Advanced Load Effect Model for Probabilistic Structural Design

M. Sýkora

In probabilistic structural design some actions on structures can be well described by renewal processes with intermittencies. The expected number of renewals for a given time interval and the probability of "on"-state at an arbitrary point in time are of a main interest when estimating the structural reliability level related to the observed period. It appears that the expected number of renewals follows the Poisson distribution. The initial probability of "on"-state is derived assuming random initial conditions. Based on a two-state Markov process, the probability of "on"-state at an arbitrary point in time then proves to be a time-invariant quantity under random initial conditions. The results are numerically verified by Monte Carlo simulations. It is anticipated that the proposed load effect model will become a useful tool in probabilistic structural design. The aims of future research are outlined in the conclusions of the paper.

Keywords: rectangular wave renewal process, probability of "on"-state, expected number of renewals, action on structures.

1 Introduction

Actions on structures are often of a time-variant nature. Special attention is in particular required when a combination of time-variant loads needs to be considered. Approaches to probabilistic structural design based on different load combination models are indicated by JCSS [1]. It appears that an advanced load effect model based on renewal processes can be suitably used to describe random load fluctuations in time, enabling sufficiently accurate estimates of the reliability level in practical applications. A great number of actions on structures can be approximated by rectangular wave renewal processes with random durations between renewals, as already recognized e.g. by Wen [2]. Models based on renewal processes with exponentially distributed durations between renewals and exponentially distributed durations of load pulses were also recommended for practical use by Iwankiewicz and Rackwitz [3, 4].

When estimating the structural reliability level related to the specified observed period T, the failure probability $P_f(0,T)$ is often assessed by the lower and upper bounds in the case of combinations of renewal processes, as indicated e.g. by Sýkora [5]. The upper bound on $P_f(0,T)$ is of great importance for practical applications. Two basic properties of the renewal process, the expected number of renewals $E[N(0,T)]$ and the probability of "on"-state $p_{on}(t)$, are needed to evaluate the upper bound, see Sýkora [5].

The probability of „on"-state was investigated by Shinozuka [6], considering a "sufficiently long" observed period T. Extension to an arbitrarily long period T, to the so-called non-stationary case, was then provided by Iwankiewicz and Rackwitz [4]. Formulas for the probability of „on"-state were derived considering various initial conditions. The present paper attempts to reinvestigate the formulas for $p_{on}(t)$ achieved by Shinozuka [6] and by Iwankiewicz and Rackwitz [3, 4]. In addition, a formula for the expected number of renewals $E[N(0,T)]$ is verified. Both the basic properties of the renewal process are investigated under random as well as given initial conditions. Special attention is paid here to the correct definition of the random initial conditions.

Initially, the expected number of renewals $E[N(0,T)]$ is shown to be independent of the initial conditions. The initial probability of „on"-state $p_{on}(0)$ is then derived under random initial conditions. A two-state Markov process developed by Madsen and Ditlevsen [7] is adopted to derive the probability of „on"-state $p_{on}(t)$ at an arbitrary point in time. It appears that $p_{on}(t)$ is a time-invariant quantity under random initial conditions. In general the paper provides a comprehensive theoretical background for practical applications of advanced load models based on renewal processes. In addition to the newly derived formulas, several results already obtained by Shinozuka [6] and Iwankiewicz and Rackwitz [4] are verified. Desirable extensions for further research are outlined.

2 Basic properties of the considered renewal process

It is further considered that the actual load process can suitably be approximated by the renewal process $S(t)$ with the following properties:

- the process is intermittent, i.e. the load may be "on"/present or "off",
- durations T_{ren} between renewals are mutually independent random variables described by an exponential distribution with the rate κ,
- durations T'_{on} of "on"-state, durations of load pulses, are also mutually independent exponential variables (rate μ). "On"-states are initiated when renewal occurs,
- load pulses do not overlap and thus the effective load pulse duration T_{on} is given by

$$T_{on} = \begin{cases} T'_{on} \,|\, T'_{on} < T_{ren} \\ T_{ren} \,|\, T'_{on} \geq T_{ren} \end{cases} \tag{1}$$

- load intensities S are mutually independent variables having an appropriate extreme distribution $\max_{T_{ren}}[S(t)]$ related to the expected duration between renewals

  $$E[T_{ren}] = 1/\kappa,$$

  see e.g. Weisstein [8],

– random conditions are primarily assumed at the initial time $t = 0$ of the observed period T, i.e. it is of a purely random nature whether the process starts in an "on"-state or an "off"-state. Some remarks on processes with the given initial conditions are also provided in the following.

The considered process $S(t)$ is indicated in Fig. 1 where the actual load history is depicted in grey. Note that random variables are further denoted by upper-case letters X (e.g. durations between renewals are referred to as T_{ren}) while lower-case letters x (e.g. t_{ren}) stand for their realizations/trials.

