where* $\mathbf{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$. *We characterize such matrices in terms of their spectra.*

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In [1], the authors study matrices of morphisms preserving the family of words coding 3-interval exchange transformations. It is well known [2–4] that matrices of morphisms preserving sturmian words (i.e. words coding 2-interval exchange transformations with the maximal possible subword complexity) form the monoid

$$\left\{\mathbf{M} \in \mathbb{N}^{2\times 2} \,\middle|\, \det \mathbf{M} = \pm 1\right\} = \left\{\mathbf{M} \in \mathbb{N}^{2\times 2} \,\middle|\, \mathbf{MEM} = \pm\mathbf{E}\right\},$$

where $\mathbf{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The result of [1] states that in the case of 3-interval exchange transformations the matrices preserving words coding these transformations and having the maximal possible subword complexity belong to the monoid

$$\left\{\mathbf{M} \in \mathbb{N}^{3\times 3} \,\middle|\, \mathbf{MEM} = \pm\mathbf{E} \text{ and } \det \mathbf{M} = \pm 1\right\},$$

where $\mathbf{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

We say that a matrix fulfilling the first condition has the so-called *MEME Property*.

Definition 1

Let $\mathbf{M} \in \mathbb{N}^{3\times 3}$. $\mathbf{M}$ *is said to have the MEME Property if*

$$\mathbf{MEM}^T = \pm\mathbf{E},$$

where $\mathbf{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

The aim of this paper is to provide a stand-alone result connected with this property. Strictly speaking, we prove another algebraic characterization of matrices having this property.

Before we give the above mentioned characterization, we need to prove the following technical lemma.

Lemma 2

Let $\mathbf{M} \in \mathbb{N}^{3\times 3}$, $\mathbf{M} = (m_{ij})_{1\le i,j\le 3}$ *be a matrix. Then* $\mathbf{M}$ *has the MEME Property if and only if there exists* $\varepsilon \in \{1, -1\}$ *such that*

$$\det\begin{pmatrix} 1 & -1 & 1 \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \varepsilon\,, \tag{1}$$

$$\det\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 1 & -1 & 1 \\ m_{31} & m_{32} & m_{32} \end{pmatrix} = -\varepsilon\,, \tag{2}$$

$$\det\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 1 & -1 & 1 \end{pmatrix} = \varepsilon\,. \tag{3}$$

Proof. Let us denote $\mathbf{K} = \mathbf{MEM}^T$. The transpose of $\mathbf{K}$ is $\mathbf{K}^T = \mathbf{M}(-\mathbf{E})\mathbf{M}^T = -\mathbf{K}$ and hence $\mathbf{K}$ is an anti-symmetric matrix. To investigate equalities of anti-symmetric matrices it suffices to consider elements k_{12}, k_{13}, k_{23}. Let us compute these relevant elements of $\mathbf{K}$.

$$\begin{aligned} K &= \begin{pmatrix} 0 & k_{12} & k_{13} \\ -k_{12} & 0 & k_{23} \\ -k_{13} & -k_{23} & 0 \end{pmatrix} = \mathbf{MEM}^T \\ &= \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{pmatrix} \\ &= \begin{pmatrix} -m_{12}-m_{13} & m_{11}-m_{13} & m_{11}+m_{12} \\ -m_{22}-m_{23} & m_{21}-m_{23} & m_{21}+m_{22} \\ -m_{32}-m_{33} & m_{32}-m_{33} & m_{31}+m_{32} \end{pmatrix}\begin{pmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{pmatrix}, \end{aligned}$$

and hence

$$\begin{aligned} k_{12} &= (-m_{12}-m_{13})m_{21} + (m_{11}-m_{13})m_{22} + (m_{11}+m_{12})m_{23}, \\ k_{13} &= (-m_{12}-m_{13})m_{31} + (m_{11}-m_{13})m_{32} + (m_{11}+m_{12})m_{33}, \\ k_{23} &= (-m_{22}-m_{23})m_{31} + (m_{21}-m_{23})m_{32} + (m_{21}+m_{22})m_{33}. \end{aligned}$$

By definition of the determinant, k_{12} is equal to the left-hand side of (1), $-k_{13}$ is equal to the left-hand side of (2), and k_{23} is equal to the left-hand side of (3). This implies $\mathbf{K} = \varepsilon\mathbf{E}$ if and only if equations (1)–(3) hold. ❑

Now we can prove the two main theorems – one providing the desired characterization in the regular case and the other in the singular case.