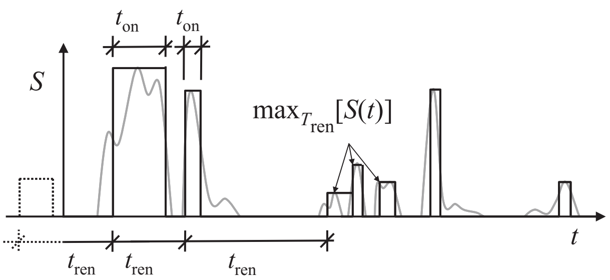

Fig. 1: Rectangular wave renewal process $S(t)$ with intermittencies

3 Random initial conditions

Special attention is paid to the correct definition of random conditions at the initial state. As the process $S(t)$ constitutes a sequence of intervals T_{ren}, random initial conditions are fulfilled when the initial point is located in an interval T_0 selected from a population of T_{ren} on a purely random basis taking into account the random properties of T_{ren} as indicated in Fig. 2. The interval T_0 (random variable) is denoted as the first renewal.

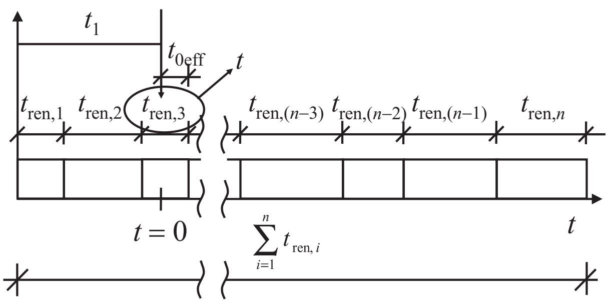

Fig. 2: Sequence of intervals T_{ren} and random selection of the first renewal T_0

To derive a cumulative distribution function (CDF) $F_{T0}(t)$ of the first renewal T_0, a sufficiently long sequence of a large number $n' \to \infty$ of trials $t_{\mathrm{ren},i} (1 \le i \le n')$ is further considered. A trial $t_0 = t_{\mathrm{ren},j} (1 \le j \le n')$ is randomly selected from the population of $t_{\mathrm{ren},i}$. Consider next that the duration of the selected trial is τ, $t_0 = t_{\mathrm{ren},j} = \tau$. By intuition, the probability $P[T_0 = t_0 = t_{\mathrm{ren},j} \in (\tau, \tau + \mathrm{d}t)]$ that the selected trial t_0 is of duration τ can be obtained as a ratio of the total duration of all $t_{\mathrm{ren},j} \in (\tau, \tau + \mathrm{d}t)$ over the total duration of all trials $t_{\mathrm{ren},i}$ from the population. Using the probability density function (PDF) of an exponential distribution $f_{T\mathrm{ren}}(t) = \kappa \,\mathrm{e}^{-\kappa t}$, the central limit theorem for a sum of n' independent random variables, see Weisstein [8], and the expected value $\mathrm{E}[T_{ren}] = 1/\kappa$, the probability becomes

$$
\begin{aligned}
&P[T_0 = t_0 \in (\tau, \tau + \mathrm{d}t)] = \\
&\lim_{n \to \infty} \frac{P\lfloor T_{\mathrm{ren}} = t_{\mathrm{ren},j} \in (\tau, \tau + \mathrm{d}t) \rfloor \times n' \times \tau}{\sum_{i=1}^{n} t_{\mathrm{ren},i}} = \\
&\lim_{n \to \infty} \frac{P\lfloor t_{\mathrm{ren},j} \in (\tau, \tau + \mathrm{d}t) \rfloor \times n' \times \tau}{n' \times E[T_{\mathrm{ren}}]} = \\
&\frac{\kappa \mathrm{e}^{-\kappa\tau} \times \mathrm{d}t \times \tau}{\frac{1}{\kappa}} = \kappa^2 \tau \mathrm{e}^{-\kappa\tau} \times \mathrm{d}t.
\end{aligned}
\tag{2}
$$

The cumulative distribution function of the first renewal T_0 is obtained from (2), as follows

$$
F_{T0}(t) = P(T_0 < t) = \int_0^t \kappa^2 t \,\mathrm{e}^{-\kappa t} \mathrm{d}t = 1 - (1 + \kappa t)\,\mathrm{e}^{-\kappa t} \tag{3}
$$

and the probability density function becomes, using (3)

$$
f_{T0}(t) = \frac{\mathrm{d}F_{T0}(t)}{\mathrm{d}t} = \kappa^2 t \,\mathrm{e}^{-\kappa t}. \tag{4}
$$

Note that CDF (3) can be suitably used in Monte Carlo simulations to achieve random initial conditions. The first renewal T_0 can either be randomly selected from a large population of samples of T_{ren}, or directly simulated using CDF (3). Simulations based on CDF (3) are clearly incomparably more efficient than "the first approach" described in the beginning of this section. More details are provided by Sýkora [5].

4 Expected number of renewals

In the following, N denotes a random number of renewals of the process $S(t)$. The expected number of renewals $\mathrm{E}[N(0,T)]$ is the essential process characteristic used to estimate the failure probability $P_{\mathrm{f}}(0,T)$. Unlike in Section 3, consider that process $S(t)$ is defined so that the first renewal starts at the initial time $t = 0$, as indicated in Fig. 3, so that the "given" initial conditions are taken into account. This assumption can be used e.g. for an imposed load model where the action starts to be "on" approximately at $t = 0$ when a new structure is put into operation. The expected number of renewals for such a process is obtained e.g. by Weisstein [8]

$$
E[N(0,T)] = T\kappa. \tag{5}
$$

The first renewal of the considered process S(t) is a "standard" exponentially distributed duration T_{ren} with CDF

$$
F_{T\mathrm{ren}}(t) = P(T_{ren} < t) = 1 - \mathrm{e}^{-\kappa t}. \tag{6}
$$

Next consider the random initial conditions again, i.e. the first renewal T_0 of the process $S(t)$ is selected purely on a random basis and the initial time point $t = 0$ is randomly located in the first renewal. The process is again a sequence of exponentially distributed durations except for the first renewal T_0. The random effective duration $T_{0\mathrm{eff}}$ of T_0 corresponds to the first renewal (6) of the process described above. The difference between these processes is indicated in Fig. 3.

Note that $T_{0\mathrm{on}}$ denotes the duration of "on"-state within the first renewal and $T_{0\mathrm{oneff}}$ the effective duration of $T_{0\mathrm{on}}$, i.e. the duration of the "on"-state involved in the observed period $(0, T)$. Random properties of the durations $T_{0\mathrm{on}}$ and $T_{0\mathrm{oneff}}$ are discussed in the following.

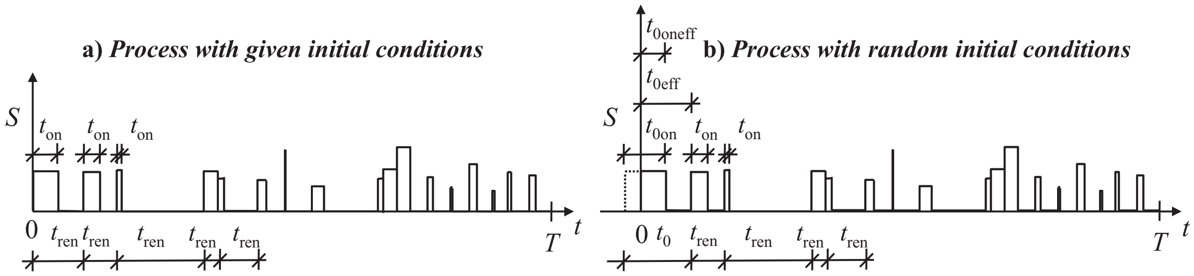

Fig. 3: Investigated processes $S(t)$

The effective duration $T_{0\mathrm{eff}}$ indicated in Fig. 3 is randomly selected from the first renewal T_0 to fulfil the random initial conditions. Given $T_0 = t_0$, the effective duration $T_{0\mathrm{eff}} \in (0, t_0)$ has the rectangular distribution $\mathrm{R}(0, t_0)$ with PDF $\mathrm{f}_{T0\mathrm{eff}}(t|T_0 = t_0) = 1/t_0$ and CDF $\mathrm{F}_{T0\mathrm{eff}}(t|T_0 = t_0) = t/t_0$. CDF $\mathrm{F}_{T0\mathrm{eff}}(t)$ of the effective duration $T_{0\mathrm{eff}}$ for an arbitrary T_0 can be derived from the sum of probabilities of two disjoint events ($t > 0$):

- T_0 is less than t, implying $T_{0\mathrm{eff}} < t$,
- T_0 is greater or equal to t while $T_{0\mathrm{eff}} < t$.

This can be rewritten as follows

$$\mathrm{F}_{T0\mathrm{eff}}(t) = \mathrm{P}(T_{0\mathrm{eff}} < t) = \mathrm{P}[(T_0 < t) \cup (T_{0\mathrm{eff}} < t < T_0)] = \mathrm{P}(T_0 < t) + \mathrm{P}(T_{0\mathrm{eff}} < t < T_0). \tag{7}$$

The first part of the right hand side of (7) is already obtained in (3). To evaluate the second part of the right hand side of (7), consideration is initially taken that $t < \tau < T_0 < t + \mathrm{d}t$. Under this assumption, $\mathrm{P}(T_{0\mathrm{eff}} < t < T_0)$ is equal to $\mathrm{P}(T_{0\mathrm{eff}} < t | t < T_0 < \tau + \mathrm{d}\tau)$ and, using CDF of $T_{0\mathrm{eff}}$ for a given t_0, arrives at

$$\mathrm{P}(T_{0\mathrm{eff}} < t | t < \tau < T_0 < \tau + \mathrm{d}\tau) = \frac{t}{\tau} \tag{8}$$