Theorem 3

Let $\mathbf{M} \in \mathbb{N}^{3\times3}$ *be a regular matrix. Then* $\mathbf{M}$ *has MEME if and only if the number* $\det\mathbf{M}$ or $-\det\mathbf{M}$ *is an eigenvalue of* $\mathbf{M}$ *with* $(1, -1, 1)$ *being an associated left eigenvector.*

Proof. $\Leftarrow$: Denote $\Delta := \det\mathbf{M}$ and let $\varepsilon\Delta$, $\varepsilon \in \{1, -1\}$ be an eigenvalue of $\mathbf{M}$. If $(1, -1, 1)$ is a left eigenvector of $\mathbf{M}$ associated to Δ then we have the following dependence between rows of $\mathbf{M}$, denoted by $\mathbf{M}_{1\bullet}, \mathbf{M}_{2\bullet}, \mathbf{M}_{3\bullet}$

$$(1,-1,1)\mathbf{M} = \mathbf{M}_{1\bullet} - \mathbf{M}_{2\bullet} + \mathbf{M}_{3\bullet} = \varepsilon\Delta(1,-1,1). \tag{4}$$

Using (4) we have

$$\Delta = \det\begin{pmatrix}\mathbf{M}_{1\bullet}\\ \mathbf{M}_{2\bullet}\\ \mathbf{M}_{3\bullet}\end{pmatrix} = \det\begin{pmatrix}\mathbf{M}_{1\bullet}\\ \mathbf{M}_{1\bullet} + \mathbf{M}_{3\bullet} - \varepsilon\Delta(1,-1,1)\\ \mathbf{M}_{3\bullet}\end{pmatrix},$$

by subtracting the first and the third row from the second row and by factoring $-\Delta$ out from the second row we obtain

$$\Delta = -\varepsilon\Delta\det\begin{pmatrix}\mathbf{M}_{1\bullet}\\ (1,-1,1)\\ \mathbf{M}_{3\bullet}\end{pmatrix},$$

which gives (2). Similarly, using the row dependence (4) in the first and in the third row of $\mathbf{M}$ provides the equalities (1) and (3). Therefore by Lemma 2 the matrix $\mathbf{M}$ has MEME.

$\Rightarrow$: Let

$$\mathbf{M}^T\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}1\\ -1\\ 1\end{pmatrix}. \tag{5}$$

We will compute the components x_1, x_2, x_3 using Cramer's rule.

$$x_1 = \frac{\det\begin{pmatrix}1 & m_{21} & m_{31}\\ -1 & m_{22} & m_{32}\\ 1 & m_{23} & m_{33}\end{pmatrix}}{\det\mathbf{M}^T} = \frac{\det\begin{pmatrix}(1,-1,1)\\ \mathbf{M}_{2\bullet}\\ \mathbf{M}_{3\bullet}\end{pmatrix}}{\det\mathbf{M}} = \frac{\varepsilon}{\det\mathbf{M}},$$

where the last equality comes by Lemma 2 from the fact that $\mathbf{M}$ has MEME. Similarly, one can compute $x_2 = -\frac{\varepsilon}{\det\mathbf{M}}$ and $x_3 = \frac{\varepsilon}{\det\mathbf{M}}$. Hence

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \frac{\varepsilon}{\det\mathbf{M}}\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}. \tag{6}$$

Substituting (6) in (5) and multiplying by $\varepsilon\det\mathbf{M}$, which is non-zero due to the regularity of $\mathbf{M}$, we obtain

$$(1,-1,1)\mathbf{M} = \varepsilon\det\mathbf{M}(1,-1,1),$$

that is, $\varepsilon\det\mathbf{M}$ is an eigenvalue of $\mathbf{M}$, $(1,-1,1)$ is an associated left eigenvector. ❑

Theorem 4.

Let $\mathbf{M} \in \mathbb{N}^{3\times3}$ *be a singular matrix. Then* $\mathbf{M}$ *has MEME if and only if* $\rho(\mathbf{M}) = \{0, \lambda_2, \lambda_3\}$, $\lambda_2\lambda_3 = \pm1$ and $(1,-1,1)$ *is a left eigenvector associated with the eigenvalue* 0.

Proof. $\Rightarrow$: Since $\det\mathbf{M} = 0$ we have $0 \in \rho(\mathbf{M})$. Let x be a left eigenvector of $\mathbf{M}$ to the eigenvalue 0, i.e. $x\mathbf{M} = 0$ and hence $x\mathbf{E} = x\mathbf{MEM}^T = 0$, that is, x is also a left eigenvector of $\mathbf{E}$, associated to 0. Obviously, $x = (1,-1,1)$.