The condition $t < \tau$ implies $\tau \in (t, \infty)$. Using (4) and (8), $\mathrm{P}(T_{0\mathrm{eff}} < t < T_0)$ can be obtained by the expectation

$$\mathrm{P}(T_{0\mathrm{eff}} < t < T_0) = \int_t^{\infty} \mathrm{P}(T_{0\mathrm{eff}} < t | t < \tau < T_0 < \tau + \mathrm{d}\tau)\mathrm{f}_{T0}(\tau)\mathrm{d}\tau = \int_t^{\infty} \frac{t}{\tau}\mathrm{f}_{T0}(\tau)\mathrm{d}\tau. \tag{9}$$

Substitution of (4) into (9) followed by the limit passage yields

$$\mathrm{P}(T_{0\mathrm{eff}} < t < T_0) = \int_t^{\infty} \frac{t}{\tau}\kappa^2 \tau \mathrm{e}^{-\kappa\tau}\mathrm{d}\tau = \kappa t\left[-\mathrm{e}^{-\kappa\tau}\right]_t^{\infty} = \kappa t \mathrm{e}^{-\kappa t}. \tag{10}$$

Substitution of (3) and (10) into (7) leads to CDF $\mathrm{F}_{T0\mathrm{eff}}(t)$

$$\mathrm{F}_{T0\mathrm{eff}}(t) = 1 - (1 + \kappa t)\mathrm{e}^{-\kappa t} + \kappa t \mathrm{e}^{-\kappa t} = 1 - \mathrm{e}^{-\kappa t}. \tag{11}$$

A comparison of CDF (6) with CDF (11) indicates that the first renewal T_{ren} of the process with the given initial conditions and the effective duration of the first renewal $T_{0\mathrm{eff}}$ of the process with random initial conditions are random variables with the same cumulative distribution functions. Since the following durations T_{ren} between renewals are the same random variables, the investigated processes inevitably have the same statistical properties. The number of renewals, therefore, remains the same for both processes. Formula (5) is thus valid for process $S(t)$ with random initial conditions.

The expected number of renewals $\mathrm{E}[N(0, T)]$ (5) of process $S(t)$ as a function of T and κ is numerically compared with the number of renewals predicted by the crude Monte Carlo simulation method (MC) in Fig. 4. Solid lines denote the results of (5) and circles ('o') denote the results of the simulations. 1000 trials of the whole processes $S(t)$ are simulated for each considered combination of T and κ, using the following procedure:

- a trial t_0 is randomly selected from a "sufficiently large" population of realizations $t_{\mathrm{ren},i} (1 \le i \le n', n' \to \infty)$ in accordance with the "first approach", described in Section 3,
- the effective duration $t_{0\mathrm{eff}}$ is obtained as a pseudorandom number in the range from 0 to t_0 (see Fig. 2),
- subsequent durations between renewals $t_{\mathrm{ren},j} (2 \le j \le n+1)$ are simulated as exponential variables with the rate κ,
- a realization of the number of renewals n is determined using the condition

$$t_0 + \sum_{j=2}^{n} t_{\mathrm{ren},j} \le T < t_0 + \sum_{j=2}^{n+1} t_{\mathrm{ren},j}. \tag{12}$$

Note that $n = 1$ if $t_0 + \le T < t_0 + t_{\mathrm{ren},2}$ and $n = 0$ if $t_0 > T$.

The number of renewals N is identified for each trial of the whole process $S(t)$ and the expected value is then determined. More details on MC verification are provided by Sýkora [5].

Fig. 4 indicates that formula (5) and MC verification lead to the same results. It is therefore concluded that formula (5) is applicable also for process $S(t)$ with random initial conditions.

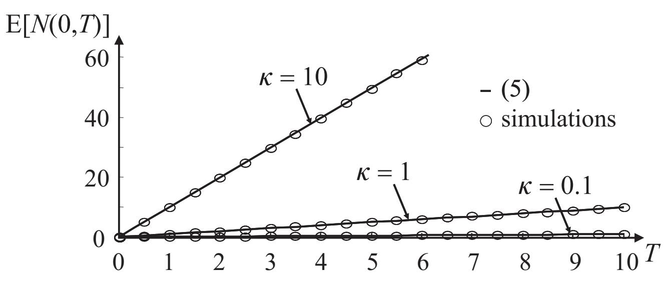

Fig. 4: Expected number of renewals $\mathrm{E}[N(0, T)]$

5 Probability of "on"-state – stationary case

In addition to the expected number of renewals $E[N(0, T)]$, the probability of "on"-state $p_{on}(t)$ of a process $S(t)$ needs to be known for applications of the renewal processes. The time variability of $S(t)$ is completely described by durations T_{ren} and T_{on} and therefore $p_{on}(t)$ can be derived from the statistical properties of T_{ren} and T_{on}. Note that the initial conditions in $t = 0$ may have an influence on the probability of "on"-state $p_{on}(t)$. Given that the process $S(t)$ is "on" at $t = 0$, $p_{on}(t)$ apparently attains 1 for $T \to 0$. However, the probability of „on"-state approaches, by intuition, a "stationary" value for large T becoming independent of the initial conditions.

Consider that the observed period is "sufficiently long" (conservatively $T > 1/\kappa$ for processes with "short" load pulses when $\kappa/\mu < 0.05$, and approximately $T > 3/\kappa$ for other processes). Note that these conditions are usually satisfied in practical applications). The probability of „on"-state then approaches the stationary value obtained by Shinozuka [6]

$$p_{on} = \frac{E(T_{on})}{E(T_{ren})}, \tag{13}$$

where $E(T_{on})$ is the expected duration of the "on"-state and $E(T_{ren}) = 1/\kappa$ denotes the expected duration between renewals.

Duration T_{on} is a truncated exponential variable, as defined in (1). If a realization of exponential duration with rate μ is less than a realization t_{ren}, $t'_{on} > t_{ren}$, duration T_{on} is truncated to $t_{on} = t_{ren}$, otherwise $t_{on} = t'_{on}$. This truncation apparently influences the expected value $E(T_{on})$. CDF $F_{Ton}(t)$ is initially derived. Two possible exclusive events can yield $T_{on} < t$:

- duration T_{ren} is less than t. This implies that duration T_{on} is less than t,
- duration T_{ren} is greater or equal to t. In this case, CDF of T_{on} in the interval $(0, t)$ remains the "standard" cumulative distribution function of the exponential distribution and the truncation has no effect here. Probability $P[T_{on} < t]$ can then be determined from CDF of the exponential distribution with rate μ.

Using CDFs of the exponential distributions with rates κ and μ, $F_{Ton}(t)$ is given for the mutually independent durations T_{on} and T_{ren} as follows

$$\begin{aligned} F_{Ton}(t) &= P(T_{on} < t) = \\ &P(T_{on} < t | T_{ren} \ge t)\, P(T_{ren} \ge t) + P(T_{ren} < t) = \\ &P(T_{on} < t)[1 - P(T_{ren} < t)] + P(T_{ren} < t) = \\ &(1 - e^{-\mu t})[1 - (1 - e^{-\kappa t})] + 1 - e^{-\kappa t} = 1 - e^{-(\kappa+\mu)t}. \end{aligned} \tag{14}$$

From (14), the expected value $E[T_{on}]$ arrives at

$$E(T_{on}) = \frac{1}{\kappa + \mu}. \tag{15}$$

Substitution of (15) into (13) yields the probability of „on"-state under stationary conditions

$$p_{on} = \frac{\kappa}{\kappa + \mu} \tag{16}$$

which is in accordance with the well-known result obtained e.g. in RCP [9].

6 Initial probability of „on"-state

Considering random initial conditions, the probability of „on"-state $p_{0on} = p_{on}(0)$ at the initial time $t = 0$ can be defined as the probability that the sum of the effective duration of the first renewal T_{0eff} and the initial duration of the "on"-state T_{0on} exceeds the duration of the first renewal T_0 (see the right hand side of Fig. 3)

$$p_{0on} = P[T_{0on}(T_0) + T_{0eff}(T_0) \ge T_0]. \tag{17}$$

Note that both durations T_{0eff} and T_{0on} are dependent on T_0. If the sum is less than T_0, the "on"-state is finished earlier than the observed period starts. The initial probability of "off"-state $p_{0off} = p_{off}(0)$, the complementary probability to p_{0on}, then follows from (17)

$$p_{0off} = P[T_{0on}(T_0) + T_{0eff}(T_0) < T_0]. \tag{18}$$

Initially consider that $T_0 = \tau$. Application of the convolution integral to (18), see e.g. Weisstein [8], yields

$$p_{0off}(\tau | T_0 = \tau) = \int_0^{\tau} F_{T0on}(\tau - \xi)\, f_{T0eff}(\xi)\, d\xi, \tag{19}$$

where $F_{T0on}(\cdot)$ is CDF of T_{0on}. The upper bound for integration in (19) is τ, since T_{0eff} is always less or equal to T_0 (see Fig. 3). Duration T_{0on} is defined in accordance with (1), i.e. T_{0on} has an exponential distribution with rate μ if $T_{0on} < T_0 = \tau$, otherwise it is truncated to $T_{0on} = \tau$. Since the integration variable $\xi \in (0, \tau)$ in (19) is always positive, $F_{T0on}(\cdot)$ is evaluated for values $0 \ge \tau - \xi \ge \tau$. Duration T_{0on} is within this interval a "standard" exponential variable with CDF $F_{T0on}(t) = 1 - e^{-\mu t}$. To satisfy the random initial conditions, the effective duration T_{0eff} has the rectangular distribution $R(0, \tau)$ with PDF $f_{T0eff}(t) = 1/\tau$. Substitution of the aforementioned functions into (19) followed by integration leads to the probability of „off"-state conditional on $T_0 = \tau$

$$\begin{aligned} p_{0off}(\tau | T_0 = \tau) &= \int_0^{\tau} \left(1 - e^{-\mu(\tau - \xi)}\right) \frac{1}{\tau}\, d\xi = \\ \frac{1}{\tau}\left[\xi - \frac{1}{\mu} e^{-\mu(\tau-\xi)}\right]_0^{\tau} &= 1 - \frac{1}{\mu\tau} + \frac{1}{\mu\tau} e^{-\mu\tau}. \end{aligned} \tag{20}$$