Concerning the other eigenvalues of $\mathbf{M}$, since the equality (4) for a singular $\mathbf{M}$ has the following form

$$(1,-1,1)\mathbf{M} = \mathbf{M}_{1\bullet} - \mathbf{M}_{2\bullet} + \mathbf{M}_{3\bullet} = 0 \tag{7}$$

the characteristic polynomial $\chi_{\mathbf{M}}(\lambda)$ will be given by the matrix

$$(\mathbf{M} - \lambda\mathbf{I}) = \begin{pmatrix}m_{11} - \lambda & m_{12} & m_{13}\\ m_{11} + m_{31} & m_{12} + m_{32} - \lambda & m_{13} + m_{33}\\ m_{31} & m_{32} & m_{33} - \lambda\end{pmatrix}.$$

Computing its determinant, the linear component of $\chi_{\mathbf{M}}(\lambda)$ is

$$-\lambda m_{33}m_{12} - \lambda m_{11}m_{33} - \lambda m_{11}m_{32} + \lambda m_{13}m_{32} + \lambda m_{13}m_{31} + \lambda m_{12}m_{31},$$

which is exactly the left hand side of (2), and hence it is equal to ±1. On the other hand, since the characteristic polynomial can be written in the form

$$\begin{aligned}\chi_{\mathbf{M}}(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\\ &= \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2\\ &\quad + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)x - \lambda_1\lambda_2\lambda_3,\end{aligned}$$

and, moreover, $\lambda_1 = \det\mathbf{M} = 0$, the linear component of $\chi_{\mathbf{M}}(\lambda)$ is also equal to $(\lambda_2\lambda_3)x$. This implies $\lambda_2\lambda_3 = \pm1$.

$\Leftarrow$: Due to the row dependency (7) we have for the left hand side of (1)

$$\det\begin{pmatrix}1 & -1 & 1\\ m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{pmatrix}$$

$$= \det\begin{pmatrix}1 & -1 & 1\\ m_{11} + m_{31} & m_{12} + m_{32} & m_{13} + m_{33}\\ m_{31} & m_{32} & m_{33}\end{pmatrix}$$

$$= \det\begin{pmatrix}1 & -1 & 1\\ m_{11} & m_{12} & m_{13}\\ m_{31} & m_{32} & m_{33}\end{pmatrix} = -\det\begin{pmatrix}m_{11} & m_{12} & m_{13}\\ 1 & -1 & 1\\ m_{31} & m_{32} & m_{33}\end{pmatrix},$$

and similarly for (3)

$$\det\begin{pmatrix}m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ 1 & -1 & 1\end{pmatrix}$$

$$= \det\begin{pmatrix}m_{11} & m_{12} & m_{13}\\ m_{11} + m_{31} & m_{12} + m_{32} & m_{13} + m_{33}\\ 1 & -1 & 1\end{pmatrix}$$

$$= \det\begin{pmatrix}m_{11} & m_{12} & m_{13}\\ m_{31} & m_{32} & m_{33}\\ 1 & -1 & 1\end{pmatrix} = -\det\begin{pmatrix}m_{11} & m_{12} & m_{13}\\ 1 & -1 & 1\\ m_{31} & m_{32} & m_{33}\end{pmatrix}.$$

Therefore the conditions (1)–(3) are equivalent for singular matrices.

Using the same argument concerning the linear component of $\chi_{\mathbf{M}}(\lambda)$ as in the previous part of the proof one can show that the conditions $\rho(\mathbf{M}) = \{0, \lambda_2, \lambda_3\}$, and $\lambda_2\lambda_3 = \pm1$ imply the equality (2) and hence $\mathbf{M}$ has MEME. ❑

In addition to the above stated algebraic characterizations of matrices having the MEME property, there is also a nice relation between the singular and the non-singular case. Let

$\mathbf{M} \in \mathbb{N}^{3\times 3}$ be a singular matrix having MEME and let us consider a matrix $\overline{\mathbf{M}}$, given by

$$\overline{\mathbf{M}} := \mathbf{M} + k\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathrm{D}},$$

where $k \in \mathbb{N}$. Since $\mathbf{DE} = \mathbf{ED}^T = 0$, we have

$$\overline{\mathbf{M}}\mathbf{E}\overline{\mathbf{M}}^T = (\mathbf{M} + k\mathbf{D})\mathbf{E}(\mathbf{M}^T + k\mathbf{D}^T)$$
$$= \mathbf{MEM}^T + k\mathbf{DEM}^T + k\mathbf{MED}^T + k\mathbf{DED}^T = \mathbf{MEM}^T = \pm\mathbf{E},$$

hence also the regular matrix $\overline{\mathbf{M}}$ has MEME. Equivalently, one can say that for each singular matrix $\mathbf{M}$ having MEME there exists a regular matrix $\overline{\mathbf{M}}$ having MEME such that their first and third row coincide, that is,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\overline{\mathbf{M}}.$$

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References

[1] Ambrož, P., Masáková, Z., Pelantová, E.: Matrices of 3iet Preserving Morphisms. Submitted to *Theor. Comp. Sci.*, 2007.

[2] Berstel, J., Séébold, P.: Morphismes de Sturm. *Bull. Belg. Math. Soc. Simon Stevin*, Vol. **1** (1994), p. 175–189.

[3] Mignosi, F., Séébold, P.: Morphismes sturmiens et r gles de Rauzy. *J. Théor. Nombres Bordeaux*, Vol. **5** (1993), p. 221–233.

[4] Séébold, P.: Fibonacci Morphisms and Sturmian Words. *Theoret. Comput. Sci.*, Vol. **88** (1991), p. 365–384.

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