The probability density function of T_0 is provided in (4). The probability of „off"-state poff for an arbitrary T_0 can be obtained by the expectation of (20), using (4)

$$\begin{aligned} p_{0off} &= E_{T0}[p_{0off}(\tau | T_0 = \tau)] = \\ &\int_0^{+\infty} \left(1 - \frac{1}{\mu\tau} + \frac{1}{\mu\tau} e^{-\mu\tau}\right) \kappa^2 \tau e^{-\kappa\tau}\, d\tau = \\ &1 - \frac{\kappa}{\mu} + \frac{\kappa^2}{\mu(\kappa + \mu)} = \frac{\mu}{(\kappa + \mu)}. \end{aligned} \tag{21}$$

Since probabilities p_{0on} and p_{0off} are mutually complementary, the initial probability of „on"-state becomes

$$p_{0on} = P[T_{0on}(T_0) + T_{0eff}(T_0) \ge T_0] = 1 - p_{0off} = \frac{\kappa}{\kappa + \mu}. \tag{22}$$

It appears that under random initial conditions, the initial probability of "on"-state p_{0on} (22) is equal to the "stationary" probability (16). This is an expected conclusion as, e.g., the probability of "on"-state of a wind action nearly always has a constant value, barely dependent on the origin $t = 0$ of the observed period $(0, T)$. the numerical study published by Sýkora [5] indicates that the result obtained in (22) is in accordance with the Monte Carlo simulations.

7 Probability of "on"-state – non-stationary case

To derive the probability of "on"-state $p_{on}(t)$ at an arbitrary point-in-time t, consider the process $S(t)$ with "short" pulses so that the probability of the duration of the "on"-state T_{on} exceeding the duration between renewals T_{ren} is negligible, $P(T_{on} > T_{ren}) \sim 0$. This process can be suitably modelled as the product of a non-intermittent rectangular wave renewal process $S_{(t)}$ and a two-state Markov process $Z(t)$, as already proposed by Madsen and Ditlevsen [7]

$$S(t) = S_{(t)} Z(t). \tag{23}$$

States of the Markov process are characterized as follows

$$Z(t) \begin{cases} 0\ldots & \text{"off"} - \text{state} \\ 1\ldots & \text{"on"} - \text{state} \end{cases} \tag{24}$$

The considered processes $S(t)$ and $Z(t)$ are indicated in Fig. 5.

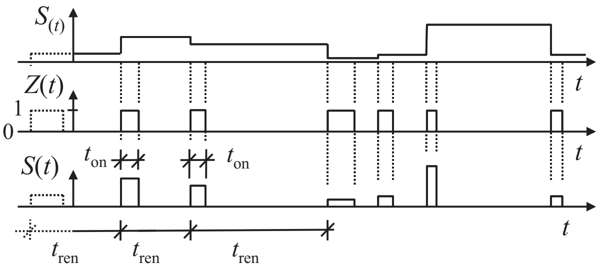

Fig. 5: Non-intermittent rectangular wave renewal process $S_{(t)}$ and two-state Markov process $Z(t)$

The renewal process $S(t)$ may be "on" $[Z(t) = 1]$ or "off" $[Z(t) = 0]$ in time t. Within an infinitely short time interval $\Delta t \to 0$, the process may jump between the "on" and "off" states or may remain in the same state. Therefore, the process $S(t)$ may again be "on" $[Z(t+\Delta t) = 1]$ or "off" $[Z(t+\Delta t) = 0]$ in $t+\Delta t$. It is further assumed that "on" and "off" states form a Markov process, and states in $t+\Delta t$ merely depend on the states in t.

Consider that the process $S(t)$ is "off" at the time point t, i.e. $Z(t) = 0$. Within Δt, the process may remain in the "off"-state $[Z(t+\Delta t) = 0]$ or a renewal may occur and an "on"-state may be initiated $[Z(t+\Delta t) = 1]$. The probability $P_{\Delta t}(N > 0)$ of the occurrence of at least one renewal within Δt is obtained e.g. by Wen [2]

$$\begin{aligned} &P_{\Delta t}(N > 0) = 1 - P_{\Delta t}(N = 0) = \\ &1 - e^{-\kappa\Delta t} = \kappa\Delta t + o(\Delta t^2) \approx \kappa\Delta t, \end{aligned} \tag{25}$$

where $o(\Delta t^2)$ denotes terms with a higher order in Δt for which $\lim\limits_{\Delta t \to 0} o(\Delta t^2) \big/ \Delta t = 0$. This implies that the transition (transfer) probability $P[Z(t+\Delta t) = 1 \,|\, Z(t) = 0]$ of a jump from the "off"-state in t into the "on"-state in $t+\Delta t$ given the "off"-state in t is

$$P[Z(t + \Delta t) = 1 | Z(t) = 0] \approx \kappa\Delta t. \tag{26}$$

The complementary transition probability $P[Z(t+\Delta t) = 0 \,|\, Z(t) = 0]$ that process $S(t)$ remains "off" becomes

$$\begin{aligned} &P[Z(t + \Delta t) = 0 | Z(t) = 0] = \\ &1 - P[Z(t + \Delta t) = 1 | Z(t) = 0] \approx 1 - \kappa\Delta t. \end{aligned} \tag{27}$$

Assuming an "on"-state at a time point $t[Z(t) = 1]$, the process may remain in the "on"-state $[Z(t+\Delta t) = 1]$ or the load pulse may be finished and an "off"-state may be initiated $[Z(t+\Delta t) = 0]$. Note that the probability of the event of an "on"-state being finished by a renewal occurrence is neglected here. This event is conditioned by $T_{on} > T_{ren}$, but only processes with "short" load pulses are investigated in this section, and thus $P[T_{on} > T_{ren}] \sim 0$. By analogy with (25), (26) and (27), the transition probability $P[Z(t+\Delta t) = 0 \,|\, Z(t) = 1]$ of a jump from the "on"-state in t into the "off"-state in $t+\Delta t$ given the "on"-state in t is

$$P[Z(t + \Delta t) = 0 | Z(t) = 1] \approx \mu\Delta t \tag{28}$$

and the complementary transition probability $P[Z(t+\Delta t) = 1 \,|\, Z(t) = 1]$ that process $S(t)$ remains "on" is

$$\begin{aligned} &P[Z(t + \Delta t) = 1 | Z(t) = 1] = \\ &1 - P[Z(t + \Delta t) = 0 | Z(t) = 1] \approx 1 - \mu\Delta t. \end{aligned} \tag{29}$$

Fig. 6 indicates all the transition probabilities (26) to (29) and Markov states.

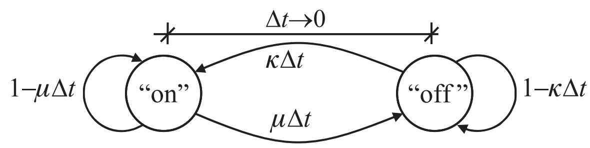

Fig. 6: Transition probabilities and Markov states

The probability of „on"-state $p_{on}(t+\Delta t)$ is obtained as

$$\begin{aligned} &p_{on}(t + \Delta t) = P[Z(t + \Delta t) = 1] = \\ &P[Z(t + \Delta t) = 1 | Z(t) = 1] \times P[Z(t) = 1] + \\ &P[Z(t + \Delta t) = 1 | Z(t) = 0] \times P[Z(t) = 0] = \\ &(1 - \mu\Delta t) \times p_{on}(t) + \kappa\Delta t \times p_{off}(t). \end{aligned} \tag{30}$$

As the probabilities of „on"-state and "off"-state are complementary, $p_{on}(t) + p_{off}(t) = 1$, (30) can be rewritten as

$$p'_{on}(t + \Delta t) = \frac{p_{on}(t + \Delta t) - p_{on}(t)}{\Delta t} = -(\kappa + \mu) p_{on}(t) + \kappa, \tag{31}$$

where $p'_{on}(t+\Delta t)$ is a derivative of the probability of „on"-state at time t. Equation (31) is in agreement with the formulas obtained by Iwankiewicz and Rackwitz [4]. It follows from (31) that the probability of „on"-state pon(t) at an arbitrary point in time t, the so-called non-stationary probability of „on"-state, is

$$p_{on}(t) = C e^{-(\kappa+\mu)t} + \frac{\kappa}{\kappa + \mu}, \tag{32}$$

where C is a constant of integration obtained from the initial conditions. Under stationary conditions, $t \to \infty$, the first term of the right hand side of (32) vanishes and $p_{on}(t)$ equals $\kappa/(\kappa+\mu)$, as already given in (16).

Considering the random initial conditions and an arbitrarily long period $(0, T)$, the initial probability of „on"-state is, in accordance with (22), $p_{on}(0) = \kappa/(\kappa+\mu)$. For $t = 0$, substitution of (22) into (32) leads to $C = 0$ and the non-stationary probability of „on“-state under random initial conditions becomes

$$p_{\text{on}}(t) = \frac{\kappa}{\kappa + \mu}. \tag{33}$$

It appears that under random initial conditions, the probability of „on“-state is time-invariant for an arbitrarily long observed period T, and process $S(t)$ is stationary and ergodic. This is a very important conclusion, probably firstly published here.

However, the initial conditions can be given in some cases. A load process $S(t)$ can then be “on” or “off” at $t = 0$. If process $S(t)$ is assumed to be “on” at $t = 0$, $p_{\text{on}}(0) = 1$, the non-stationary probability of „on“-state (32) reads

$$p_{\text{on}}(t) = \frac{\kappa}{\kappa + \mu} + \frac{\mu}{\kappa + \mu} e^{-(\kappa+\mu)t}. \tag{34}$$

If the load process $S(t)$ is “off” at $t = 0$, $p_{\text{on}}(0) = 0$, the non-stationary probability of „on“-state (32) writes

$$p_{\text{on}}(t) = \frac{\kappa}{\kappa + \mu} - \frac{\kappa}{\kappa + \mu} e^{-(\kappa+\mu)t} \tag{35}$$

Probabilities (34) and (35) are identical with those already achieved by Iwankiewicz and Rackwitz [4]. Note that in the referenced paper the random initial conditions are approximated as $p_{\text{on}}(0) = p_{\text{off}}(0) = 0.5$. Formula (32) then yields

$$p_{\text{on}}(t) = \frac{\kappa}{\kappa + \mu} - \frac{\kappa - \mu}{2(\kappa + \mu)} e^{-(\kappa+\mu)t}. \tag{36}$$

The difference between the initial conditions applied to derive formula (33) in present paper and (36) published by Iwankiewicz and Rackwitz [4] is obvious, and needs no further comment.

The probability of „on“-state $p_{\text{on}}(t)$ (33) is further numerically evaluated using the MC simulations. For the number of trials $n' = 10^5$, “on”- and “off”-states are identified at selected times t and the probability of „on“-state $p_{\text{on}}(t)$ is then determined. Fig. 7 indicates the probabilities $p_{\text{on}}(t)$ for three alternatives:

- *alt. A*: the time variability of the considered process is defined by $\kappa_A = 1$ and $\mu_A = 0.1$. Formula (33) predicts $p_{\text{on,A}} = 0.91$. It is thus foreseen that process A is nearly always active,
- *alt. B*: rates $\kappa_B = 1$ and $\mu_B = 1$ and $p_{\text{on,B}} = 0.5$. Process B is sometimes “on” and sometimes “off”,
- *alt. C*: rates $\kappa_C = 1$ and $\mu_C = 10$ and $p_{\text{on,C}} = 0.09$. Process B has large intermittencies and short load pulses.

Note that processes A and B hardly fulfil the condition for a process with short pulses. It can be easily shown that $P_A(T_{\text{ren}} < T_{\text{on}}) = \kappa_A/(\kappa_A + \mu_A) = 0.91$ and $P_B(T_{\text{ren}} < T_{\text{on}}) = 0.5$. The condition is perhaps satisfied for process C only. The dashed lines in Fig. 7 indicate the results obtained by (36), solid lines by (33) and circles ‘o’ by simulations.

The MC simulations are in agreement with (33). The probability of „on“-state $p_{\text{on}}(t)$ proves to be time-invariant when random initial conditions are satisfied. The non-stationary probability of „on“-state (36) provided in the literature fails to describe alternatives A and C for lower t. It appears that for a larger time (in this case approximately $t > 5$), the “non-stationary” effects involved in (36) vanish, and formula (36) corresponds well to the MC simulations.

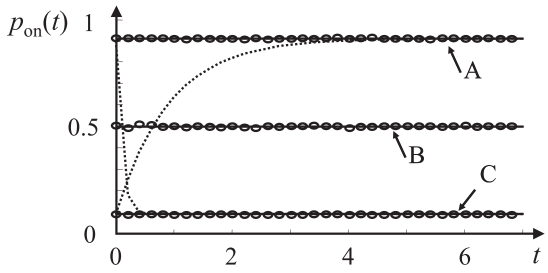

Fig. 7: Probability of “on”-state $p_{\text{on}}(t)$

As already mentioned, formulas (33), (34) and (35) for $p_{\text{on}}(t)$ are derived assuming “short” load pulses, $P(T_{\text{on}} > T_{\text{ren}}) \sim 0$. It is foreseen that deriving of the probability of „on“-state for a process with general load pulses is a much more difficult task. However, the preliminary numerical results partly presented in Fig. 7 for alternatives A and B indicate that the derived formulas remain valid for processes with arbitrarily long pulses.

8 Concluding remarks

It is indicated that the cumulative distribution function $F_{T0}(t)$ of the first renewal T_0 obtained in the paper can suitably be used to simulate random initial conditions of the renewal process. The expected number of renewals $E[N(0, T)]$ of the renewal process considered here is shown to be independent of the initial conditions. Using the newly derived initial probability of „on“-state $p_{\text{on}}(0)$ and a two-state Markov process, the probability of „on“-state $p_{\text{on}}(t)$ at an arbitrary point in time then proves to be time-invariant under correctly defined random initial conditions.

It is foreseen that the investigated model based on renewal processes will become a useful tool in probabilistic structural design, particularly in applications of time-variant reliability analysis. As the formula obtained for $p_{\text{on}}(t)$ is derived considering a process with “short” load pulses, future research should focus on deriving $p_{\text{on}}(t)$ for a general process with arbitrarily long pulses.

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Ing. Miroslav Sýkora, Ph.D.
e-mail: sykora@klok.cvut.cz

Department of Structural Reliability

Czech Technical University in Prague
Klokner Institute,
Šolínova 7
166 08 Prague 6, Czech Republic

